Resuming perturbative invariants of hyperbolic knots

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Based on **arXiv:2410.20973** joint with C. Wheeler

Observables in gauge theory and gravity

IPhT, Université Paris-Saclay, 11 December 2025

Plan of the talk

1. Quick overview of Borel summation

2. Perturbative invariants for hyperbolic knots

3. Homology theory relative to the dilogarithm

4. Conclusions

Borel summability

Borel resuming a Gevrey-1 divergent $\tilde{\Upsilon}$ series allows constructing an analytic function, the Borel sum, which is asymptotic to $\tilde{\Upsilon}$

$$\tilde{\Upsilon} \in \mathbb{C}[\![\tau^{-1}]\!] \xleftarrow{\text{Asymptoics}} \textbf{???}$$

Borel summability

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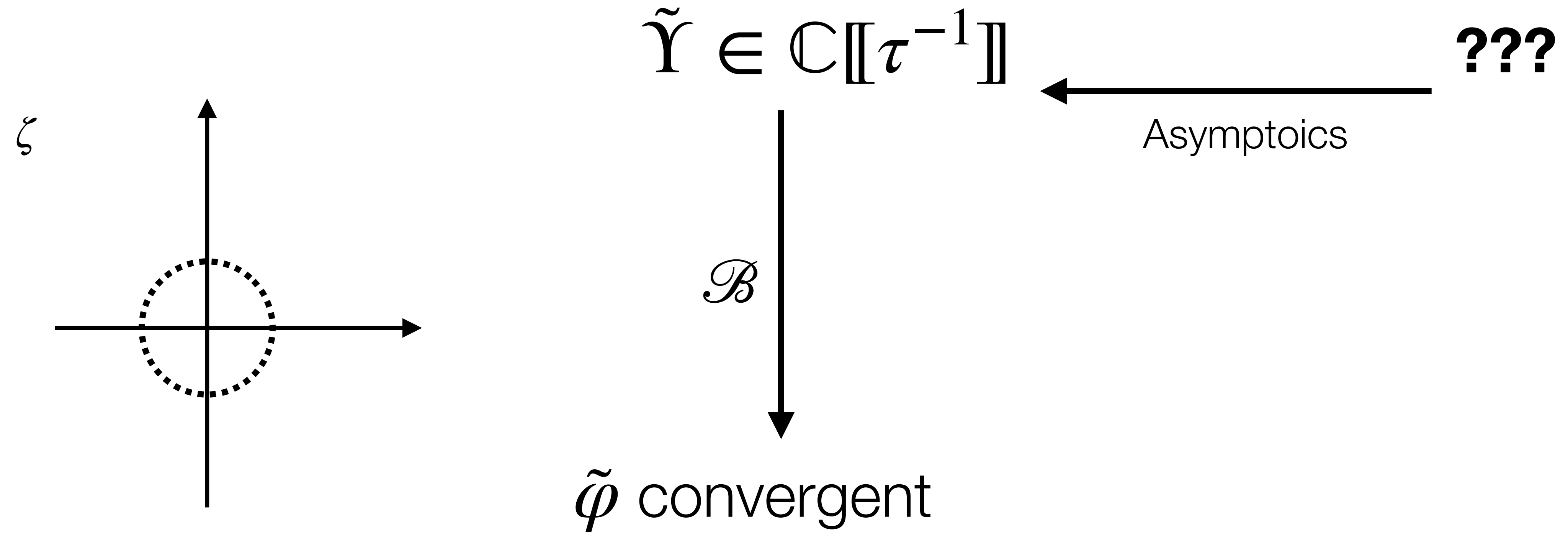

The **Borel transform** $\mathscr{B}: \mathbb{C}[[\tau^{-1}]] \mapsto \mathbb{C}[[\zeta]]$ is defined as $\mathscr{B}: \sum_{n=1}^{\infty} a_n \tau^{-n} \longmapsto \sum_{n=1}^{\infty} a_n \frac{\zeta^{n-1}}{(n-1)!}$

A divergent series is **Gevrey-1** if and only if its Borel transform is convergent

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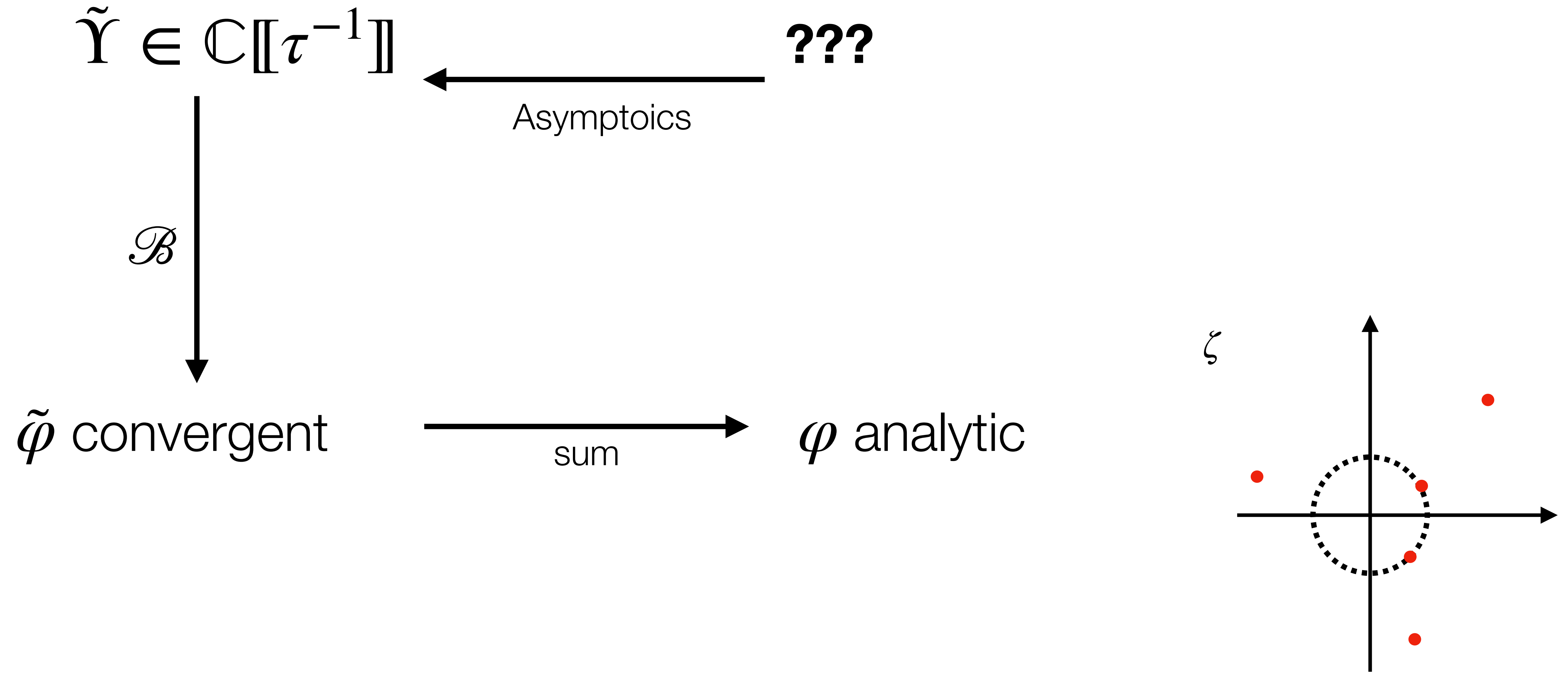

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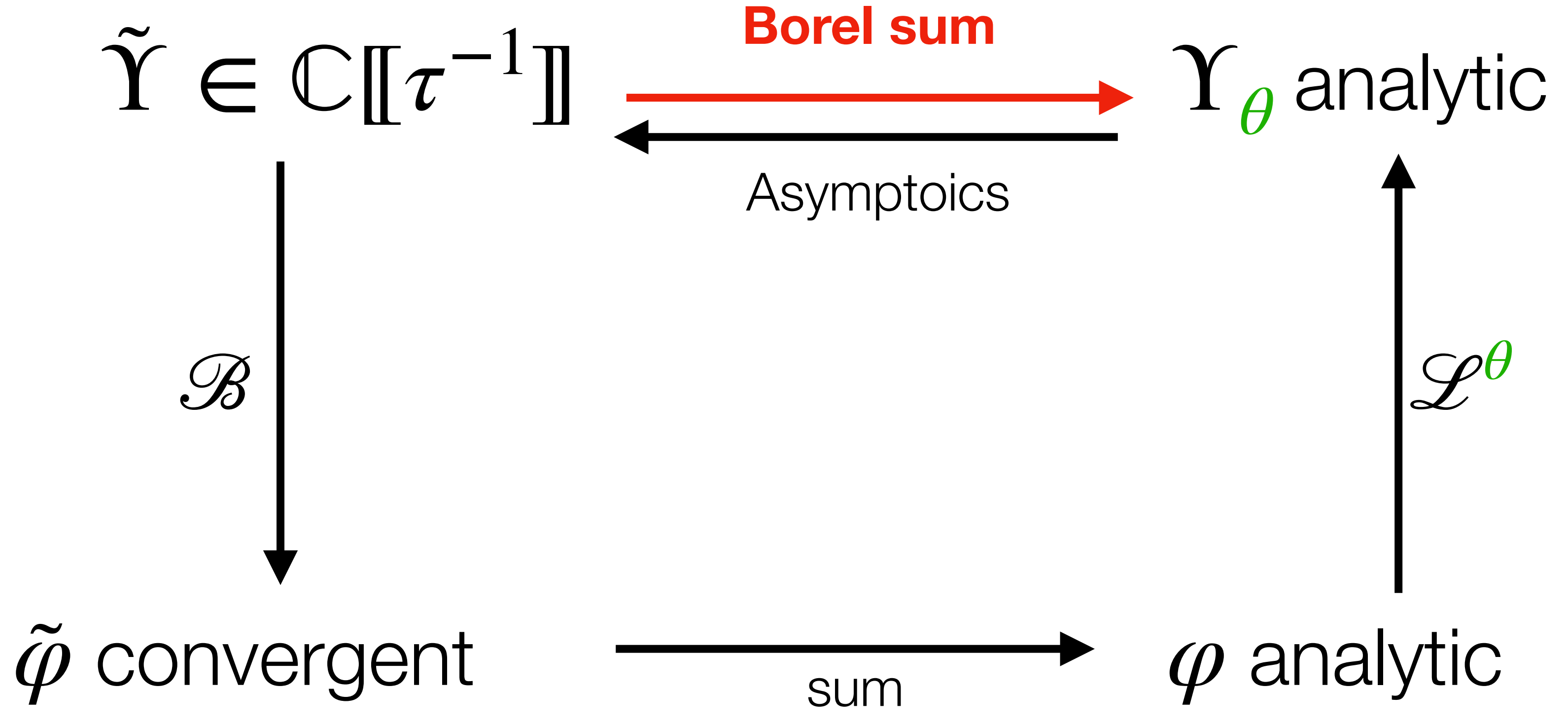
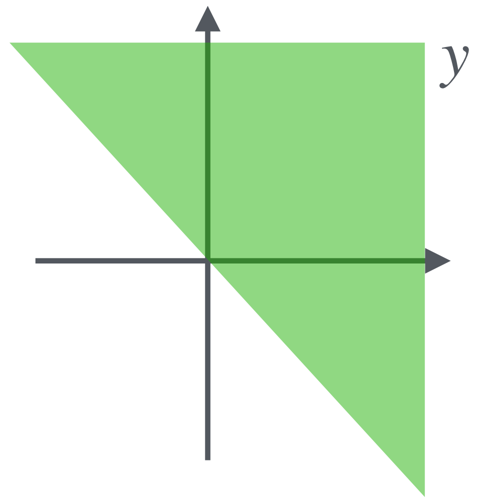
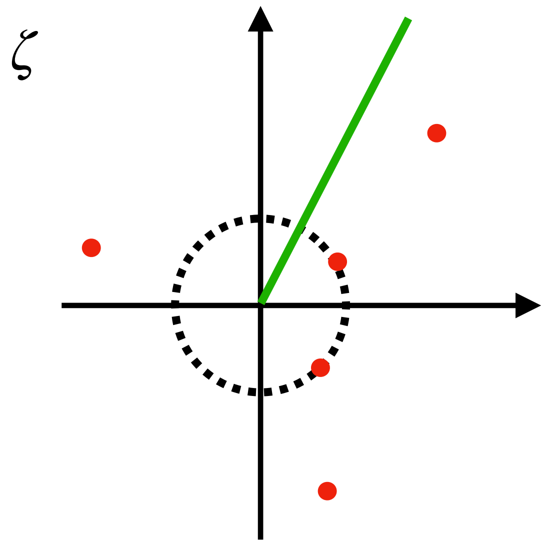




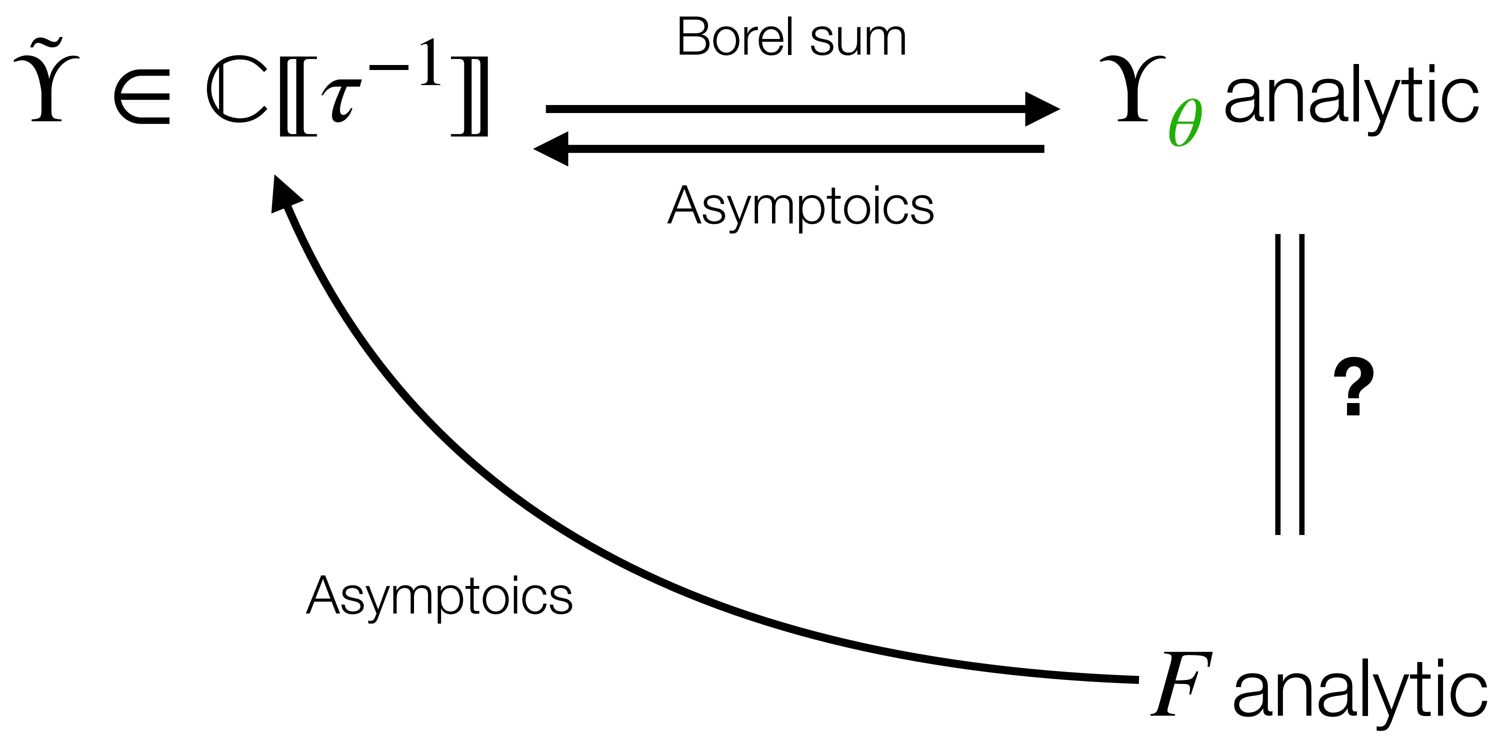


The **Laplace transform** in the direction θ is $\mathscr{L}^\theta\varphi(\tau) := \int_0^{e^{i\theta}\infty} e^{-\tau\zeta}\varphi(\zeta)\, d\zeta$

Varying the direction θ, the function Υ_θ might jump: $\Upsilon_{\theta_+} - \Upsilon_{\theta_-} = S\,\Upsilon_{\theta_-}$ for some constant $S \in \mathbb{C}$ called the **Stokes constant**

Effectiveness of Borel summability

An analytic function F, whose asymptotic expansion is a divergent series $\tilde{\Upsilon}$, is **Borel regular** if it agrees with the Borel sum of $\tilde{\Upsilon}$



Thm **[Nevanlinna 1918]** An analytic function F is Borel regular if and only if it is uniformly Gevrey asymptotic to a divergent series $\tilde{\Upsilon}$ in a domain of opening angle $\geq \pi$

The q-Pochhammer symbol $(q\,z;q)_\infty$, $q = \exp(2\pi \mathrm{i}\,\tau)$ is not Borel regular as $\tau \to 0$ with $\mathrm{Im}(\tau) > 0$

Perturbative invariants for hyperbolic knots

Observables as exponential integrals

Sometimes, in mathematical physics, the quantities of interest are of the following form

$$\int e^{-\tau f(\mathbf{z})} \varphi(\mathbf{z}, \tau) \, d\mathbf{z}$$

$f\colon X \to \mathbb{C}$ is an **algebraic map** on the complex **n-dimensional** variety X

$\varphi(\mathbf{z}, \tau)$ is analytic in $\mathbf{z} \in X$ and $\tau \in \mathbb{C}$

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Then one can ask:

What is a **good contour** of integration?

What is the **behaviour for large** τ ?

Thimble integrals

Assuming

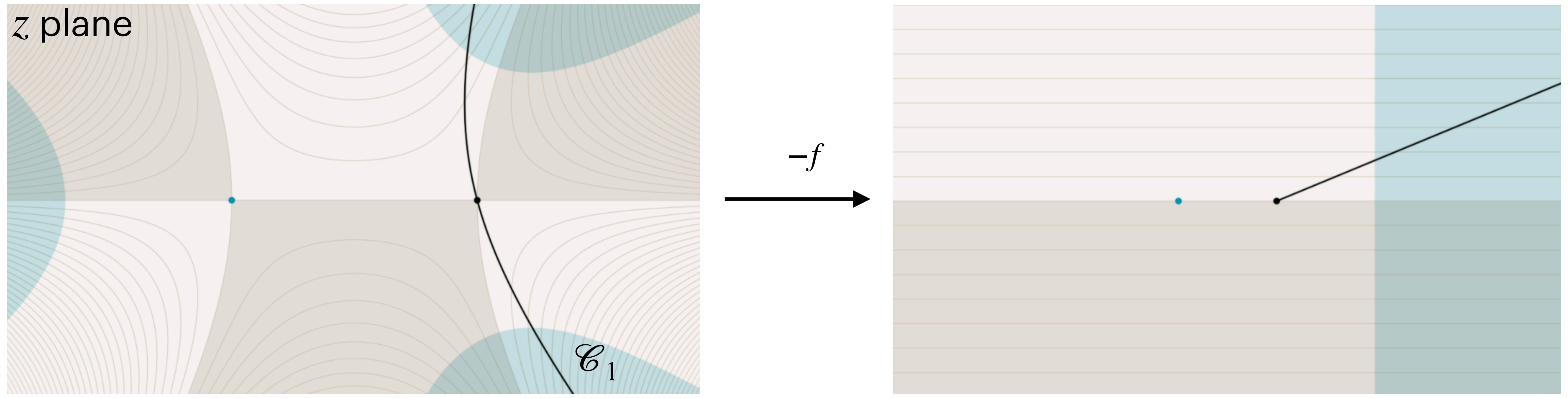
$$\varphi(\mathbf{z},\tau) \sim \tilde{\varphi}_0(\mathbf{z}) + \frac{\tilde{\varphi}_1(\mathbf{z})}{\tau} + \frac{\tilde{\varphi}_2(\mathbf{z})}{\tau^2} + \dots \text{ with } \tilde{\varphi}_j(\mathbf{z}) \text{ being analytic in } \mathbf{z} \in X,$$

a natural contour of integration is a **thimble** $\mathscr{C}_p$ through a critical point p of the potential function f. Indeed, the **thimble integral**

$$F_p(\tau) = \int_{\mathscr{C}_p} e^{-\tau f(\mathbf{z})}\, \varphi(\mathbf{z},\tau)\, d\mathbf{z}$$

is a well-defined **analytic function of τ**

Example: the Airy function $\quad \mathrm{Ai}(\tau) = \int_{\mathscr{C}_1} e^{-\tau\left(\frac{z^3}{3} - z\right)}\, dz$



Effectiveness of Borel summation for thimble integrals

The asymptotic behaviour of $F_p(\tau)$ for large τ is a **divergent power series**

- If the function $\varphi(\mathbf{z}, \tau)$ **does not depend on** τ, then the thimble integral $F_p(\tau)$ **agrees with the Borel resummation** of its asymptotics for large τ **[VF-Fenyes 23, Kontsevich-Soibelman 24]**

The thimbles represent **homology classes relative to** f **:** $\mathscr{C}_p \in H_n(X, f)$

- If the function $\varphi(\mathbf{z}, \tau)$ has an **asymptotic divergent series for large** τ that for every fixed $\mathbf{z} \in X$ is **Borel summable**, then the thimble integral $F_p(\tau)$ **agrees with the Borel resummation** of its asymptotic expansion for large τ **[Andersen-VF-Kontsevich-Wheeler** ***in progress*****]**

The thimbles represent **homology classes relative to** f **with coefficients in a sheaf** $\mathscr{V}$ **:** $\mathscr{C}_p \in H_n(X, f; \mathscr{V})$

Observables as exponential integrals with multivalued potential

Often, in mathematical physics, the quantities of interest are of the following form

$$\int e^{-\tau f(\mathbf{z})}\, \varphi(\mathbf{z},\tau)\, d\mathbf{z}$$

$f\colon X \dashrightarrow \mathbb{C}$ is a **multivalued function** on the complex **n-dimensional** variety X

$\varphi(\mathbf{z},\tau)$ is analytic in $\mathbf{z} \in X$ and $\tau \in \mathbb{C}$

Then one can ask:

What is a **good contour** of integration?

What is the **behaviour for large** τ ?

Thimble integrals with multivalued potential

Assuming

$$\varphi(\mathbf{z},\tau) \sim \tilde{\varphi}_0(\mathbf{z}) + \frac{\tilde{\varphi}_1(\mathbf{z})}{\tau} + \frac{\tilde{\varphi}_2(\mathbf{z})}{\tau^2} + \ldots \text{ with } \tilde{\varphi}_j(\mathbf{z}) \text{ being analytic in } \mathbf{z} \in X,$$

a natural contour of integration is a **thimble** $\mathscr{C}_p$ through a critical point p of the holomorphic function $f^*: \Sigma \to \mathbb{C}$ from a Riemann surface Σ. Indeed, the **thimble integral**

$$G_p(\tau) = \int_{\mathscr{C}_p} e^{-\tau f^*(\mathbf{z})} \varphi(\mathbf{z},\tau)\, d\mathbf{z}$$

is a well-defined **analytic function of** τ

Effectiveness of Borel summation for thimble integrals with multivalued potential

The asymptotic behaviour of $G_p(\tau)$ for large τ is a **divergent power series**

- Independently of the asymptotic behaviour of the function $\varphi(\mathbf{z}, \tau)$ for large τ, the thimble integral $G_p(\tau)$ does **not always agree with the Borel resummation** of its asymptotics for large τ

The thimbles represent **homology classes with coefficients in a sheaf** $\mathscr{V}$ **:** $\mathscr{C}_p \in H_n(X; \mathscr{V})$

- There might be cycles not associated with critical values of the potential f

[Andersen-VF-Kontsevich-Wheeler *in progress***]**

Examples

- Scattering phase in 2D string theory **[Alexandrov-Kaushik 25]**
- Hemisphere partition functions in GLSM for hypersurfaces in $\mathbb{P}^N$ **[Knapp-Romo-Scheidegger 16]**
- Exact WKB **[Aoki-Kawai-Takei 01]**
- Feynman integrals in Baykov representation **[Angius-Cacciatori-Massidda 25]**
- Fermionic spectral traces from the mirror curve of toric Calabi-Yau 3-folds **[Kashaev-Mariño 15]**
- Andersen-Kashaev state integrals **[Andersen-Kashaev 13]**
- …

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- **Andersen-Kashaev state integrals** **[Andersen-Kashaev 13]**
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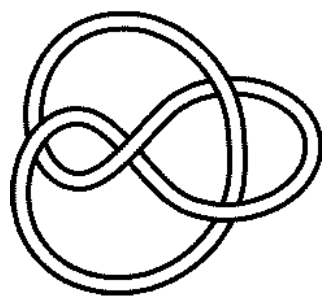
Andersen-Kashaev (AK) state integrals of hyperbolic knots

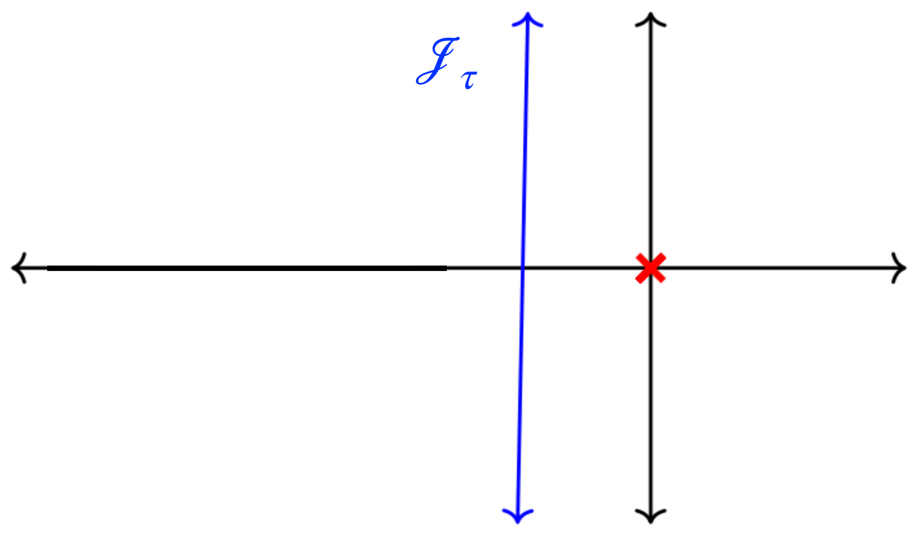
Given a hyperbolic knot K, the **AK state integral** I_K is the partition function of a 3d Teichmüller TQFT, which conjecturally describes $\mathrm{SL}_2(\mathbb{C})$ Chern-Simons theory on $S^3 \setminus K$ and conjecturally for large τ

$$I_K(\tau) \sim \exp(-\mathrm{Vol}(S^3 \setminus K)\,\tau)\ldots$$ **AK Volume conjecture**

For the figure-eight knot 4_1, the AK integral is



$$I_{4_1}(\tau) \;=\; e^{\frac{\pi i}{4} - \frac{\pi i}{6\tau} + \frac{\pi i \tau}{6}} \int_{\mathcal{J}_\tau} \Phi(z\,\tau;\tau)^2\, \mathbf{e}\Big(\frac{1}{2} z(z\tau + \tau + 1)\Big) dz$$

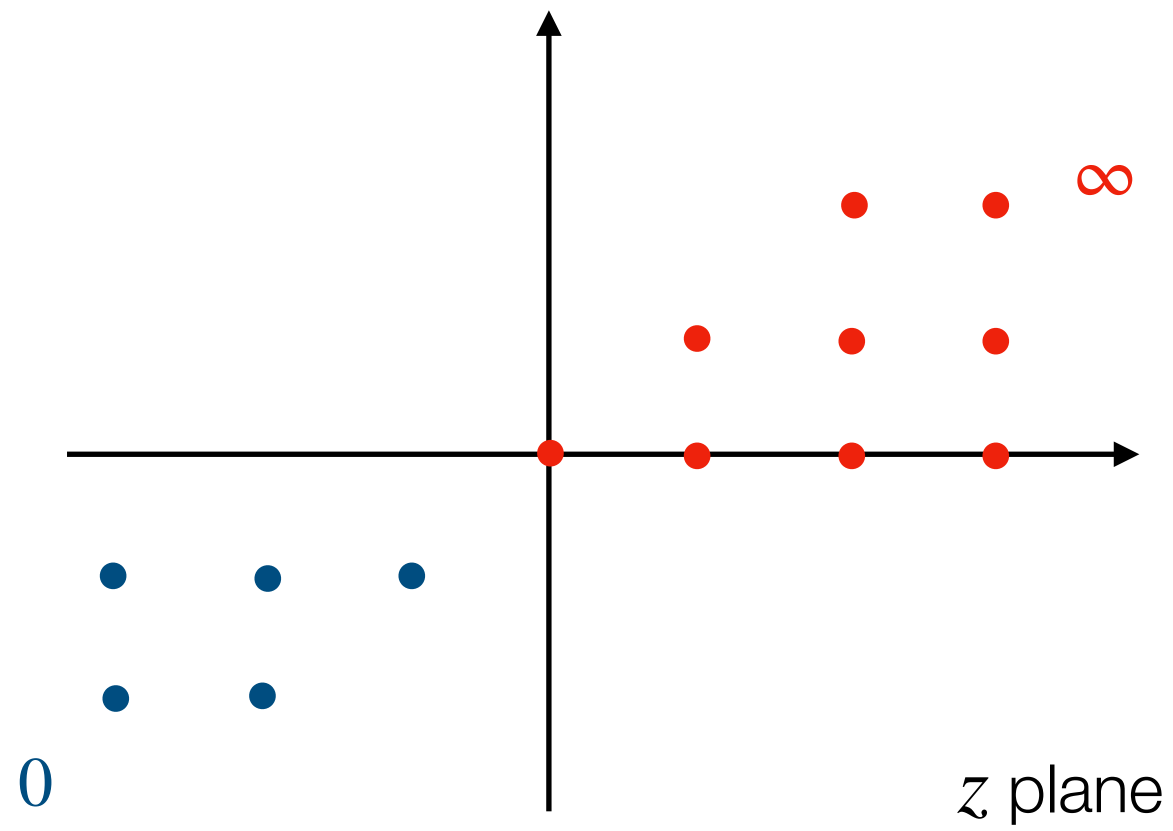


where $\Phi(z;\tau)$ is **Faddeev's quantum dilogarithm** and $\mathbf{e}(x) = \exp(2\pi i x)$

Faddeev's quantum dilogarithm

Faddeev's quantum dilogarithm is defined as

$$\Phi(z;\tau) = \exp\left(\int_{\mathrm{i}\sqrt{\tau}\mathbb{R}+\varepsilon\sqrt{\tau}} \frac{\mathbf{e}((z+1+\tau)w/\tau)}{(\mathbf{e}(w)-1)(\mathbf{e}(w/\tau)-1)} \frac{dw}{w} \right)$$



It is a meromorphic function of $\tau \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$ and $z \in \mathbb{C}$

Its **asymptotic expansion** for large τ

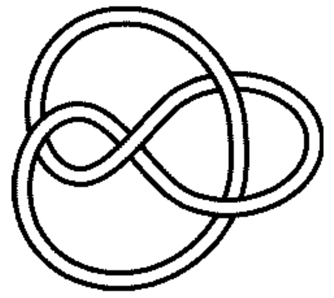
$$\tilde{\Psi}(z,\tau) = \mathbf{e}\left(\frac{\pi \mathrm{i}}{4} - \frac{\tau}{24} - \frac{1}{24\tau} - \frac{\tau}{(2\pi\mathrm{i})^2}\mathrm{Li}_2(\mathbf{e}(z)) - \sum_{k=1}^{\infty} (2\pi\mathrm{i})^{k-2} \frac{B_k}{k!} \mathrm{Li}_{2-k}(\mathbf{e}(z))\tau^{1-k} \right)$$

and $\mathrm{Li}_2(\mathbf{e}(z))$ **is multivalued**, with branch points at $z \in \mathbb{Z}$ and monodromy $2\pi i m \log(z) + (2\pi\mathrm{i})^2 n$, with $m, n \in \mathbb{Z}$

Perturbative invariants of hyperbolic knots

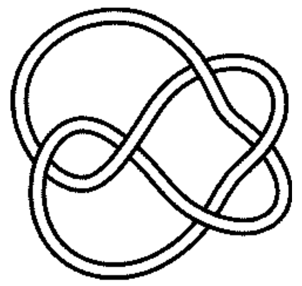
Given a **hyperbolic knot** K, the **perturbative invariants** $\tilde{\Upsilon}_K$ defined by **[Dimofte-Garoufalidis 13]** and proved to be topological invariants by **[Garoufalidis-Strozer-Wheeler 22]** are constructed from the data of an ideal triangulation of the knot complement (the so-called Neumann-Zagier data)

For example, for the simplest hyperbolic knots 4_1 and 5_2 the formal invariant $\tilde{\Upsilon}_K$ is given as follows



$\mathbf{4_1}$: $(A = 1\,, B = 2)$

$$\tilde{\Upsilon}_K(\tau) \;:=\; \int \tilde{\Psi}(z;\tau)^B \exp\!\left(\frac{A}{2}z^2\tau\right) dz \;=\; \int \sum_{k=0}^{\infty} a_k(z)\,\tau^{-k}\,\mathbf{e}\big(V(z)\tau\big)\,,$$



$\mathbf{5_2}$: $(A = 2\,, B = 3)$

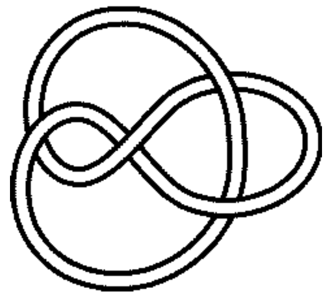
where $V(z) = B\,\dfrac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2} + \dfrac{B}{24} + \dfrac{A}{2}z^2$

For every critical point $x = \exp(z_{\mathrm{crit}})$ of the function V, we get a formal power series $\tilde{\Upsilon}_{K,x}(\tau)$ by doing formal Gaussian integration

In particular, the **critical points** x are solutions of $(1-x)^B = x^A$ and correspond to the $\mathrm{SL}_2(\mathbb{C})$ **geometric flat connections**

P vs NP invariants of hyperbolic knots

For the hyperbolic knots 4_1 and 5_2 we have two sets of invariants



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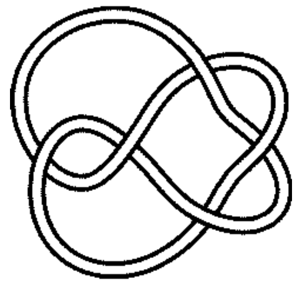

$\mathbf{5_2}$: $(A = 2\,, B = 3)$

Perturbative invariants

$$\tilde{\Upsilon}_K(\tau) := \int \tilde{\Psi}(z;\tau)^B\, \mathbf{e}\Big(\frac{A}{2} z^2 \tau\Big)\, dz$$

AK state integrals

$$I_{K,m,\ell}(\tau) := \int_{\mathscr{J}_{\tau,\ell}} \Phi((z-\ell)\tau;\tau)^B\, \mathbf{e}\Big(\frac{A}{2} z(z\tau + \tau + 1) + mz\tau\Big) dz\,,$$

with $m = 0,\ldots, A-1$ et $\ell \in \mathbb{Z}$

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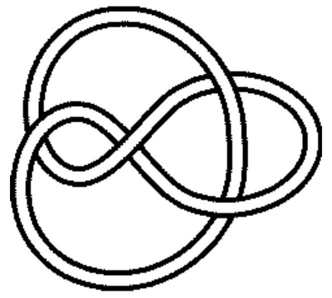

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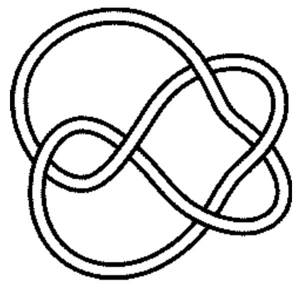

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with $m = 0,\ldots, A-1$ et $\ell \in \mathbb{Z}$

$$\tilde{\Psi}(z;\tau) \;\underset{\text{asymptotics}}{\overset{\text{Borel sum}}{\rightleftarrows}}\; \Phi(z;\tau)$$

Thm **[Kashaev-Garoufalidis 20]** Faddeev's quantum dilogarithm agrees with the Borel sum of its asymptotic expansion for large τ

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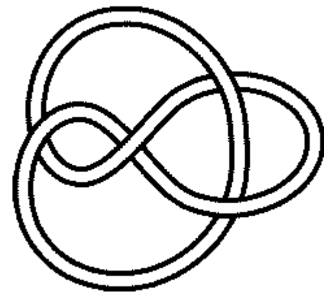

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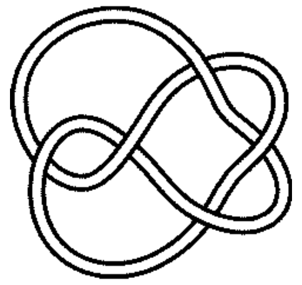

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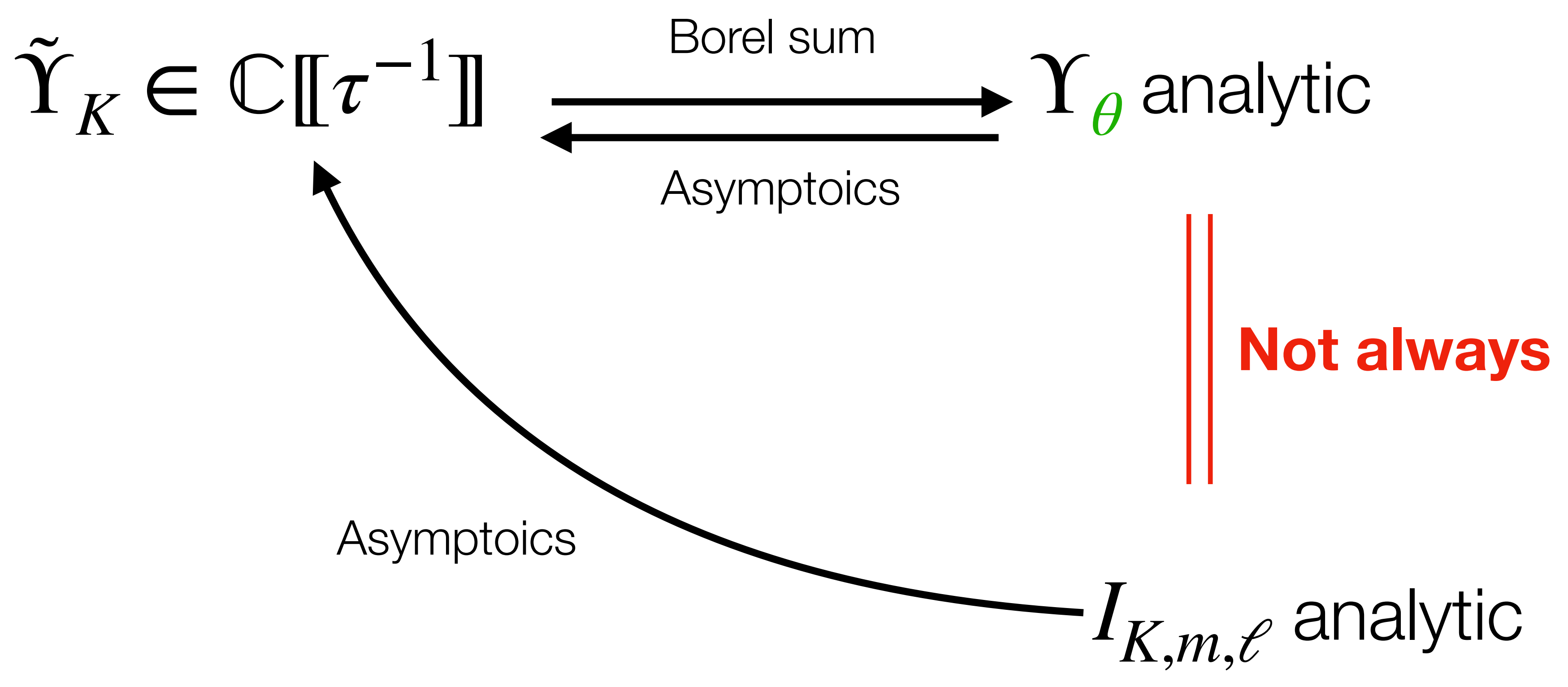
with $m = 0,\ldots, A-1$ et $\ell \in \mathbb{Z}$

$$\tilde{\Psi}(z;\tau) \underset{\text{asymptotics}}{\overset{\text{Borel sum}}{\rightleftarrows}} \Phi(z;\tau)$$

Conj **[Garoufalidis-Gu-Mariño 21]** For every critical point x, the Borel resummation of the perturbative invariants $\tilde{\Upsilon}_{K,x}$ is a linear combination of the AK integral I_K and its descendants $I_{K,m,\ell}$

Main result

Thm **[VF-Wheeler 24]** The conjecture by GGM21 holds for the knots 4_1 and 5_2



Main result

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The functions $I_{K,m,\ell}$ **are not thimble integrals**; indeed, the contours $\mathscr{J}_{\tau,\ell}$ are not of steepest descent for the function

$$V(z) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z))}{(2\pi \mathrm{i})^2} + \frac{B}{24} + \frac{A}{2} z(z+1)$$

However, they form a basis for a ***relative*** **homology with coefficients**, which contains the class of the thimbles

Hence, we define an **algorithm to decompose the thimble** into state integral contours $\mathscr{J}_{\tau,\ell}$, allowing us to compute the Stokes constants

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V **is multivalued**

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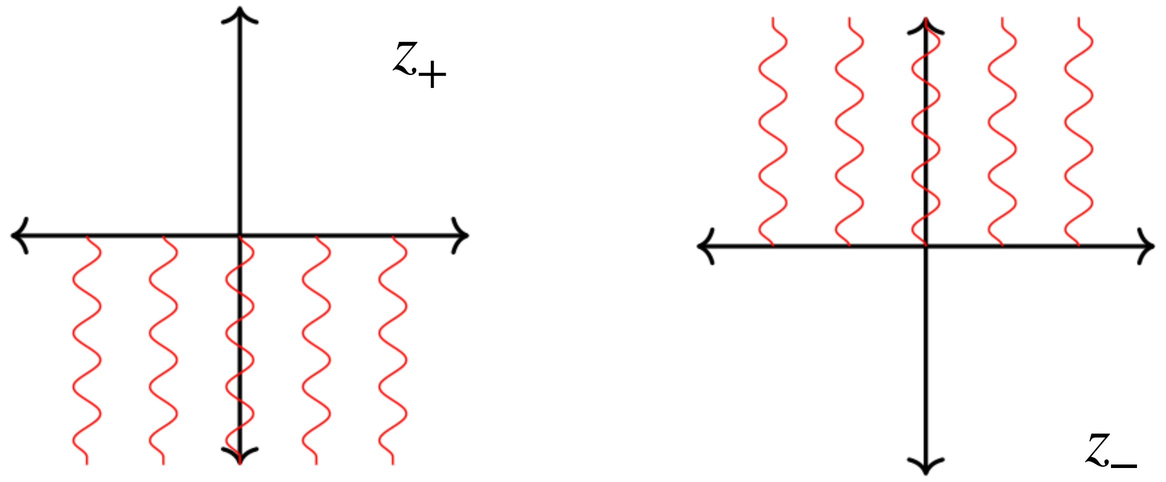
Hence, we define an **algorithm to decompose the thimble** into state integral contours $\mathcal{J}_{\tau,\ell}$, allowing us to compute the Stokes constants

The **system of coefficients** is given by the **Stokes phenomenon** of the **Faddeev's dilogarithm**

A homology theory relative to the dilogarithm

A homology theory for thimbles associated with Li_2

The Riemann surface of V



Fix a branch of $\mathrm{Li}_2(\mathbf{e}(z))$ and restrict the function V to the Riemann surface Σ

$$V(z_+, m, n) = B\,\frac{\mathrm{Li}_2(\mathbf{e}(z_+))}{(2\pi \mathrm{i})^2} + \frac{B}{24} + \frac{A}{2}z_+^2 + m\,z_+ + n\,,$$

$$\Lambda(z_-, m, n) = B\frac{\mathrm{Li}_2(\mathbf{e}(-z_-))}{-(2\pi i)^2} + \frac{B}{12} - \frac{B}{2}\left(z_- - \frac{1}{2}\right)^2 + \frac{A}{2}z_-^2 + m\,z_- + n\,,$$

where $m = 0,\ldots, A-1$ and $n \in \mathbb{Z}$

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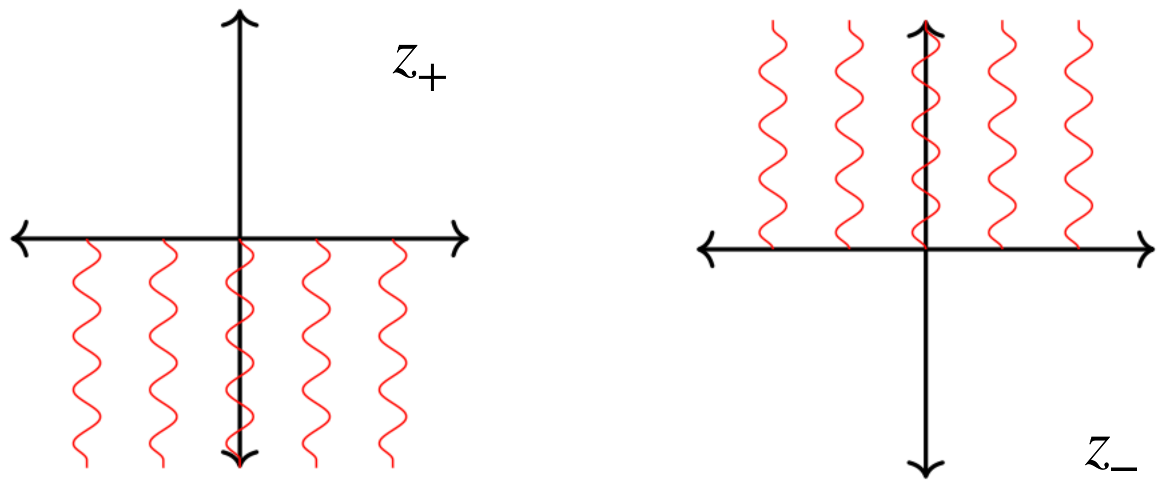

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where $m = 0,\ldots, A-1$

Critical points of the function $V : \Sigma \to \mathbb{C}/\mathbb{Z}$ are solutions of an algebraic equation $x^A = (1-x)^B$ and $x = \mathbf{e}(z)$

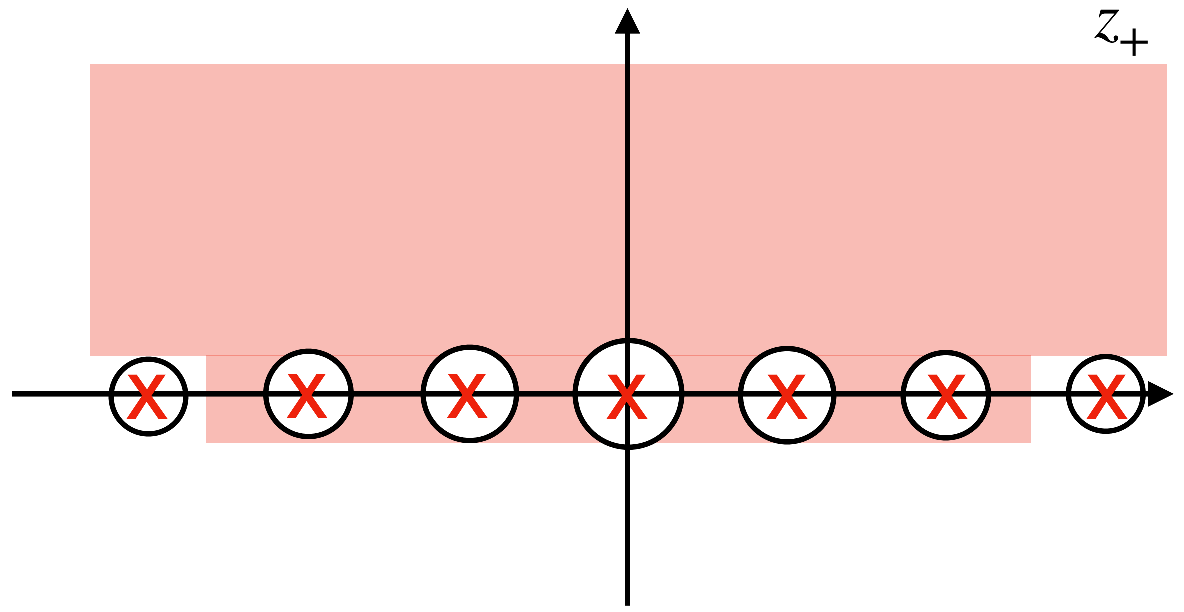
A homology theory for thimbles associated with Li_2

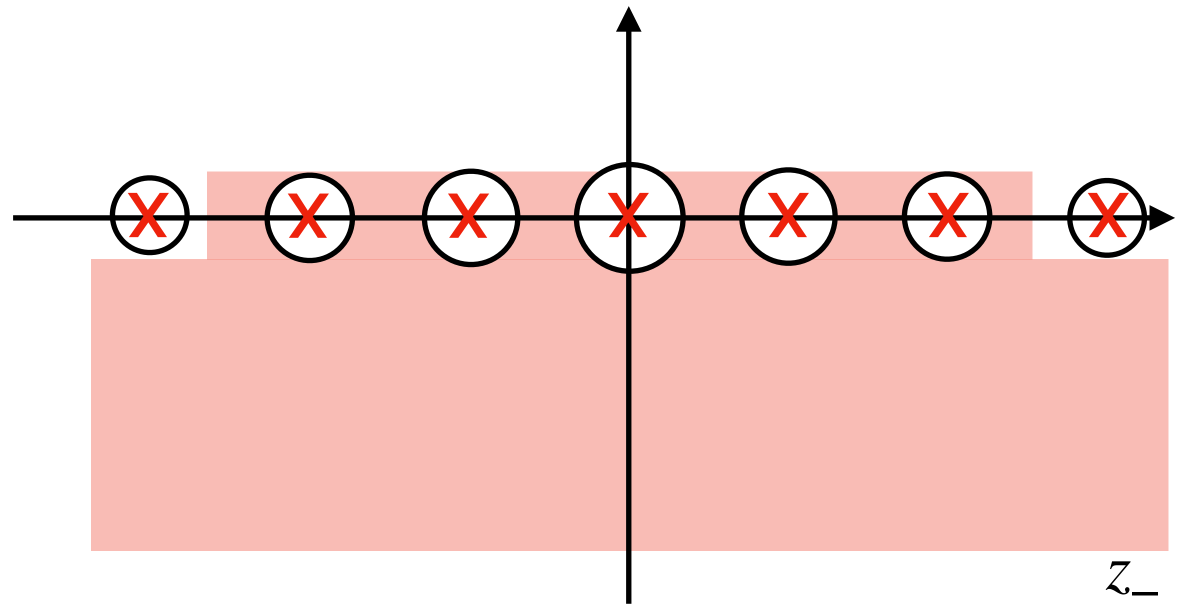
A relative homology for $V\colon \Sigma \to \mathbb{C}/\mathbb{Z}$

For z large enough, the function V (resp. Λ) is dominated by the **Gaussian term** $f_+(z) = \frac{A}{2}z^2$ (resp. $f_-(z) = \frac{(A-B)}{2}z^2$)

For generic directions $\vartheta \in [0,2\pi)$, the steepest descent contours of V (resp. Λ) are ϵ-bounded away from the branch points

Define the surface $X_{M,\epsilon} \subset \Sigma$ by gluing different coordinate charts





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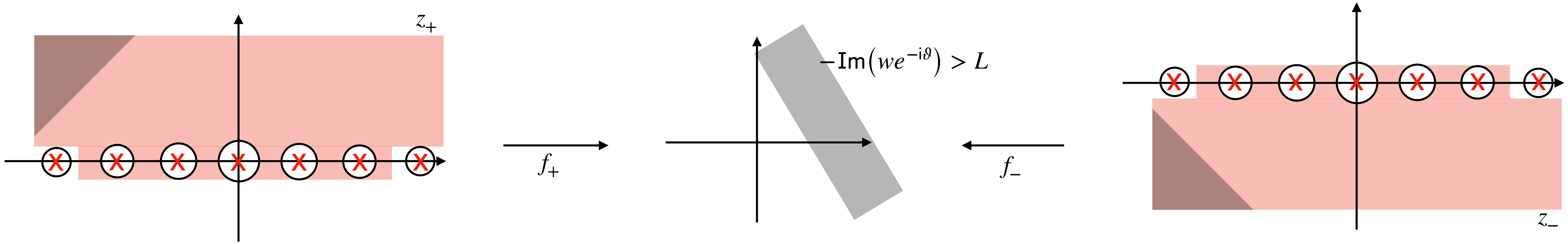


Proposition **[VF-Wheeler 24]** The relative homology $H_1(X_{M,\epsilon}, f)$ is finite-dimensional, and a basis is given by the state integrals contours $\mathcal{J}_{\ell,\tau}$ with $\arg(\tau) = \vartheta$

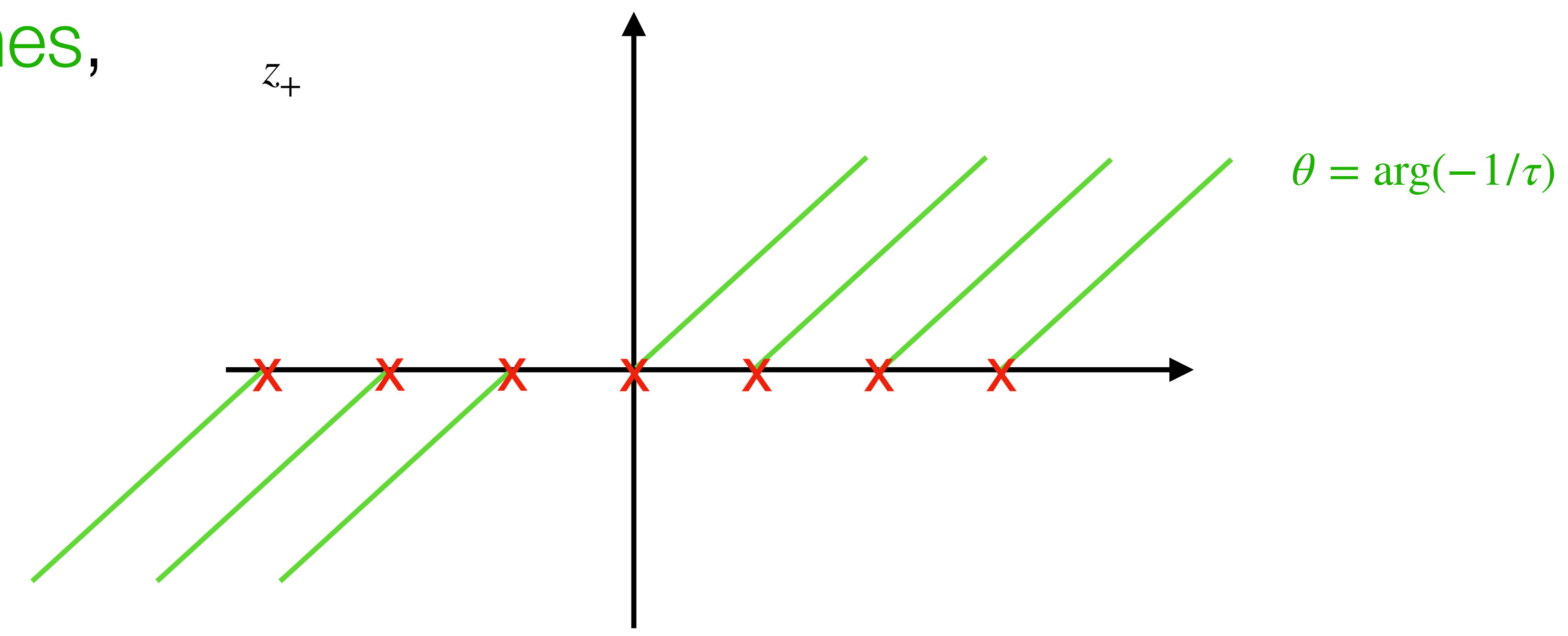
A homology theory for thimbles associated with Li_2

The system of coefficients

The integral defining the formal series $\tilde{\Upsilon}_K$ depends on τ both the exponential term $\mathbf{e}(\tau\, V(z))$ and the 1-form

$$\exp\Big(-\mathrm{Li}_1(\mathbf{e}(z)) - \sum_{k=2}^{\infty} (2\pi \mathrm{i})^{k-1} \frac{B_k}{k!} \mathrm{Li}_{2-k}(\mathbf{e}(z))\, \tau^{1-k} \Big)\, dz \in \Omega^1(\Sigma)[\![\tau^{-1}]\!]$$

The Faddeev's quantum dilogarithm $\Phi(z;\tau)$ jumps crossing these green lines,



Build a sheaf $\mathscr{V} \to \Sigma$ that includes these contributions (sheaf of resurgent structure), and define the sheaf homology $H_1(\Sigma, \mathscr{V})$
[Andersen-VF-Kontsevich-Wheeler *in progress***]**

Conclusions

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Perturbative topological invariants of the hyperbolic knots 4_1 and 5_2 are described by **one-dimensional** integrals $\tilde{\Upsilon}_{K,x}$

- Their Borel sum $\Upsilon_{K,x,\vartheta}$ is a **thimble integral** for the **multivalued function** V
- The thimbles give classes in a **relative homology theory with coefficients**
- The **Stokes jumps** of Faddeev's quantum dilogarithm define the **sheaf of coefficients**
- A **basis** for the homology is given by AK state integrals and their descendants

- Extend the result to higher-dimensional integrals to include more examples of hyperbolic knots and beyond the dilogarithm function **[Andersen-VF-Kontsevich-Wheeler** *in progress***]**
- Application to Feynman integrals for the computation of master integrals

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Thank you for your attention