Gravitational-Wave Observables: PM expansion vs soft theorems

Observables in Gauge Theory and Gravity, IPhT, *Saclay*

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Based on:

- 2306.16488: Report on the gravitational eikonal

  Paolo Di Vecchia, CH, Rodolfo Russo, Gabriele Veneziano
- 2312.07452, 2402.06361: Analysis of the NLO waveform

  Alessandro Georgoudis, CH, Rodolfo Russo
- 2407.04128: Logarithmic soft theorems and soft spectra

  Francesco Alessio, Paolo Di Vecchia, CH
- 2506.20733: Nonlinear Gravitational Memory in the Post-Minkowskian Expansion

  Alessandro Georgoudis, Vasco Goncalves, CH, Julio Parra-Martinez

Outline

Introduction

Warm-Up: Elastic Eikonal and Deflection Angle

Eikonal Operator and Gravitational Waveform

- Soft Limit

Soft Theorems, Soft Energy Spectrum

Nonlinear Memory from Amplitudes

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Two-Body Problem: Analytical Approximation Methods

- Post-Newtonian (PN): expansion
  "for small G and small v"

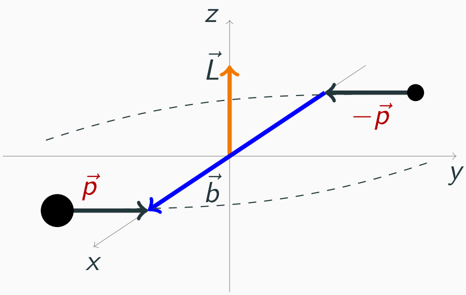
$$\frac{Gm}{rc^2} \sim \frac{v^2}{c^2} \ll 1\,.$$

- Post-Minkowskian (PM): expansion
  "for small G"

$$\frac{Gm}{rc^2} \ll 1\,, \qquad \text{generic } \frac{v^2}{c^2}\,.$$

- Self-Force: expansion
  in the near-probe limit $m_2 \ll m_1$ or

$$m = m_1 + m_2\,, \qquad \nu = \frac{m_1 m_2}{m^2} \ll 1\,.$$



- Soft limit: expansion
  in the limit of small frequencies

$$\omega \ll \frac{v}{r}\,.$$

General Relativity from Scattering Amplitudes

Key Idea: Extract the PM gravitational dynamics from scattering amplitudes.

- Weak-coupling expansion $\leftrightarrow$ PM expansion

Weak-coupling:	$\mathcal{A}_0 = \mathcal{O}(G)$	$\mathcal{A}_1 = \mathcal{O}(G^2)$	$\mathcal{A}_2 = \mathcal{O}(G^3)$	$\cdots$	state of the art
PM:	1PM	2PM	3PM	$\cdots$	5PM

[Plefka et al.'24, Bern et al.'24,'25]

- Lorentz invariance $\leftrightarrow$ generic velocities
- Study scattering events, then export to bound trajectories (V_{eff}, analytic continuation...) [Kälin, Porto '19; Saketh, Steinhoff, Vines, Buonanno '21; Cho, Kälin, Porto '21]

Classical Soft Theorems

[Sahoo, Sen '18; Saha, Sahoo, Sen '19; Sahoo '20; Sahoo, Sen '21; Sahoo, Krishna '23]

- Universal constraints on the soft expansion $\omega \to 0$ of the gravitational waveform:

$$\tilde{w}^{\mu\nu} = -\frac{i}{\omega}\,\omega^{2iGE\omega}\sum_{n=0}^{\infty}\frac{1}{n!}\,(-i\omega\,\log\omega)^n\,a_n^{\mu\nu} + \text{subleading logs}$$

- Explicitly, for $\frac{1}{\omega}$, $\log\omega$, $\omega(\log\omega)^2$,

$$a_0^{\mu\nu} = \sum_a \frac{p_a^\mu p_a^\nu}{p_a\cdot n}\,,\qquad a_1^{\mu\nu} = G\sum_{a,b}\frac{\tau_{ab}^{(\eta)}\,p_a^\mu}{p_a\cdot n}\,n_\rho p^\rho_{[b}p^\nu_{a]}\,,\qquad a_2^{\mu\nu} = G^2\sum_{a,b,c}\frac{\tau_{ab}^{(\eta)}\,\tau_{ac}^{(\eta)}}{p_a\cdot n}\,n_\rho p^\rho_{[b}p^\mu_{a]}\,n_\sigma p^\sigma_{[c}p^\nu_{a]}$$

and $\tau_{ab}^{(\eta)}$ is a simple function of the **invariants** $\sigma_{ab} = -\eta_a\eta_b p_a\cdot p_b/(m_a m_b)$ (with $\eta_a = +$ if the hard state is outoing, -1 if it is incoming)

$$\tau_{ab}^{(\eta)} = |\eta_a + \eta_b|\,\tau_{ab}\,,\qquad \tau_{ab} = -\frac{\sigma_{ab}(\sigma_{ab}^2 - \frac{3}{2})}{(\sigma_{ab}^2-1)^{3/2}}\quad \text{for GR}\,.$$

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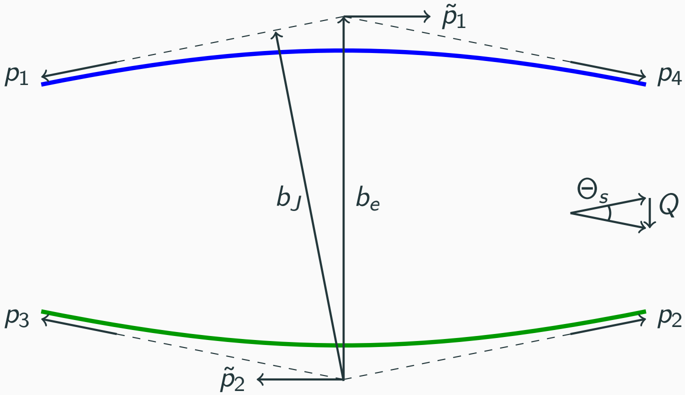
Kinematics of Classical Post-Minkowskian (PM) Scattering

$$\tilde{p}_1^\mu = m_1 \tilde{u}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$

$$\tilde{p}_2^\mu = m_2 \tilde{u}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$

$$Q^\mu = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$

$$b_e^\mu = b_J^\mu - \left(\frac{\check{v}_1^\mu}{2m_1} - \frac{\check{v}_2^\mu}{2m_2}\right) Qb$$

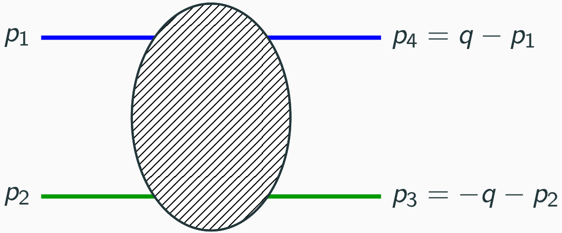


In this way, $v_1 \cdot b_J = v_2 \cdot b_J = 0$ and $\tilde{u}_1 \cdot b_e = \tilde{u}_2 \cdot b_e = 0$. Classical PM regime:

$$\frac{Gm^2}{\hbar} \underset{CL}{\gg} 1\,, \qquad \frac{Gm}{b} \underset{\mathrm{PM}}{\ll} 1\,, \qquad \boxed{\frac{\hbar}{m} \ll Gm \ll b} \qquad \sigma = \frac{1}{\sqrt{1-v^2}} \geq 1 \text{ (generic)}.$$

Kinematics of the Elastic $2 \to 2$ Amplitude

$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$\boxed{q^\mu} = p_1^\mu + p_4^\mu = -p_2^\mu - p_3^\mu$$



Defining velocities by $p_1^\mu = -m_1 v_1^\mu$, $p_2^\mu = -m_2 v_2^\mu$

$$\boxed{\sigma} = -v_1 \cdot v_2 = \frac{1}{\sqrt{1 - v^2}}$$

with v the speed of either object as measured by the other one.

Dual velocities: $v_1^\mu = \sigma \check{v}_2^\mu + \check{v}_1^\mu$, $v_2^\mu = \sigma \check{v}_1^\mu + \check{v}_2^\mu$ obey $\check{v}_i \cdot v_j = -\delta_{ij}$.

The Elastic Eikonal

- From q to b: Fourier transform $[q \sim \mathcal{O}(\frac{\hbar}{b})]$

$$\tilde{\mathcal{A}}^{(4)}(b) = \frac{1}{4Ep} \int \frac{d^{D-2}q}{(2\pi)^{D-2}} \, e^{ib\cdot q} \mathcal{A}^{(4)}(q) \,, \qquad \boxed{1 + i\tilde{\mathcal{A}}^{(4)}(b) = e^{2i\delta(b)}}$$

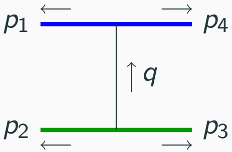
with $2\delta = 2\delta_0 + 2\delta_1 + 2\delta_2 + \cdots \sim \frac{Gm^2}{\hbar} \left(\log b + \frac{Gm}{b} + \left(\frac{Gm}{b}\right)^2 + \cdots \right)$

- From b to Q: stationary-phase approximation $[Q \sim \mathcal{O}(p \cdot \frac{Gm}{b})]$

$$\int d^{D-2}b \, e^{-ib\cdot Q} e^{i2\delta(b)} \implies Q_\mu = \frac{\partial \, \mathrm{Re} \, 2\delta}{\partial b_e^\mu}$$

Tree-Level Amplitude and 1PM Impulse

- Tree-level amplitude in $D = 4 - 2\epsilon$ dimensions



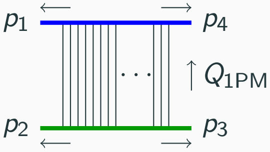
$$\mathcal{A}_0^{(4)}(q) = \frac{32\pi G m_1^2 m_2^2 (\sigma^2 - \frac{1}{2-2\epsilon})}{q^2} + \cdots$$

$$\tilde{\mathcal{A}}_0^{(4)}(b) = \frac{4 G m_1 m_2 (\sigma^2 - \frac{1}{2-2\epsilon})}{2\sqrt{\sigma^2 - 1}} \frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}} .$$

- Matching to the eikonal exponentiation [Kabat, Ortiz '92; Bjerrum-Bohr et al. '18]

$$e^{2i\delta_0} \underset{\text{“small } G\text{”}}{\longrightarrow} 1 + i\tilde{\mathcal{A}}_0^{(4)} \implies 2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} .$$

- From $2\delta_0$, we obtain the leading-order deflection

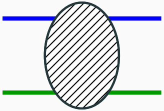


$$Q_{1\text{PM}} = -\frac{\partial 2\delta_0}{\partial b} = \frac{4 G m_1 m_2 \left(\sigma^2 - \frac{1}{2}\right)}{b\sqrt{\sigma^2 - 1}}$$

$$\Theta_{1\text{PM}} = \frac{4 G E \left(\sigma^2 - \frac{1}{2}\right)}{b(\sigma^2 - 1)} .$$

Impulse from the Eikonal Phase up to One Loop

[Collado, Di Vecchia, Russo '19]

$1 + i\,\mathrm{FT}$  $\sim e^{2i\delta}\,, \qquad 2\delta = 2\delta_0 + 2\delta_1 + \cdots \qquad Q_\mu = \dfrac{\partial 2\delta}{\partial b_e^\mu}$

- Tree level: $i\tilde{\mathcal{A}}_0 = 2i\delta_0$, so

$$2\delta_0 = \tilde{\mathcal{A}}_0^{(4)} = \frac{2Gm^2\nu(\sigma^2 - \frac{1}{2-2\epsilon})}{\sqrt{\sigma^2-1}}\frac{\Gamma(-\epsilon)}{(\pi b^2)^{-\epsilon}}\,, \qquad Q^\mu_{1\mathrm{PM}} = -\frac{4Gm^2\nu(\sigma^2-\frac{1}{2})}{b\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

- One loop: By the unitarity, $i\tilde{\mathcal{A}}_1 - \frac{1}{2!}(2i\delta_0)^2 = i\,\mathrm{Re}\,\tilde{\mathcal{A}}_1 = 2i\delta_1$, so

$$2\delta_1 = \mathrm{Re}\,\tilde{\mathcal{A}}_1^{(4)} = \frac{3\pi G^2 m^3 \nu\left(5\sigma^2-1\right)}{4b\sqrt{\sigma^2-1}}\,, \qquad Q^\mu_{2\mathrm{PM}} = -\frac{3\pi G^2 m^3 \nu\left(5\sigma^2-1\right)}{4b^2\sqrt{\sigma^2-1}}\frac{b_e^\mu}{b}\,.$$

The 3PM Eikonal in General Relativity [Di Vecchia, CH, Russo, Veneziano '20, '21]

[Related work at 3PM: Bern, Cheung, Roiban, Shen, Solon, Zeng '19; Damour '20; Herrmann, Parra-Martinez, Ruf, Zeng '21, Bjerrum-Bohr, Damgaard, Planté, Vanhove '21; Brandhuber, Chen, Travaglini, Wen '21]

- Eikonal phase:

$$\mathrm{Re}\, 2\delta_2 = \frac{4G^3 m_1^2 m_2^2}{b^2}\left[\frac{s\left(12\sigma^4 - 10\sigma^2 + 1\right)}{2m_1 m_2\left(\sigma^2-1\right)^{\frac{3}{2}}} - \frac{\sigma\left(14\sigma^2+25\right)}{3\sqrt{\sigma^2-1}} - \frac{4\sigma^4 - 12\sigma^2 - 3}{\sigma^2-1}\,\mathrm{arccosh}\sigma\right] + \mathrm{Re}\, 2\delta_2^{\mathrm{RR}}\,,$$

$$\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} = \frac{G}{4} Q_{1\mathrm{PM}}^2 \mathcal{I}(\sigma)\,, \quad \mathcal{I}(\sigma) \equiv \frac{2(8-5\sigma^2)}{3(\sigma^2-1)} + \frac{2\sigma\left(2\sigma^2-3\right)}{(\sigma^2-1)^{3/2}}\,\mathrm{arccosh}\,\sigma\,.$$

- Infrared divergent exponential suppression:

$$\mathrm{Im}\, 2\delta_2 = \frac{1}{\pi}\left[-\frac{1}{\epsilon} + \log(\sigma^2-1)\right]\mathrm{Re}\, 2\delta_2^{\mathrm{RR}} + \cdots$$

- $\mathrm{Re}\, 2\delta_2^{\mathrm{RR}}$ contributes half-odd-PN corrections (odd in velocity) to $\Theta_{3\mathrm{PM}}$

Smoothness and Universality of $\mathrm{Re}\, 2\delta_2$ at High Energy [Di Vecchia, CH, Russo, Veneziano '20, '21]

At high energy, as $\sigma \to \infty$ and $s \sim 2m_1 m_2 \sigma$, i.e. in the massless limit:

- The *complete* eikonal phase is smooth, although the conservative and radiation-reaction parts separately diverge like $\log \sigma$
- Its expression is the same in $\mathcal{N} = 8$ supergravity and in GR,

$$\mathrm{Re}\, 2\delta_2 \sim Gs \frac{\Theta_s^2}{4}\,, \qquad \Theta_s \sim \frac{4G\sqrt{s}}{b}$$

in agreement with [Amati, Ciafaloni, Veneziano '90].

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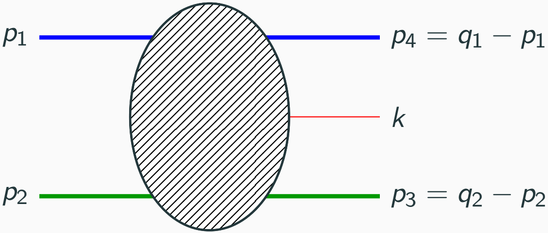
$$\bar{p}_1^\mu = \frac{1}{2}(p_4^\mu - p_1^\mu)$$
$$\bar{p}_2^\mu = \frac{1}{2}(p_3^\mu - p_2^\mu)$$
$$\boxed{q_1^\mu} = p_1^\mu + p_4^\mu$$
$$\boxed{q_2^\mu} = p_2^\mu + p_3^\mu$$
$$0 = q_1^\mu + q_2^\mu + k^\mu$$



More invariants, besides q_1^2, q_2^2, also

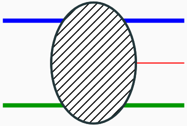
$$\boxed{\sigma} = -v_1 \cdot v_2\,, \qquad \boxed{\omega_1} = -v_1 \cdot k\,, \qquad \boxed{\omega_2} = -v_2 \cdot k\,.$$

We denote by E, ω the total energy and the graviton frequency in the CoM frame,

$$E = \sqrt{-(p_1 + p_2)^2}\,, \qquad \omega = \frac{1}{E}(p_1 + p_2) \cdot k = \frac{1}{E}(m_1\omega_1 + m_2\omega_2)\,, \qquad \alpha_{1,2} = \frac{\omega_{1,2}}{\omega}\,.$$

$2 \to 3$ **Amplitude up to One Loop**

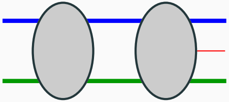
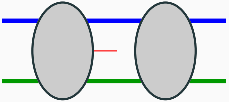
[Brandhuber et al. '23; Herderschee, Roiban, Teng 23; Elkhidir, O'Connell, Sergola, Vazquez-Holm '23] [Georgoudis, CH, Vazquez-Holm '23]

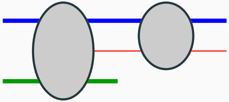
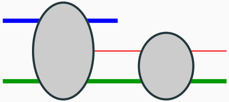
$\mathcal{A} =$  $= \mathcal{A}_0 + \mathcal{A}_1 + \cdots$

with $\mathcal{A}_0$ the tree-level amplitude, and

$$\mathcal{A}_1 = \mathcal{B}_1 + \frac{i}{2}(s + s') + \frac{i}{2}(c_1 + c_2)\,.$$

where $\mathcal{B}_1 = \mathrm{Re}\,\mathcal{A}_1$ and the unitarity cuts can be depicted as follows,

$s =$  $s' =$ 

$c_1 =$  $c_2 =$ 

Inelastic Final State [Di Vecchia, CH, Russo, Veneziano '22]

[cf. Kosower, Maybee, O'Connell '18; Damgaard, Planté, Vanhove '21; Cristofoli et al. '21]

Eikonal Exponentiation of Graviton Exchanges + Coherent Radiation:

$$e^{2i\hat{\delta}(b_1,b_2)} = e^{i\,\mathrm{Re}\,2\delta(b)} e^{i\int_k \left[\tilde{W}(k)a^\dagger(k)+\tilde{W}^*(k)a(k)\right]} .$$

- Final state, schematically:

$$|\mathrm{out}\rangle = e^{2i\hat{\delta}(b_1,b_2)}|\mathrm{in}\rangle$$

- Unitarity:

$$\langle\mathrm{out}|\mathrm{out}\rangle = \langle\mathrm{in}|\mathrm{in}\rangle = 1$$

- The asymptotic metric fluctuation $h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ sourced by the scattering (the waveform) is expressed formally as

$$h_{\mu\nu}(x) = \sqrt{32\pi G}\,\langle\mathrm{out}|\hat{H}_{\mu\nu}(x)|\mathrm{out}\rangle \sim \frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi} + (\mathrm{c.c.})$$

where $\kappa = \sqrt{8\pi G}$, r is the distance from the observer and U the retarded time. Normalization $\tilde{W}^{\mu\nu} = \kappa\,\tilde{w}^{\mu\nu}$.

The Waveform Kernel

[Caron-Huot, Giroux, Hannesdottir, Mizera '23] [Georgoudis, CH, Russo, '23, '24]

- Radiation kernel in the "incoming" variables, [Caron-Huot, Giroux, Hannesdottir, Mizera '23]

$$W = \mathcal{A}_0 + \left[\mathcal{B}_1 + \tfrac{i}{2}\left(s_1 - s_2\right) + \tfrac{i}{2}\left(c_1 + c_2\right)\right] .$$

- Working with "eikonal" variables, this simplifies (up to an overall phase) to,

$$W = \mathcal{A}_0 + \left[\mathcal{B}_1 + \tfrac{i}{2}\left(c_1 + c_2\right)\right] .$$

- Tree level: $\mathcal{A}_0$ is a relatively simple rational function [Luna, Nicholson, O'Connell, White '17]
- One loop: We isolate the even and odd parts of $\mathcal{B}_1$ under $\omega_{1,2} \mapsto -\omega_{1,2}$,

$$\mathcal{B}_1 = \mathcal{B}_{1O} + \mathcal{B}_{1E} ,$$

and

$$\mathcal{B}^{(h)}_{1O} = \pi G E \omega \, \mathcal{A}_0 , \qquad \mathcal{B}^{(i)}_{1O} = -\frac{\sigma\left(\sigma^2 - \frac{3}{2}\right)}{(\sigma^2 - 1)^{3/2}} \pi G E \omega \, \mathcal{A}_0$$

while $\mathcal{B}_{1E}$ and c_1, c_2 represent new one-loop data.

Infrared Divergences Revisited

[Georgoudis, CH, Russo '24] [Bini et al. '24]

- IR divergences due to c_1, c_2,

$$\frac{i}{2}\,c_1 = 2iGm_1\omega_1\left(-\frac{1}{2\epsilon}+\log\frac{\omega_1}{\mu}\right)\mathcal{A}_0+\frac{i}{2}\,c_1^{(\mathrm{reg})}$$

exponentiate in momentum space,

$$W = e^{-\frac{i}{\epsilon}\,GE\omega}\left[\mathcal{A}_0+\mathcal{B}_1+\tfrac{i}{2}\,\mathcal{C}\right] = e^{-\frac{i}{\epsilon}\,GE\omega}\,W^{\mathrm{reg}}\,,$$

where $\frac{i}{2}\,\mathcal{C} = \sum_{a=1,2}\left(2iGm_a\omega_a\log\frac{\omega_a}{\mu}+\frac{i}{2}c_a^{(\mathrm{reg})}\right)$

- Multiplication by the overall phase $e^{-i\omega\delta U}$ $\leftrightarrow$ time translation by δU
- Cancel the divergence by redefining the origin of retarded time [Goldberger, Ross '10]

$$h_{\mu\nu}(x) \sim \frac{4G}{\kappa r}\int_0^\infty e^{-i\omega U}\,\tilde{W}^{\mathrm{reg}}_{\mu\nu}(\omega n)\,\frac{d\omega}{2\pi}+(\mathrm{c.c.})$$

Regions in the Soft Limit

[Georgoudis, CH, Russo '23]

Letting $k^\mu = \omega\, n^\mu$, we target non-analytic pieces as $\omega \to 0$, i.e. $\boxed{\omega \ll b^{-1}}$

$$\tilde{W} = \tilde{W}^{[\omega^{-1}]} + \tilde{W}^{[\log\omega]} + \tilde{W}^{[\omega^0]} + \tilde{W}^{[\omega(\log\omega)^2]} + \tilde{W}^{[\omega\log\omega]} + \dots$$

- Region 1: $\boxed{\omega \ll q_\perp \sim b^{-1}}$ The amplitude simplifies and FT become elementary,

$$\int \frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}\,(q_\perp^2)^\nu\, e^{ib\cdot q_\perp} = \frac{4^\nu}{\pi^{1-\epsilon}}\,\frac{\Gamma(1+\nu-\epsilon)}{\Gamma(-\nu)(b^2)^{1+\nu-\epsilon}}$$

- Region 2: $\boxed{\omega \sim q_\perp \ll b^{-1}}$ FT turns into an ordinary integral. At tree level,

$$I_{i_1 i_2} = \int \frac{d^{2-2\epsilon}q_\perp}{(2\pi)^{2-2\epsilon}}\,\frac{1}{\left(q_\perp^2 + \frac{\omega^2\alpha_2^2}{\sigma^2-1}\right)^{i_1}\left((q_\perp - n_\perp)^2 + \frac{\omega^2\alpha_1^2}{\sigma^2-1}\right)^{i_2}}$$

$$I_{10} = \frac{\Gamma(\epsilon)}{(4\pi)^{1-\epsilon}}\left(\frac{\alpha_2^2\omega^2}{\sigma^2-1}\right)^{-\epsilon} \qquad I_{01} = \frac{\Gamma(\epsilon)}{(4\pi)^{1-\epsilon}}\left(\frac{\alpha_1^2\omega^2}{\sigma^2-1}\right)^{-\epsilon} \qquad I_{11} = \frac{\sqrt{\sigma^2-1}}{4\pi\alpha_1\alpha_2\omega^2}\,\mathrm{arccosh}\,\sigma$$

Universal Terms ω^{-1}, $\log\omega$, $\omega(\log\omega)^2$

- Leading $1/\omega$ soft term (memory effect in time domain) [matches Weinberg '64; Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega^{-1}]} = \frac{i\kappa Q}{b\omega\tilde{\alpha}_1^2\tilde{\alpha}_2^2}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)(2\tilde{\alpha}_1\tilde{\alpha}_2 b_e\cdot\varepsilon + b_e\cdot n(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon + \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon))$$

- Subleading $\log\omega$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\log\omega]} = \kappa\frac{2Gm_1m_2\sigma(2\sigma^2-3)}{\tilde{\alpha}_1\tilde{\alpha}_2(\sigma^2-1)^{3/2}}(\tilde{\alpha}_1\tilde{u}_2\cdot\varepsilon - \tilde{\alpha}_2\tilde{u}_1\cdot\varepsilon)^2\log\left(\frac{\omega b\, e^{\gamma}}{2\sqrt{\sigma^2-1}}\right)$$
$$+ 2iGE\omega\,\tilde{W}_0^{[\omega^{-1}]}\log\omega$$

- Sub-subleading $\omega(\log\omega)^2$ soft term [matches Sahoo, Sen '18; '21]

$$\tilde{W}^{[\omega(\log\omega)^2]} = 2iGE\omega\tilde{W}_0^{[\log\omega]}\log\omega$$

- Non-universal ω^0 piece of the tree-level result,

$$\tilde{W}_0^{[\omega^0]} = \kappa(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2 \left[\frac{G m_1 m_2 \sigma (2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} \log(\tilde{\alpha}_1 \tilde{\alpha}_2) - \frac{2 G m_1 m_2 (2\sigma^2 - 1)}{\tilde{\mathcal{P}} \sqrt{\sigma^2 - 1}} \right]$$
$$+ \frac{4 G m_1 m_2}{\tilde{\mathcal{P}}} \left[\frac{(\tilde{\alpha}_1 \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \tilde{u}_1 \cdot \varepsilon)^2}{\tilde{\alpha}_1 \tilde{\alpha}_2 \tilde{\mathcal{P}}} \left(g_3 \operatorname{arccosh} \sigma + g_2 \log \frac{\tilde{\alpha}_1}{\tilde{\alpha}_2} \right) \right.$$
$$\left. + \frac{2\sigma^2 - 1}{2 b^2 \tilde{\alpha}_1^2 \sqrt{\sigma^2 - 1}} g_1 \right] + i b_2 \cdot n \, \omega \tilde{W}_0^{[\omega^{-1}]} .$$

- For this one, when expanding for small frequencies, both regions in the Fourier integral are needed.

[Georgoudis, CH, Russo '23]

- Tree-level $\omega \log \omega$ piece [matches Ghosh, Sahoo '21]

$$\tilde{W}_0^{[\omega \log \omega]} = \kappa \frac{2iGm_1 m_2 \sigma(2\sigma^2 - 3)}{\tilde{\alpha}_1 \tilde{\alpha}_2 (\sigma^2 - 1)^{3/2}} (\tilde{\alpha}_1 \, \tilde{u}_2 \cdot \varepsilon - \tilde{\alpha}_2 \, \tilde{u}_1 \cdot \varepsilon)$$
$$\times \left[\tilde{\alpha}_1 \tilde{\alpha}_2 \, b_e \cdot \varepsilon + \tilde{\alpha}_2 (b_1 \cdot n)(\tilde{u}_1 \cdot \varepsilon) - \tilde{\alpha}_1 (b_2 \cdot n)(\tilde{u}_2 \cdot \varepsilon)\right] \omega \log \omega$$

- Non-universal one-loop $\omega \log \omega$ piece. $\mathcal{B}_{1E}$ does not contribute.

$$\frac{i}{2}(\tilde{c}_1 + \tilde{c}_2)^{[\omega \log \omega]} = iGE \left[-\frac{1}{\epsilon} + \log \frac{\alpha_1 \alpha_2}{\mu_{\rm IR}^2}\right] \omega \tilde{W}_0^{[\log \omega]} + 2iGE\omega \log \omega \, \tilde{W}_0^{[\omega^0]} + i\tilde{\mathcal{M}}_1^{[\omega \log \omega]}$$

with

$$i\tilde{\mathcal{M}}_1^{[\omega \log \omega]} = i\kappa\omega \log \omega \, G^2 \, m_1^2 m_2 \frac{2\sigma(\alpha_1 \, u_2 \cdot \varepsilon - \alpha_2 \, u_1 \cdot \varepsilon)^2}{(\sigma^2 - 1)^{3/2} \tilde{\mathcal{P}}}$$
$$\times \left[\frac{2\sigma^2 - 3}{\tilde{\mathcal{P}}} \left(f_3 \frac{\operatorname{arccosh} \sigma}{(\sigma^2 - 1)^{3/2}} + f_2 \frac{1}{\alpha_2} \log \frac{\alpha_1}{\alpha_2}\right) - \frac{f_1}{\alpha_2 (\sigma^2 - 1)}\right] + (1 \leftrightarrow 2)\,.$$

Comparison with Predictions from MPM Formalism

[Georgoudis, CH, Russo '23]

- The result for the $\omega \log \omega$ term was given explicitly in the PN expansion using the Multipolar post-Minkowskian (MPM) formalism in [Bini, Damour, Geralico '23], a **mismatch** was found with the amplitude-based result starting at 2.5PN ($\sim 1/c^5$)
- Agreement is restored by the following supertranslation [Veneziano, Vilkovisky '22]

$$U \mapsto U - T(n)\,, \qquad T(n) = 2G(m_1\alpha_1 \log \alpha_1 + m_2\alpha_2 \log \alpha_2)$$

  or more precisely

$$\delta_T h_{AB} = -T(n)\,\partial_u h_{AB} + r\,[2D_A D_B - \gamma_{AB}\Delta]\,T(n)$$

  where only the first term on the RHS (the non-static one) matters.
  Here, $n^\mu = (1, \hat{n})$, $e_A^\mu = \partial_A n^\mu$, $h_{AB} = r^2 e_A^\mu e_B^\nu h_{\mu\nu}$, $\gamma_{AB} = e_A \cdot e_B$, $\Delta = D_A D^A$.
- Confirmed beyond the soft limit

[Georgoudis, CH, Russo '24] [Bini, Damour, De Angelis, Geralico, Herderschee, Roiban, Teng '24]

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Emitted Energy-Momentum

[Georgoudis, CH, Russo '24]

- The insertion of $\hat{P}^\alpha$ measures the emitted energy-momentum $\langle \text{out}|\hat{P}^\alpha|\text{out}\rangle = P^\alpha$,

$$P^\alpha = \int k^\alpha \rho(k)\, \widetilde{dk}\,, \qquad \widetilde{dk} = 2\pi\theta(k^0)\,\delta(k^2)\,\frac{d^D k}{(2\pi)^D}$$

where the spectral emission rate ρ is given by

$$\rho = \tilde{w}^{\text{TT}*}_{\mu\nu}\,\tilde{w}^{\text{TT}\mu\nu} = \tilde{W}^*_{\mu\nu}\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma}\right)\tilde{W}_{\rho\sigma}$$

Note the equivalence between the two expressions, with

$$\tilde{w}^{TT}_{\mu\nu} = \Pi^{\text{TT}}_{\mu\nu\rho\sigma}\tilde{W}^{\rho\sigma}\,, \qquad k_\mu \tilde{W}^{\mu\nu}(k) = 0\,.$$

- We can choose the TT projector to be space-like in the CoM frame, so that

$$\kappa^2 P^0 \equiv \kappa^2 E_{\text{rad}} = G\int_0^\infty \frac{d\omega}{\pi}\oint \frac{d\Omega}{2\pi}\,\omega^2\,\tilde{w}^{\text{TT}*}_{ab}\,\tilde{w}^{\text{TT}}_{ab}\,,$$

Zero-Frequency Limit (ZFL) of the Energy Spectrum

[Di Vecchia, CH, Russo, Veneziano '22]

- Inserting the leading soft theorem, $\tilde{w}^{\mu\nu} \simeq -\frac{i}{\omega}\, a_0^{\mu\nu}$ with $a_0^{\mu\nu} = \sum_a p_a^\mu p_a^\nu / p_a \cdot n$ and performing the angular integrals, one finds

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{2G}{\pi} \sum_{a,b} m_a m_b \left(\sigma_{ab}^2 - \tfrac{1}{2}\right) \eta_a \eta_b \, \frac{\operatorname{arccosh} \sigma_{ab}}{\sqrt{\sigma_{ab}^2 - 1}}$$

- **PM expansion** $Q = 2p \sin\frac{\Theta}{2} \ll p \sim m_{1,2}$

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{GQ^2}{\pi}\mathcal{I}(\sigma) - \frac{GQ^4}{\pi m_1 m_2}\left[\frac{3}{2}\frac{\operatorname{arccosh}\sigma}{(\sigma^2-1)^{\frac{5}{2}}} + \frac{\sigma}{2}\frac{2\sigma^2-5}{(\sigma^2-1)^2} + \frac{2}{5}\frac{m_1^2+m_2^2}{m_1 m_2}\right] + \mathcal{O}(G^6)\,.$$

- Ultrarelativistic limit $m_{1,2} \ll Q = 2p\sin\frac{\Theta}{2}$ (so $\sqrt{s} = E \simeq 2p$) [Addazi, Bianchi, Veneziano '19]

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{4G}{\pi}\left[Q^2 \log\left(\frac{s}{Q^2} - 1\right) - s\log\left(1 - \frac{Q^2}{s}\right)\right]$$

- The small-$m_{1,2}$ of the PM expansion is singular, while the $\Theta \ll 1$ limit is smooth in the ultrarelativistic regime $\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} \simeq \frac{Gs\Theta^2}{\pi}\log\frac{4e}{\Theta^2}$

Nonlinear Memory Contribution?

[Alessio, Di Vecchia, CH '24]

- Take outgoing gravitons into account by

$$\sum_a \mapsto \sum_{a_m} + \int_k \rho(k)$$

where a_m runs over massive states, $\rho(k)$ is the distribution of emitted gravitons.

- This is the operation that gives the nonlinear memory effect,

$$a_0^{\mu\nu} \mapsto a_0^{\mu\nu} + \delta a_0^{\mu\nu}\,, \qquad \delta a_0^{\mu\nu} = \int_k \rho(k)\, \frac{k^\mu k^\nu}{k\cdot n}\,.$$

- For the ZFL of the energy spectrum, it gives

$$\delta\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = -\frac{4G}{\pi}\int_k \rho(k) \sum_a p_a\!\cdot\! k \log\left(-\eta_a \frac{p_a\cdot k}{m_a\Lambda}\right) \simeq -\frac{4G}{\pi}\int_k \rho(k) Q\!\cdot\! k \log\frac{\tilde{u}_1\cdot k}{\tilde{u}_2\cdot k}$$

- The $\mathcal{O}(G^5)$ **vanishes** since $\rho(k)$ is invariant under $b\cdot k \to -b\cdot k$ to leading order in G owing to the reality of the tree-level amplitude.

Log-Resummed Waveform

[Alessio, CH, Di Vecchia '24]

- Considering an elastic $2 \to 2$ hard process, let us define

$$E = (p_1+p_2)\cdot n\,, \quad B^{\mu\nu}(p_1,p_2) = (p_1+p_2)\cdot n\left(\frac{p_1^\mu p_1^\nu}{p_1\cdot n}+\frac{p_2^\mu p_2^\nu}{p_2\cdot n}\right) - (p_1^\mu+p_2^\mu)(p_1^\nu+p_2^\nu)\,.$$

- Then, the known soft theorems [Sahoo, Sen '18: '21] for $\ell = 0,1,2$ reduce to (define $h(\sigma) = \sigma(2\sigma^2-3)/(\sigma^2-1)^{3/2}$)

$$a_\ell^{\mu\nu} = \frac{1}{E}(-GEh(\sigma))^\ell\left[B^{\mu\nu}(p_1,p_2) - (-1)^\ell B^{\mu\nu}(p_3,p_4)\right]$$

- We **conjecture** that this expression generalizes to any $\ell \geq 0$.
- Frequency-domain resummation

$$\tilde{w}^{\mu\nu} = -\frac{i}{E\omega}\,\omega^{2iGE\omega}\left[\omega^{iGE\omega h(\sigma)}B^{\mu\nu}(p_1,p_2) - \omega^{-iGE\omega h(\sigma)}B^{\mu\nu}(p_3,p_4)\right] + \cdots$$

- Proofs: (1) at Newtonian level as $p_\infty \to 0$ for generic GM/bp_∞^2 [Alessio, CH, Di Vecchia '24]
  (2) in the near-probe limit $\nu \to 0$ [Fucito, Morales, Russo '24]
- Cross-check: 2PN approximation up to $\mathcal{O}(G^3)$ [Bini, Damour, Geralico '24]

Log-Resummed Soft Spectra

[Alessio, Di Vecchia, CH '24]

- The resummed waveform in the soft limit gives universal results for the "leading logs" (LL) of the type $(\omega \log \omega)^n$ in the energy emission spectrum $dE/d\omega$.
- In the CoM frame we find, expanding for small deflections $Q \to 0$,

$$\left(\frac{dE}{d\omega}\right)_{\rm LL} = \left[1 - \cos\left(2GEh(\sigma)\omega \log \omega\right)\right] \frac{2G}{\pi} \mathcal{H}(m_1, m_2, \sigma) + \cos\left(2GEh(\sigma)\omega \log \omega\right) \frac{GQ^2}{\pi} \mathcal{I}(\sigma) + \cdots$$

fixing $G^{2n+1}(\omega \log \omega)^{2n}$ for $n = 1, 2, \ldots$ and $G^{2n+3}(\omega \log \omega)^{2n}$ for $n = 0, 1, 2, \ldots$ (see the additional material for the functions $\mathcal{H}(m_1, m_2, \sigma)$ and $\mathcal{I}(\sigma)$).

- In the ultrarelativistic limit instead

$$\left(\frac{dE}{d\omega}\right)_{\rm LL} = \frac{4G}{\pi} \left[\sin(2G\sqrt{s}\,\omega \log \omega)\right]^2 s + \frac{4G}{\pi} \cos(4G\sqrt{s}\,\omega \log \omega) \left[Q^2 \log\left(\frac{s}{Q^2} - 1\right) - s \log\left(1 - \frac{Q^2}{s}\right)\right] + \cdots$$

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Linear and Nonlinear Memory

[Zel'Dovich, Polnarev '74; Braginsky, Thorne '87; Christodoulou '91; Thorne '92]

- Let us focus on the leading order for small frequencies,

$$\tilde{w}_{\mu\nu} \sim -\frac{i}{\omega} F_{\mu\nu} + \mathcal{O}(\log \omega)\,, \text{ as } \omega \to 0$$

- Our first goal is to deduce from the amplitude-based waveform that $F^{\mu\nu} = f^{\mu\nu} + \delta F^{\mu\nu}$ (linear+nonlinear memory)

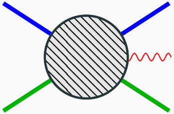
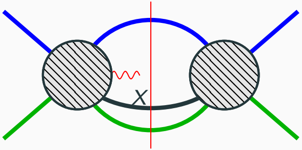
$$f^{\mu\nu} = \sum_{a=1}^{4} \frac{p_a^\mu p_a^\nu}{p_a \cdot n}\,, \qquad \delta F^{\mu\nu} = \int \widetilde{dk}\, \rho(k)\, \frac{k^\mu k^\nu}{k \cdot n}$$

to leading nontrivial order, i.e. $\mathcal{O}(G^3)$.

- The second goal is to compute $\delta F^{\mu\nu}$ explicitly at this order, for two-body scattering.

Waveform vs Memory

The small-frequency limit of the PM waveform

$W^{\mu\nu} =$  $- i \sum_X$ 

to be compared with the small-deflection expansion of $f^{\mu\nu}$, [Herrmann, Parra-Martinez, Ruf, Zeng '21]

$$p_1^\mu + p_4^\mu = -Q\,\frac{b_e^\mu}{b_e} - Q_\parallel\,\check{v}_2^\mu + \mathcal{O}(G^4)\,,$$

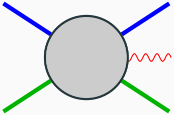
$$p_2^\mu + p_3^\mu = +Q\,\frac{b_e^\mu}{b_e} - Q_\parallel\,\check{v}_1^\mu + \mathcal{O}(G^4)\,,$$

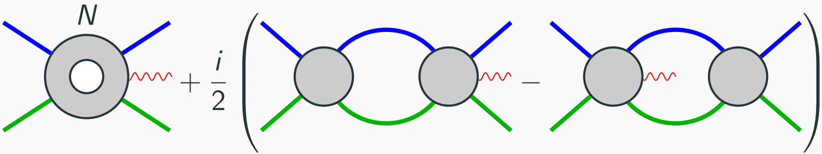
where $\check{v}_{1,2}^\mu = (\sigma v_{2,1}^\mu - v_{1,2}^\mu)/(\sigma^2 - 1)$ for small Q, where

$$\varepsilon_\mu\, f^{\mu\nu} \varepsilon_\nu = Q\,S_1^{(\varepsilon)} + Q^3\,S_3^{(\varepsilon)} + Q_\parallel\,T_3^{(\varepsilon)} + \mathcal{O}(G^4)\,,$$

and $\delta F^{\mu\nu}$.

Tree Level and One Loop

$W_0^{\mu\nu} \sim$ 

$W_1^{\mu\nu} \sim$ 

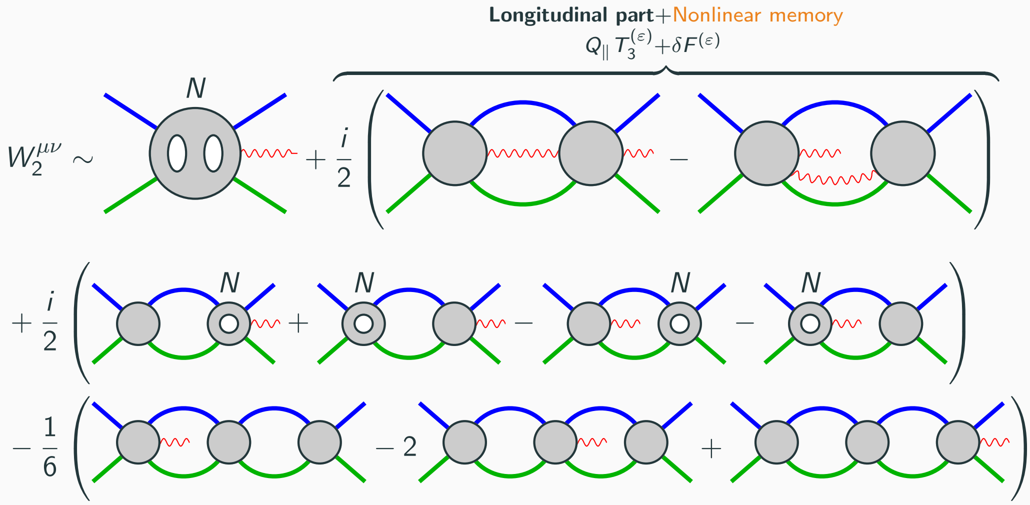
where $S = e^{iN}$, agree with

$$\varepsilon_\mu f^{\mu\nu} \varepsilon_\nu = Q\, S_1^{(\varepsilon)} + \cdots$$

Only linear memory

Two loops

[Georgoudis, Goncalves, CH, Parra-Martinez '25]



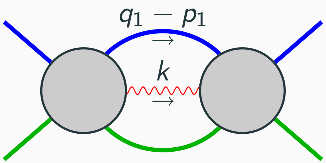
The rest reproduces the **transverse part** $Q\, S_1^{(\varepsilon)} + Q^3\, S_3^{(\varepsilon)}$.

- Phase-space integral of a product of Fourier transforms [HARD]

$$\delta F^{\mu\nu} = \int \widetilde{dk}\, \rho(k)\, \frac{k^\mu k^\nu}{k \cdot n}\,, \qquad \rho = 8\pi G\, \tilde{w}_{\mu\nu} \left(\eta^{\mu\rho}\eta^{\nu\sigma} - \tfrac{1}{D-2}\,\eta^{\mu\nu}\eta^{\rho\sigma}\right) \tilde{w}^*_{\rho\sigma}$$

- Fourier transform of a two-loop convolution [EASIER]

$$\delta F^{\mu\nu} = \mathrm{FT}_4 \int_{q_1} \int_k \frac{k^\mu k^\nu}{k \cdot n}$$



,

- Still a complicated function of $\sigma = 1/\sqrt{1-v^2}$ and θ, ϕ entering via $n^\mu = (1, \sin\theta\cos\phi, \sin\theta\sin\phi, \cos\phi)$.

Calculating the Nonlinear Memory Contribution

[Georgoudis, Goncalves, CH, Parra-Martinez '25]

- Consider the projection on spin-weighted spherical harmonics

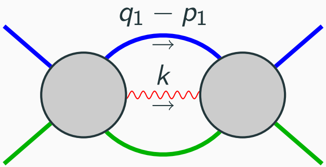
$$\delta F^{\ell m} = \oint Y_{\pm 2}^{\ell m *}\, \delta F_{\mu\nu}\, \varepsilon_{\pm}^{\mu} \varepsilon_{\pm}^{\nu}\, d\Omega$$

- This gives the following simple result for generic ℓ, m [Blanchet, Damour '92] ($\mathcal{N}_2^{\ell m}$ is a normalization factor)

$$\delta F^{\ell m} = -\frac{2\pi}{\ell(\ell-1)} \mathcal{N}_2^{\ell m} Y_{\mu_1 \cdots \mu_\ell}^{\ell m *}\, \mathrm{FT}_4[\mathbb{I}^{\mu_1 \cdots \mu_\ell}]$$

- We need to calculate [EASY]

$$\mathbb{I}^{\mu_1 \cdots \mu_\ell} = \int_{q_1} \int_k \frac{k^{\mu_1} \cdots k^{\mu_\ell}}{(-t \cdot k)^{\ell - 1}}$$



Nonlinear Memory Multipoles

[Georgoudis, Goncalves, CH, Parra-Martinez '25]

- The result for each ℓ, m is simple and retains an exact dependence on σ:

$$\delta F^{\ell m} = \frac{G^3 \pi^2 m_1^2 m_2^2}{b^3 (\sigma^2 - 1)^{\frac{3}{2}}} \, i^\ell \mathcal{N}_2^{\ell m} \mathcal{F}^{\ell m}(\sigma) + \mathcal{O}(G^4) \, ,$$

where, letting $f_j^{\ell m}(\sigma)$ denote polynomials in σ,

$$\mathcal{F}^{\ell m}(\sigma) = (\sigma^2 - 1)^{-\frac{\ell}{2}} \left[f_1^{\ell m} + f_2^{\ell m} \log\left(\tfrac{\sigma+1}{2}\right) + f_3^{\ell m} \frac{\operatorname{arccosh} \sigma}{\sqrt{\sigma^2 - 1}} \right]$$

- For instance,

$$f_2^{22} = \frac{1}{8} (\sigma^2 - 1)^2 \left(35\sigma^4 + 420\sigma^3 - 510\sigma^2 + 292\sigma + 67\right)$$

- Properties: $\delta F^{\ell m} = 0$ if $\ell + m$ is odd (no "V" multipoles), $\delta F^{\ell (-m)} = (-)^\ell \delta F^{\ell m}$ (symmetry of $\rho(k)$ under $b \mapsto -b$ at leading order)
- Check: Matches direct PN calculation of [Wiseman, Will '91] and up to 9PN.

Summary and Outlook

- The **eikonal** provides a tool to calculate scattering observables, including the **impulse**, the **waveform** and the emitted **energy and angular momentum**.
- PM expansion and soft theorems provide nicely **complementary** approaches. Combining them, we have calculated the NNLO PM waveform to leading order in the soft limit (**nonlinear memory**).

For the future:

- Amplitude derivation of the log-resummed waveform?
- NNNLO waveform? Memory involves square of LO + NLO waveform
- Solving the high-energy puzzle?

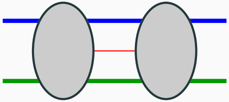
ADDITIONAL MATERIAL

Unitarity and Analyticity Fix the Radiation-Reaction Contribution

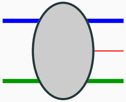
[Di Vecchia, CH, Russo, Veneziano '21]

- Unitarity determines the imaginary part of the two-loop eikonal,

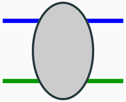
$$2\,\mathrm{Im}\,2\delta_2 = \mathrm{FT}$$



- IR divergence comes from low frequencies, use the soft graviton theorem:



$$\sim \sqrt{8\pi G}\sum_a \frac{p_a^\mu p_a^\nu}{p_a\cdot k}$$



as $k^\alpha \to 0$

- Then, using the natural upper cutoff $\omega^* \simeq \frac{v}{b}$, we find

$$\mathrm{Im}\,2\delta_2 = \frac{G}{2\pi}\left[-\frac{1}{2\epsilon} + \log\sqrt{\sigma^2-1}\right] Q_{1\mathrm{PM}}^2\,\mathcal{I}(\sigma) + \cdots$$

- By analyticity, $i\log(1-\sigma^2-i0) = i\log(\sigma^2-1)+\pi$, hence

$$\mathrm{Re}\,2\delta_2^{\mathrm{RR}} = \lim_{\epsilon\to 0}\left[-\pi\epsilon\,\mathrm{Im}\,2\delta_2\right] = \frac{G}{4}\,Q_{1\mathrm{PM}}^2\,\mathcal{I}(\sigma)\,.$$

A Bound on the PM Expansion

[Di Vecchia, CH, Russo, Veneziano '22]

- The explicit nonperturbative expression for the ZFL reads

$$\left(\frac{dE}{d\omega}\right)_{\text{ZFL}} = \frac{4G}{\pi}\Bigg\{2m_1 m_2\left(\sigma^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma}{\sqrt{\sigma^2-1}} - 2m_1 m_2\left(\sigma_Q^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\sigma_Q}{\sqrt{\sigma_Q^2-1}}$$
$$+\sum_{a=1,2}\left[\frac{m_a^2}{2} - m_a^2\left(\left(1+\tfrac{Q^2}{2m_a^2}\right)^2-\tfrac{1}{2}\right)\frac{\operatorname{arccosh}\left(1+\frac{Q^2}{2m_a^2}\right)}{\sqrt{\left(1+\frac{Q^2}{2m_a^2}\right)^2-1}}\right]\Bigg\}.$$

- Note the presence of branch points at (recall also $Q \sim \sqrt{m_1 m_2 \sigma}\,\Theta$)

$$Q^2 = -4m_a^2$$

corresponding to the t-channel thresholds (outside the physical region)

- The PM expansion converges for [D'Eath '76; Kovacs, Thorne '77,'78]

$$\sigma \lesssim \frac{1}{\Theta^2}$$

NLO, $\omega^2(\log\omega)^2$, in the Energy Spectrum [Alessio, Di Vecchia, CH '24]

- We specify a reference frame: center-of-mass
- Inserting the $\log\omega$ and $\omega(\log\omega)^2$ soft theorems in the expression for the spectrum, we obtain a general prediction for the $\omega^2(\log\omega)^2$ contribution
- **PM expansion** $Q = 2p\sin\frac{\Theta}{2} \ll p \sim m_{1,2}$

$$\left(\frac{dE}{d\omega}\right)_{\omega^2(\log\omega)^2} = (GEh(\sigma))^2\,\frac{G}{\pi}\,\mathcal{H}(m_1,m_2,\sigma) + 2\,(GEh(\sigma))^2\,\frac{GQ^2}{\pi}\,\mathcal{I}(\sigma) + \mathcal{O}(G^6)$$

- Ultrarelativistic limit $m_{1,2} \ll Q = 2p\sin\frac{\Theta}{2}$ (so $\sqrt{s} = E \simeq 2p$) [Sahoo, Sen '19]

$$\left(\frac{dE}{d\omega}\right)_{\omega^2(\log\omega)^2} = s\,\frac{16G^3}{\pi}\left[s + 2s\log\left(1-\frac{Q^2}{s}\right) - 2Q^2\log\left(\frac{s}{Q^2}-1\right)\right]$$

- For $\Theta \ll 1$, one finds $\left(\frac{dE}{d\omega}\right)_{\omega^2(\log\omega)^2} = \frac{16G^3}{\pi}\,s^2\left[1 - \frac{2Q^2}{s}\log\frac{s}{Q^2} + \cdots\right]$.
- Again, (only) when a log appears, the UR limit of the PM expansion is singular

The functions $\mathcal{H}(m_1, m_2, \sigma)$ and $\mathcal{I}(\sigma)$ appearing in the low-frequency spectrum

$$\mathcal{H}(\sigma, m_1, m_2) = \left[2(s - m_1 m_2 \sigma) + \frac{m_2^2(2m_1\sigma + m_2)}{m_1\sqrt{\sigma^2 - 1}}\,\ell_1 + \frac{m_1^2(2m_2\sigma + m_1)}{m_2\sqrt{\sigma^2 - 1}}\,\ell_2\right],$$

with

$$\ell_1 = \log\left(\frac{x\,(m_1 x + m_2)}{m_2 x + m_1}\right), \qquad \ell_2 = \log\left(\frac{x\,(m_2 x + m_1)}{m_1 x + m_2}\right), \qquad x = \sigma - \sqrt{\sigma^2 - 1}\,,$$

while

$$\mathcal{I}(\sigma) = \frac{2}{\sigma^2 - 1}\left[\frac{8 - 5\sigma^2}{3} + \frac{\sigma(2\sigma^2 - 3)\,\mathrm{arccosh}\,\sigma}{\sqrt{\sigma^2 - 1}}\right].$$