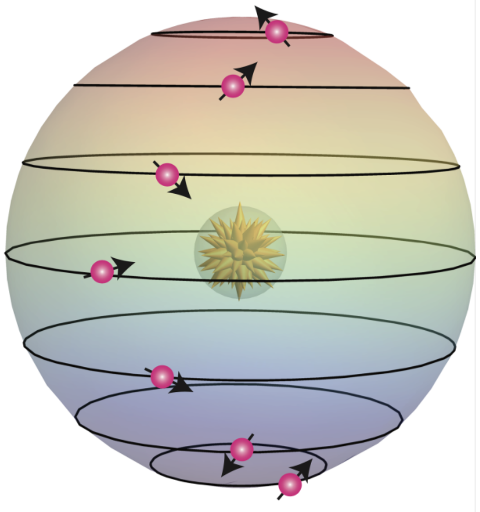
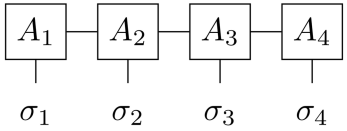


(Tensor network) Simulations of 2+1D CFTs through the Fuzzy sphere



Loïc Herviou

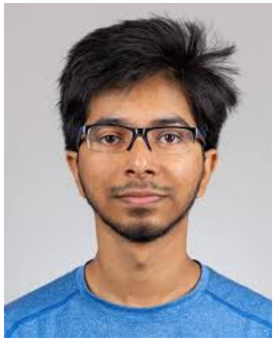
CPTGA
October 8th 2025

Laüchli et al, SciPost Phys. 19, 076 (2025)
Dey et al, arXiv:????.???? x2

The people

Andreas Laüchli (PSI/EPFL)

Slava Rychkov (IHES)



Arjun Dey (PSI)

Christopher Mudry (PSI)

Patrick Wilhem (PSI)

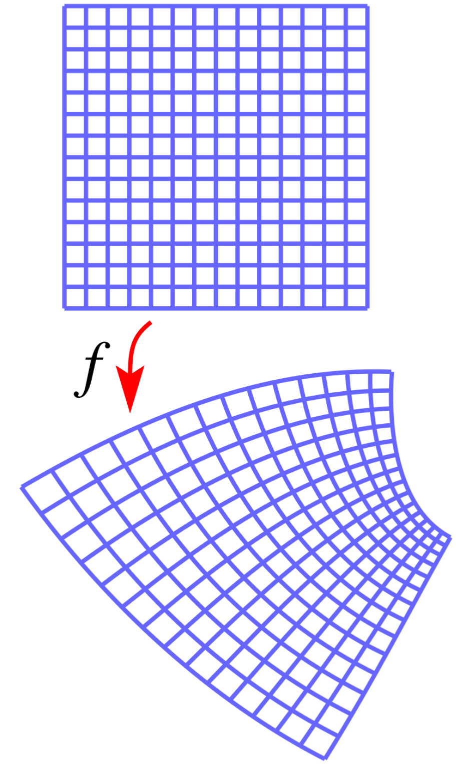
Conformal Field Theories (CMT POV)

What is a CFT?

Quantum field theory describing a critical point

Scale-invariant theory with conformal invariance

Conformal transformation = angle preserving transformations



From wikipedia

Why are they crucial in condensed matter?

Universality: emergent low-energy behavior independent of microscopic details

Classification of gapless phases of matter

Exact solvability in 1+1d

Key concepts

Power-law correlations (scale invariance) $\langle \mathcal{O}(r)\mathcal{O}(0)\rangle \approx r^{-2\Delta_{\mathcal{O}}}$

Conformal dimension indicates relevance $\Delta_{\mathcal{O}} \leq \Delta_c = d+1$

Primary operators vs descendant operators $\mathcal{O},\, \Delta \to \partial^n \mathcal{O},\, \Delta + n$

Operator product expansion $\mathcal{A}(x)\mathcal{B}(0) \to_{x\to 0} \sum_{\mathcal{O}} f_{\mathcal{A}\mathcal{B}\mathcal{O}} \mathcal{O}(\frac{x}{2})$

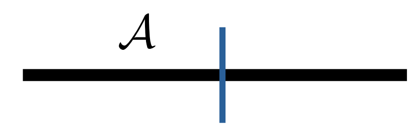
Conformal Field Theories in 1+1d

Conformal invariance in 1+1d

The conformal group is infinite $\qquad z \to \dfrac{az+b}{cz+d}$

(Minimal) models classified by their central charge

Central charge $\leftrightarrow$ effective degrees of freedom $\qquad S = \dfrac{c}{6}\log l_{\mathcal{A}}$



Example 1: the free boson (c=1)

Physical realization: Free fermions, Hubbard models, Heisenberg spin-½ chains…

Key features: $\qquad S = \int dx d\tau \frac{1}{K}(\partial_x\phi)^2 + K(\partial_x\theta)^2 \qquad \langle\phi(x)\phi(0)\rangle \approx \log x$

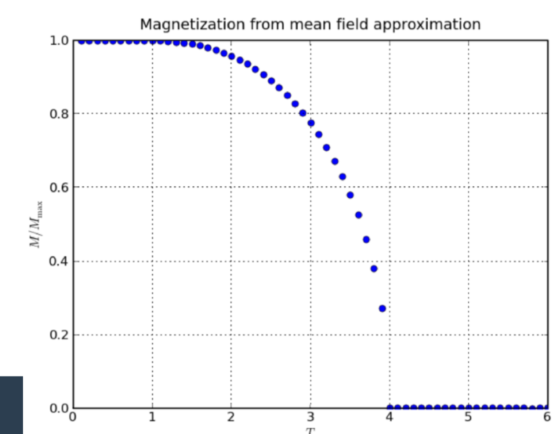
$$\langle n(x)n(0)\rangle \approx \frac{A}{x^2} + \frac{B\cos 2k_F x}{x^{2K}}$$

Example 2: the Ising model (c=0.5)

Physical realization: transverse Ising models, Kitæv chains…

Key features: simplest non-trivial CFT with 3 primary operators

Identity, $I, \Delta = 0$ $\qquad$ Spin, $\sigma, \Delta = 1/8$ $\qquad$ Energy, $\varepsilon, \Delta = 1$



Conformal Field Theories: 2+1d

Conformal invariance in 2+1d

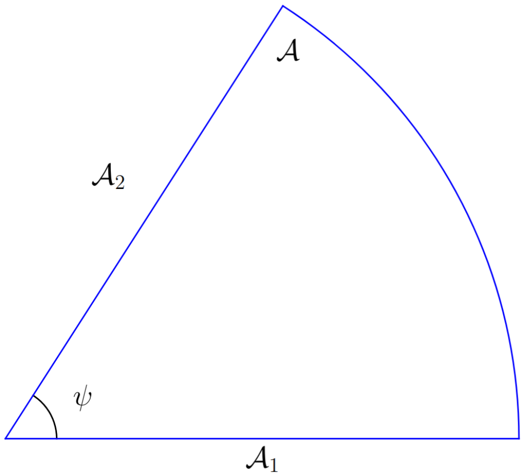
Finite-dimensional algebra...

Central charge dœs not fully classify models

Central charge intervene in corner contributions to the entropy

Nothing is solvable...

$$S_{\mathrm{vN}} = \alpha P_{\mathcal{A}} - a(\psi)\log l_{\mathcal{A}} + \dots$$



Example 1: the free (Dirac) fermion (c=N)

Physical realization: graphene (but), transitions in 2D TI, surface states of 3D TI

Key features: describe gapless chiral fermions, (often) unstable to interactions

Example 2: the 3D Ising model (c=0.53...)

Physical realization: liquid-gaz critical point, Curie point of uniaxial ferromagnet etc

Key features: the “Ising model” of 2+1d CFTs.

Conformal dimensions/critical exponents are well-known numerically

Bueno et al, Phys. Rev. Lett. 115, 021602 (2015)

Challenges in numerical methods

Direct simulations

Core concept: use a method of your choice to get the groundstate

Key outputs: explicit forms for the two-point functions $\langle\phi(x)\phi(0)\rangle$

Major difficulties: Strongly interacting theories (exp. Hilbert space, sign problem...)

Rely on finite-size scaling and fitting

Conformal bootstrap

Core concept: use fundamental CFT axioms to derive rigorous constraints on CFT data

Keys outputs: rigorous bounds on scaling dimensions and OPE cœfficients.

Major difficulties: Exponentially harder for correlators with more operators.

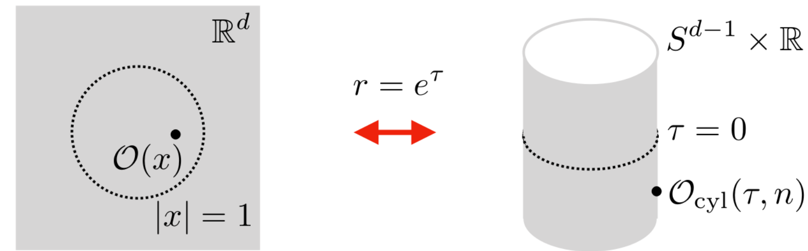
Extracting detailed data is harder than getting bounds.

Others

Large N or ε expansion => imprecise, perturbative results

State-operator correspondence

Weyl transformation



$$ds^2 = dr^2 + r^2 dn^2 \qquad ds^2_{\text{cyl}} = d\tau^2 + dn^2 = r^{-2} ds^2_{\mathbb{R}^d}$$

Exact mapping from $\mathbb{R}^d$ to $\mathbb{R} \times$ d $\mathcal{S}_d$

On the plane, dilatations act simply $\quad |\Phi\rangle = \Phi(0)|0>$

$$D|\Phi\rangle = \Delta_\Phi |\Phi\rangle$$

On the sphere, dilatations map to CFT Hamiltonian $\quad D \to H_{\text{CFT}}$

Taking into account global factors $\quad E_\Phi = v\dfrac{\Delta_\Phi}{R} + E_0 + \ldots$

Example: 1+1d CFT

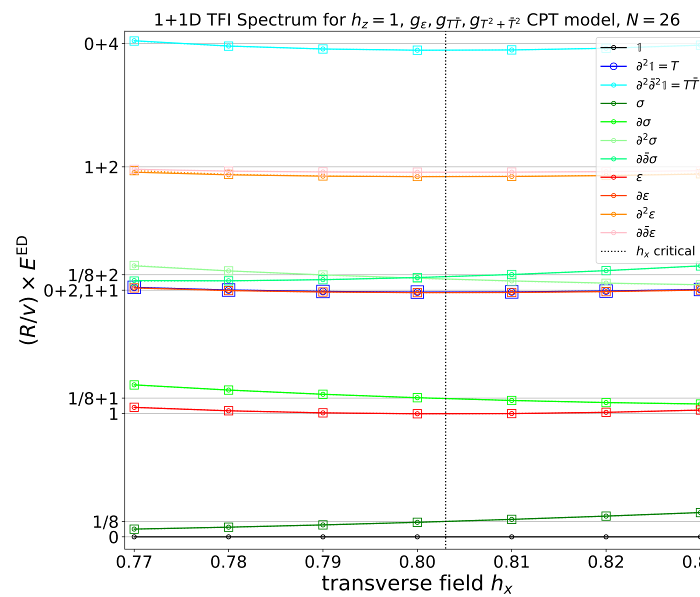
Weyl transformation: $\quad z = e^{\tau + ir}$

Dilatation becomes translations in time

$$z \to \lambda z \Leftrightarrow \tau \to \tau + \log \lambda$$

$$D \to H_{\text{CFT}} = L_0 + L_0^\dagger + \ldots$$

$$\Delta_\Phi = h + \overline{h} \qquad\qquad K \propto h - \overline{h}$$



$$H_{\text{TFI}} = J\sum_i \sigma_i^z \sigma_{i+1}^z - h_z \sum_i \sigma_i^z - h_x \sum_i \sigma_i^x$$

State-operator in 2d in practice?

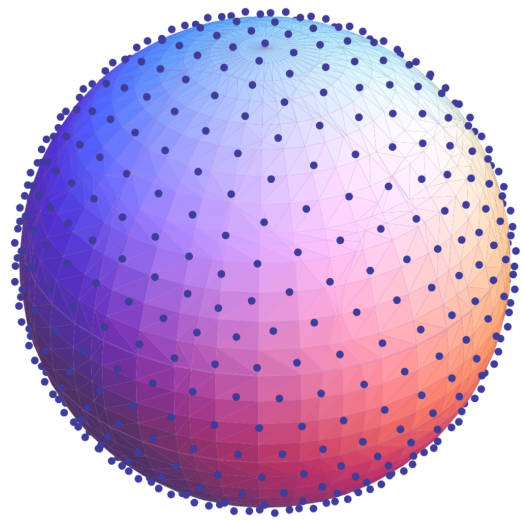
Conceptual challenges

Crucial importance of the sphere: large cylinders or planes are not relevant

Continuous models are **hard**

Lattices models: how to pave a sphere with a 2d lattice?

Isocahedron (12 sites)

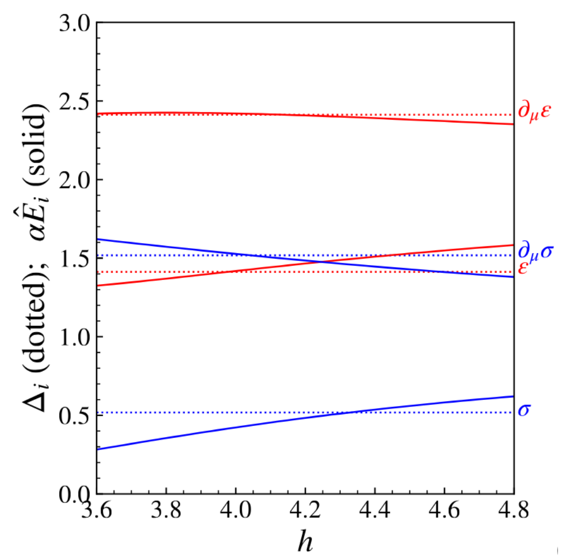


Icosahedral shell

Hard to solve (proper 2D)

Weakly broken SO(3) invariance

Significant finite-size effects



Brower et al, PoS 2012 (061), Lao et al, SciPost Phys. 15, 243 (2023)

Fuzzy sphere: from the integer QHE

Integer quantum Hall effect

2d electron gaz in a normal magnetic field $H_{\mathrm{LL}} = \frac{(\vec{p} + e\vec{A})^2}{2m} \Leftrightarrow \hbar\omega_c(n + \frac{1}{2})$

Macroscopic degeneracy of each Landau level $N_\phi \approx \frac{BS}{\phi_0}$

Fully occupied Landau level = gapped bulk with edge conductance

$$\sigma = n\frac{e^2}{\hbar}$$

But on a sphere!

Still well-defined Landau levels $E = \hbar\omega_c(n + \frac{1}{2} + \frac{n(n+1)}{2s_0}), s_0 = \frac{\Phi}{2\phi_0}$

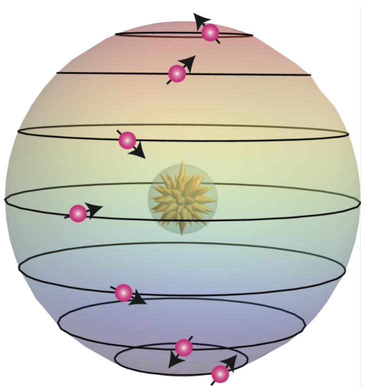
No edges = no transport

Orbital basis: (modified) spherical harmonics $Y_{s_0, s_0+n, m}(\theta, \phi)$

Fully occupied Landau levels = fully SO(3) invariant ground state

Large magnetic field = single Landau level projection, rest if frozen

$$Y_{s_0, s_0, m}(\theta, \phi) = \cos^{s_0+m}\frac{\theta}{2}\sin^{s_0-m}\frac{\theta}{2}e^{im\phi}$$



Taken from Simons center

Haldane, PRL51 (1983), Greiter, PRB 83 (2011)

Fuzzy sphere: to effective spin models

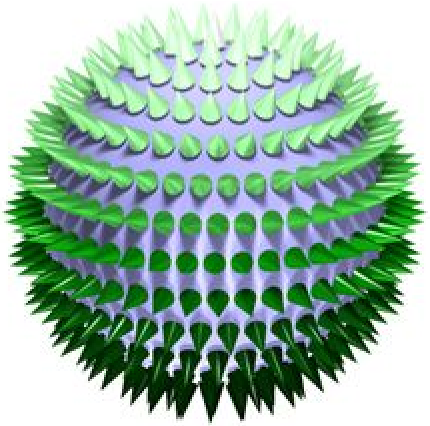
Quantum Hall ferromagnets

What about the remaining spin degrees of freedom?

In experiments, weak Zeeman splitting

Coulomb interactions induce a weak ferromagnet

Plenty of possible exciting physics



Skyrmionic spin-state

Hamiltonian engineering

Core idea: modify standard Hamiltonians to induce a transition for the spins

Recipe: Write your Hamiltonian of choice $-J\iint S^{\alpha}(\vec{r}_1)K_{\alpha,\beta}(\vec{r}_1-\vec{r}_2)S^{\beta}(\vec{r}_2)+h_{\alpha}\int S^{\alpha}(\vec{r})$

Convert your spins into fermionic operator $\vec{S}(\vec{r})=\Psi^{\dagger}_{\sigma}(\vec{r})\vec{S}_{\sigma,\sigma'}\Psi_{\sigma'}(\vec{r})$

Project the fermionic field onto a single Landau level

$$\Psi_{\sigma}(\vec{r})=\sum_{n}\sum_{m}Y_{s_0,s_0+n,m}(r)c_{n,m,\sigma}\Rightarrow\Psi_{\sigma}(\vec{r})=\sum_{m=-s_0}^{s_0}Y_{s_0,s_0,m}(r)c_{n,m,\sigma}$$

Profit? $\{\Psi(\vec{r}_1),\Psi^{\dagger}(\vec{r}_2)\}\neq\delta(\vec{r}_1-\vec{r}_2)$ Fuzzy!

Zhu et al, PRX 13 (2023)

The Fuzzy Ising model

Quantum Ising

$$\iint n(\vec{r}_1)K_{00}(\vec{r}_1-\vec{r}_2)n(\vec{r}_2) \quad -J\iint S^z(\vec{r}_1)K_{zz}(\vec{r}_1-\vec{r}_2)S^z(\vec{r}_2) + h_x\int S^x(\vec{r})$$

Ferromagnetic regime Ising transition Paramagnetic regime

Fuzzy Ising

$$K_{00}(\vec{r}) = K_{zz}(\vec{r}) \Rightarrow \lambda V_0 + \mu V_1 = (\lambda + \mu\nabla^2)\delta(\vec{r})$$

First Haldane pseudopotentials

$$H_{00} = \frac{1}{2}\sum_{m_1,m_2,m=-s}^{s} V_{m_1,m_2,m_2-m,m_1+m}(c^\dagger_{m_1}c_{m_1+m})(c^\dagger_{m_2}c_{m_2-m})$$

Stabilization of the FQHE

$$H_{zz} = -\frac{1}{2}\sum_{m_1,m_2,m=-s}^{s} V_{m_1,m_2,m_2-m,m_1+m}(c^\dagger_{m_1}\sigma^z c_{m_1+m})(c^\dagger_{m_2}\sigma^z c_{m_2-m})$$

Ising interaction

$$H_{t,x} = -h\sum_{m=-s}^{s} c^\dagger_m\sigma^x c_m$$

Transverse field

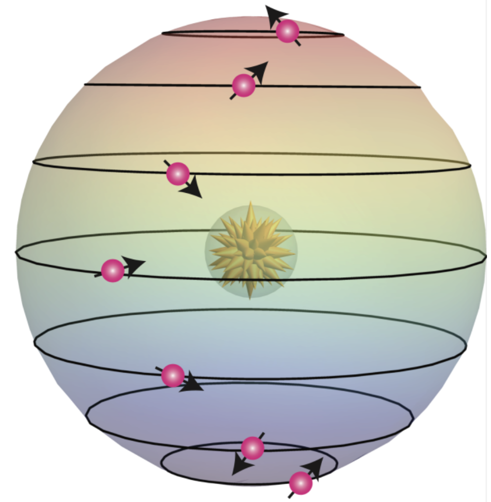
Tensor networks on a sphere?

Tensor networks on a sphere

2D tensor networks would go back to our original problems

Haldane orbitals $Y_{s0,\,s0,\,m}$ have a natural 1d structure

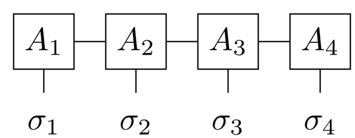
Use usual matrix product states algorithms

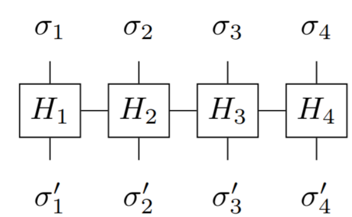


MPS in a nutshell (thanks Gilles and Jeanne)

Represent quantum states as a product of tensors

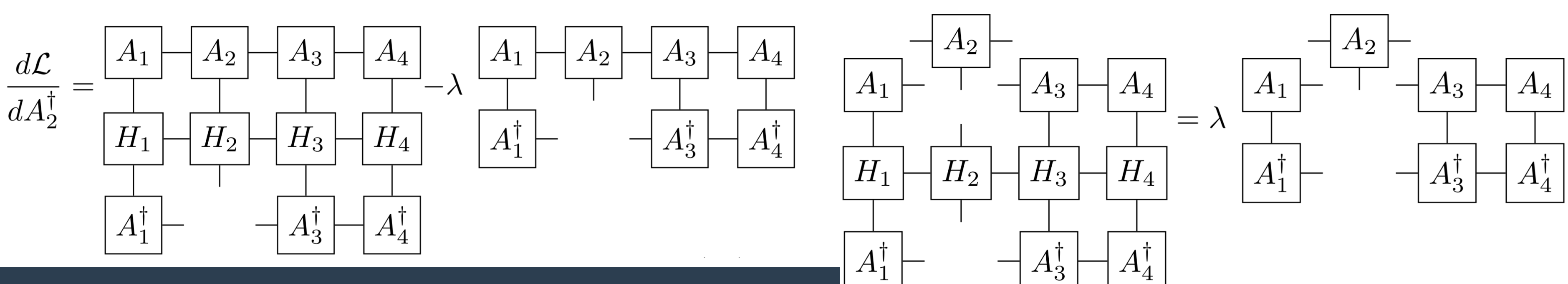
$$\langle \sigma_1 \ldots | \Psi \rangle = \sum_{m_1,\ldots}^{\chi} A^{\sigma_1}_{m_1} A^{\sigma_2}_{m_1,m_2} \ldots$$



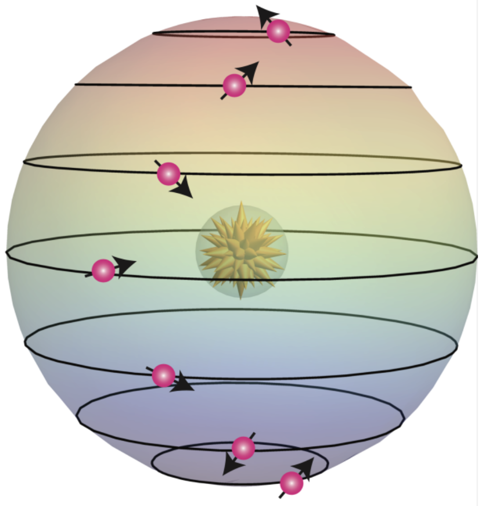


Number of parameters $\log \chi \sim S_{\mathrm{vN}}$ Perfect for 1d gapped, ok for 1d cft, so-so for 2d

Systematic variational algorithm: (finite) DMRG $\mathcal{L} = \langle \Psi | H | \Psi \rangle - \lambda \langle \Psi | \Psi \rangle$



Technical “challenges” for MPS on the sphere



No free lunch theorem: 2D model

Area of contact between successive orbitals $\propto$ perimeter

Surface of the sphere $\propto$ number of orbitals

Exponential complexity typical of 2D $\quad S_{vN} \propto R \propto \sqrt{N}$

Open question: reducing the prefactor?

Heavy Hamiltonians

$$H_{00} = \frac{1}{2} \sum_{m_1, m_2, m=-s}^{s} V_{m_1, m_2, m_2-m, m_1+m} (c^\dagger_{m_1} c_{m_1+m})(c^\dagger_{m_2} c_{m_2-m})$$

Quasi-local Hamiltonians are still in practice very-long range

Solution: compression of the Hamiltonian (similar to MPS compression)

Symmetries

Large entanglement forces us to use highly symmetric models + rotation invariance

Local updates are blocked (Coulomb blockade) => a naive algorithm is not ergodic

Solution: noisy DMRG. For the future, implementation of non-Abelian symmetries?

Back to the Fuzzy Ising model

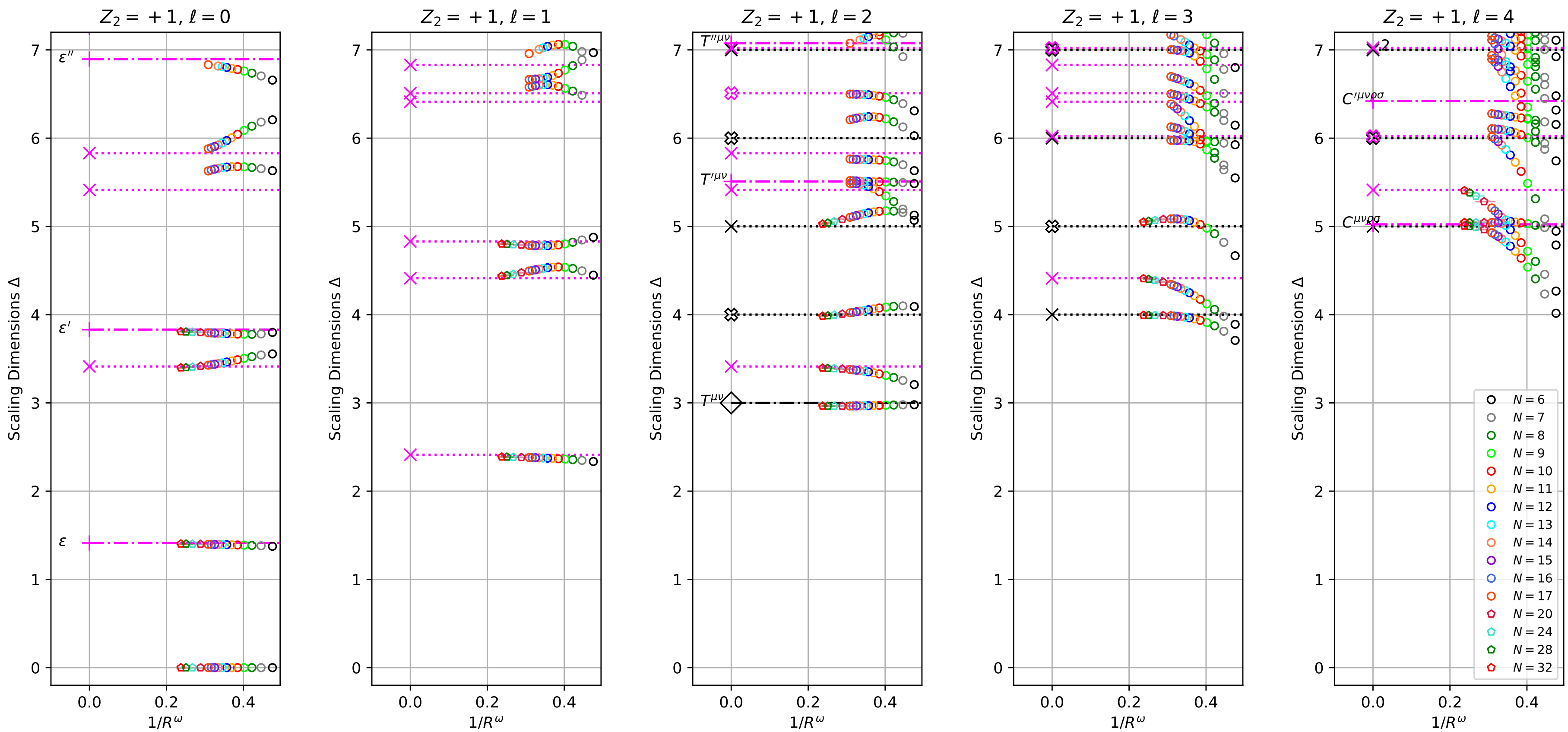
What we do

Project a simple model on the Fuzzy sphere to get a regularized complex models

Using exact diagonalization, find the critical point

ED + DMRG: get a (large) set of low-energy states

DMRG: get excited states by including $\mu|GS\rangle\langle GS|$ in the Hamiltonian



Conformal perturbation expansion

Small systems = perturbation theory

Due to finite-size effects, need to include perturbations $\mathcal{S}_{\mathrm{EFT}} = \mathcal{S}_{\mathrm{CFT}} + \sum_{\mathcal{V}} G_{\mathcal{V}} \int d^d x \mathcal{V}(x)$

While relevant operators can be tuned away, irrelevant operators $G_{\mathcal{V}} \sim 1$

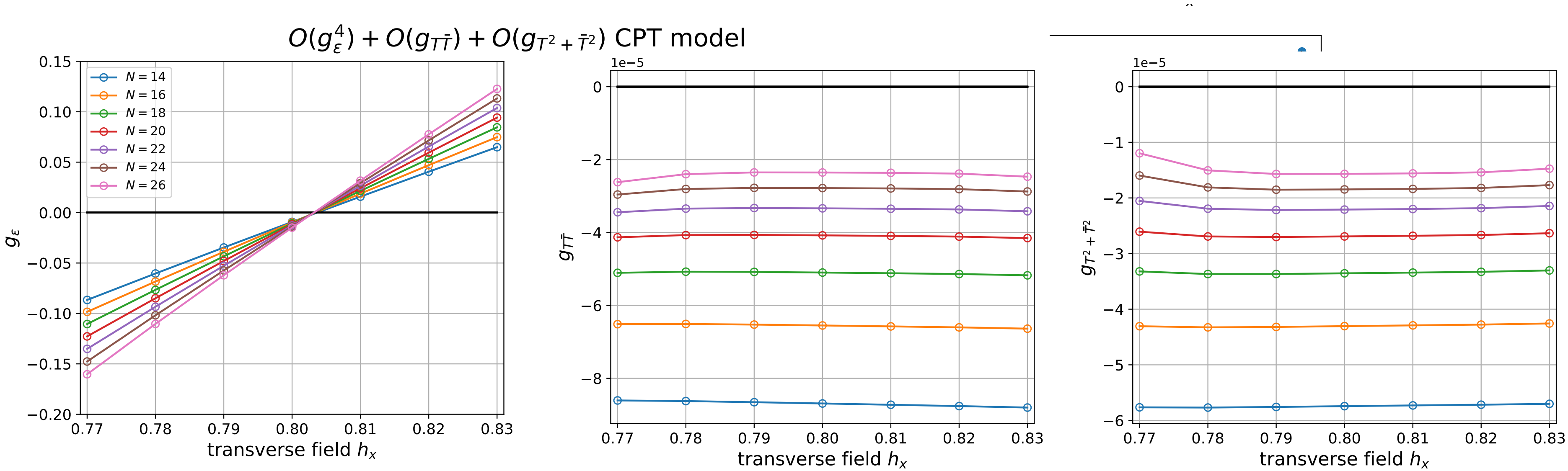
Operator expectation values need to be adjusted

$$\langle \Phi(0)\Phi(r) \rangle_{\mathrm{EFT}} = \langle \Phi(0)\Phi(r) \rangle_{\mathrm{CFT}} - \sum_{\mathcal{V}} G_{\mathcal{V}} \int d^d x \langle \Phi(0)\Phi(r)\mathcal{V}(x) \rangle_{\mathrm{CFT}} + \dots$$

After a bit of algebra, once on the sphere

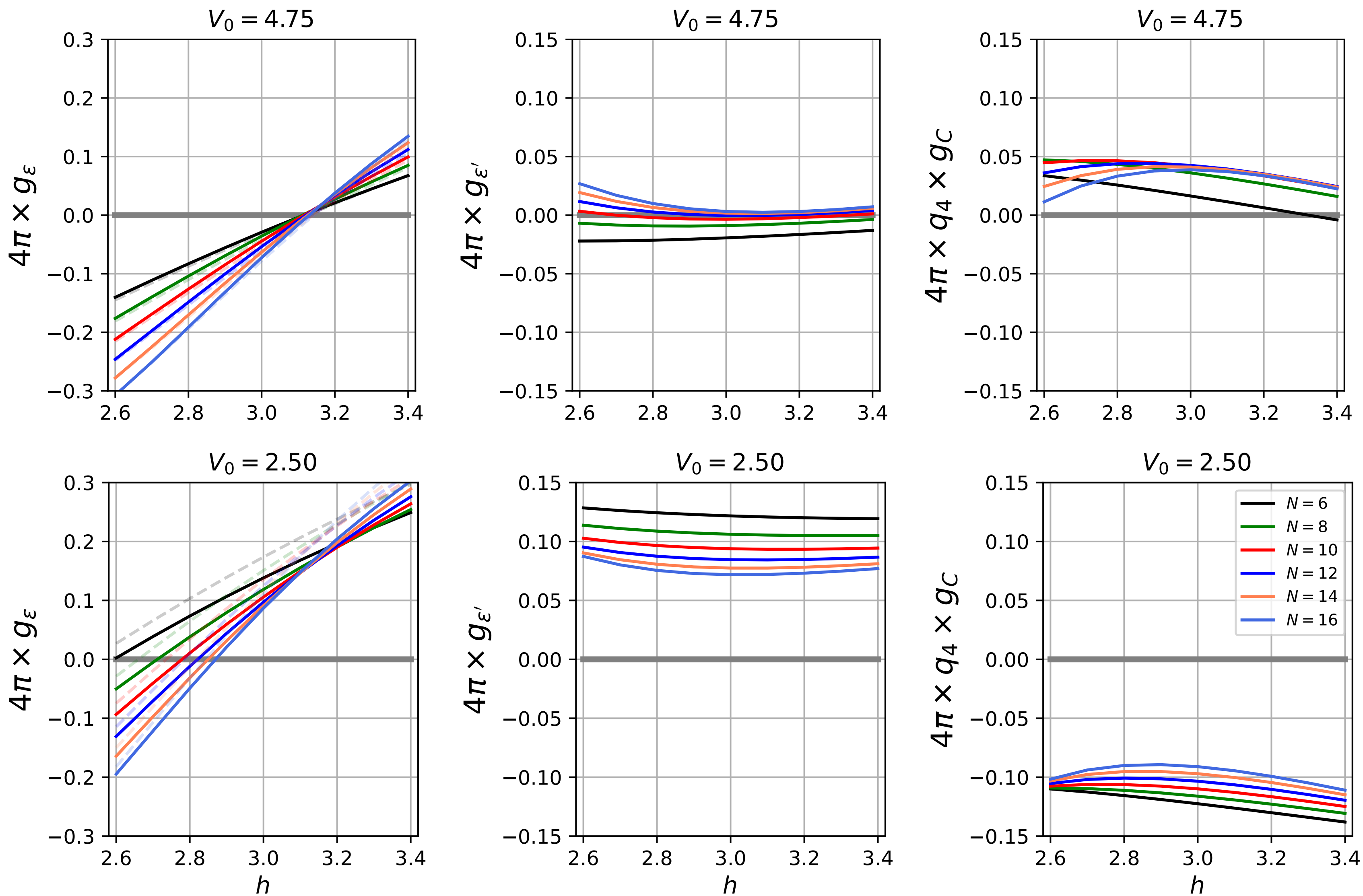
Primary fields $\delta E_{\mathcal{O}} = G_{\mathcal{V}} R^{d-\Delta_{\mathcal{V}}} \mathrm{Vol}(\mathbb{S}^{d-1}) f_{\mathcal{O}\mathcal{O}\mathcal{V}}$

Descendants $\delta E_{\partial\mathcal{O}} = \left(1 + \frac{\Delta_{\mathcal{V}}(\Delta_{\mathcal{V}} - 3)}{6\Delta_{\mathcal{O}}}\right) \delta E_{\mathcal{O}}$



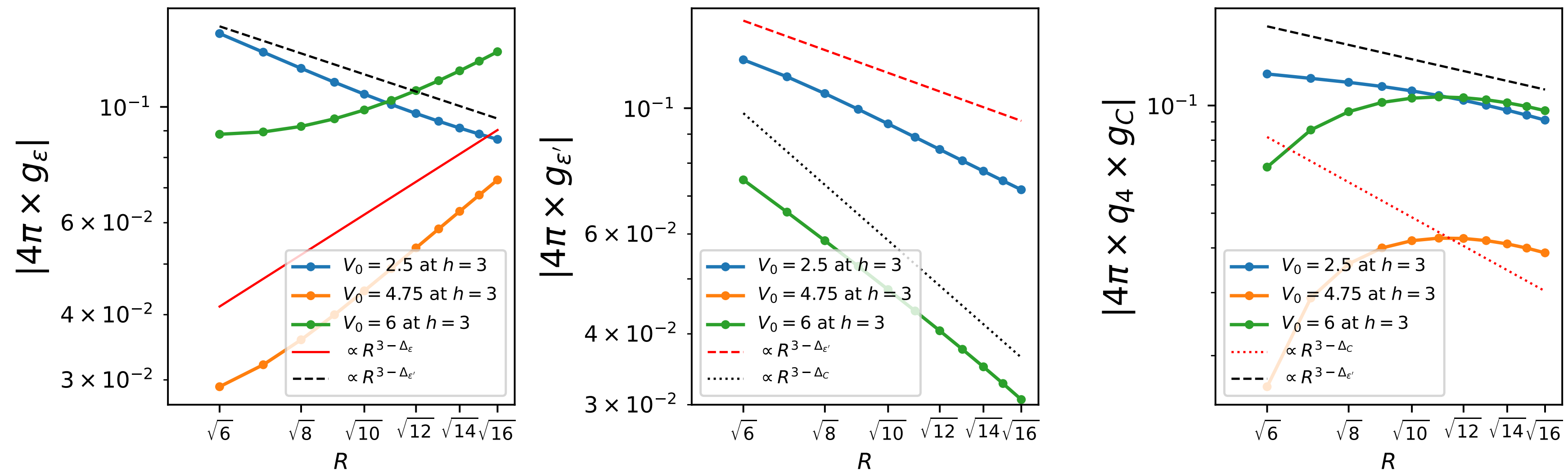
Conformal perturbation expansion

Estimate the perturbation by fitting δE_σ and its descendants

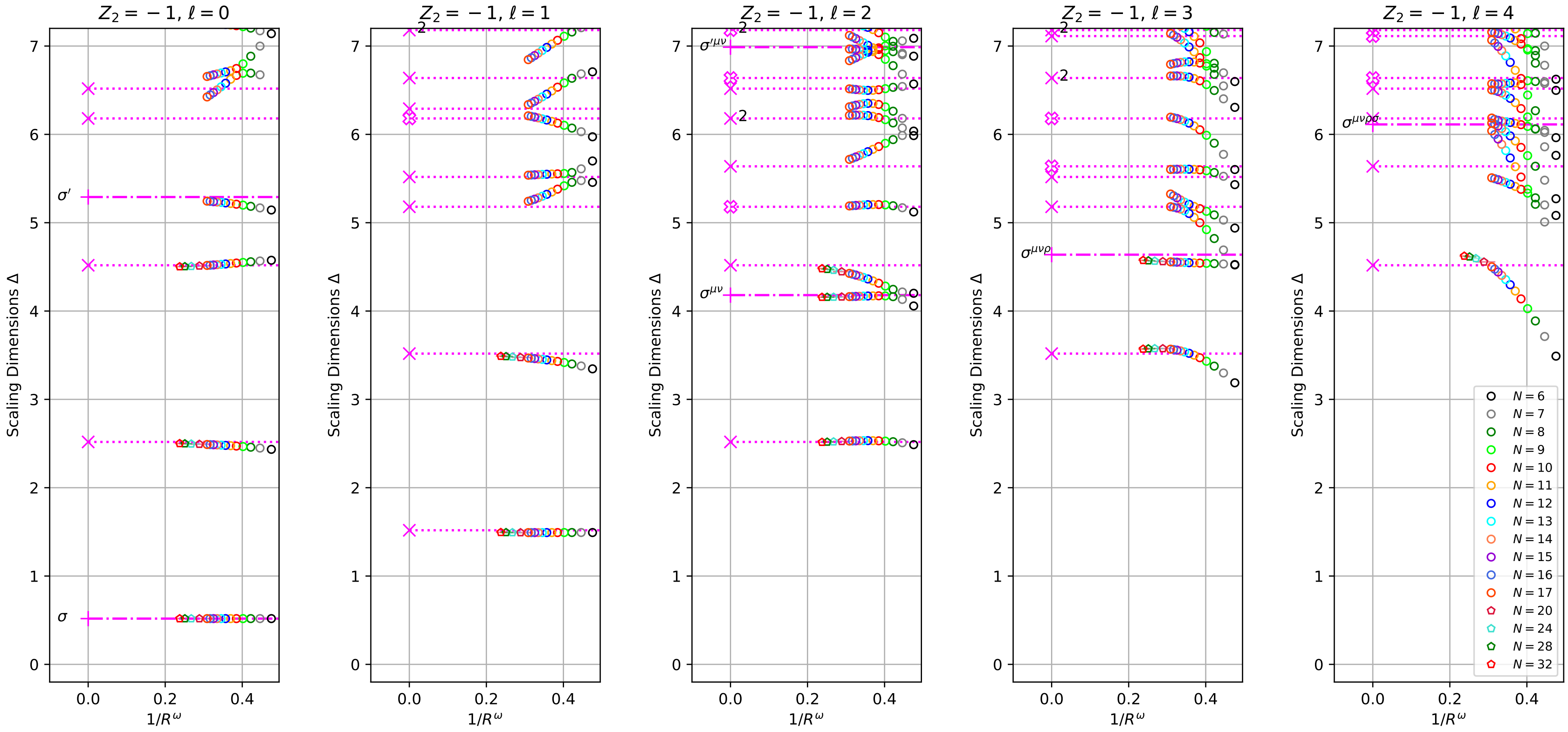


Conformal perturbation expansion

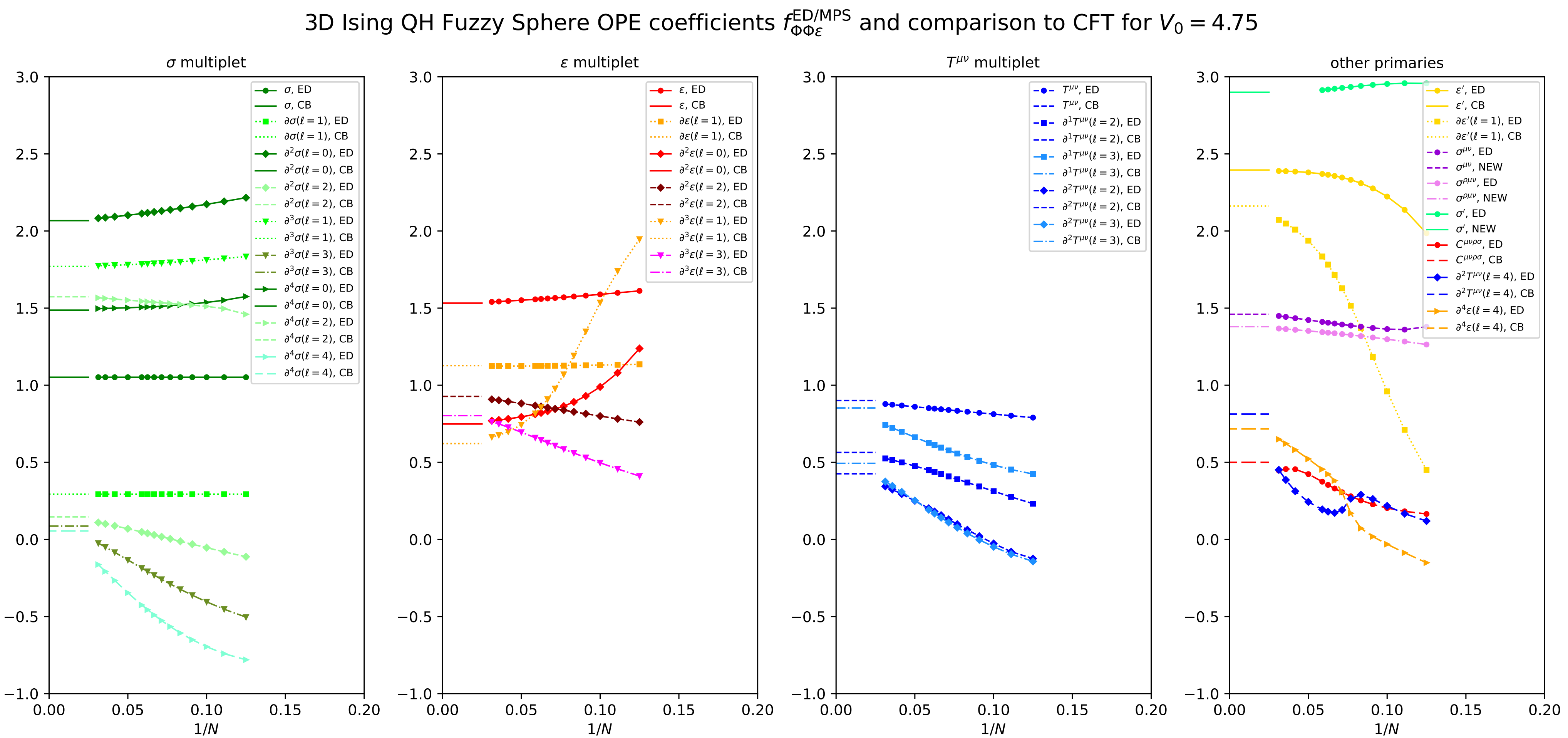
Estimate the perturbation by fitting δE_σ and its descendants



Some results for the Ising models



Some results for the Ising models



Generalization of the approach: O(N) models

O(N) Wilson-Fisher universality class

Kinetic energy O(N) ferromagnetism

$$\hat{H}_N = \frac{1}{2I}\sum_i \hat{\mathbf{L}}_i^2 - J\sum_{\langle ij\rangle} \hat{\mathbf{n}}_i\cdot\hat{\mathbf{n}}_j,$$

N-dimensional quantum rotors on a regular lattice

Key concepts: 1) diagonalize the kinetic term $\hat{\mathbf{L}}^2$

$$N=2: \quad e^{im\theta},\ m\in\mathbb{Z} \qquad E_{\text{kin}} = E_0 m^2$$

$$N=3: \quad Y_{l,m},\ l\in\mathbb{N},\ -l\le m\le l \qquad E_{\text{kin}} = E_0 l(l+1)$$

2) truncate the basis with a minimal number of irreps

$$N=2: \quad e^{im\theta},\ m\in\{0,1,-1\} \Rightarrow 3 \text{ orbitals} \qquad N=3: \quad Y_{l,m},\ l\in\{0,1\},\ -l\le m\le l \Rightarrow 4 \text{ orbitals}$$

3) rewrite H in the truncated basis (pseudospin) and apply our recipe

A few comments

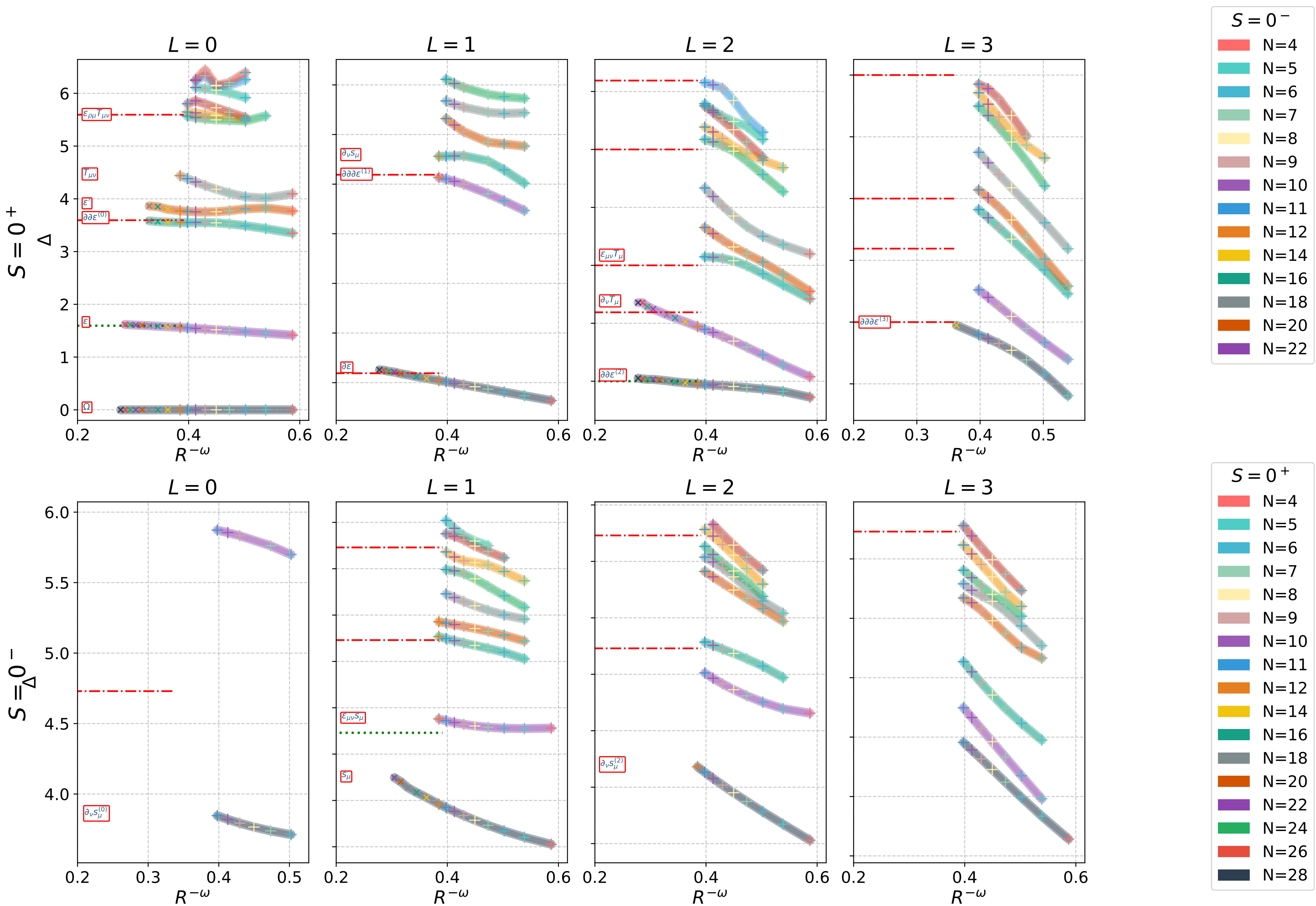
Obviously, any $O(N_1)\times O(N_2)$ models can be straightforwardly studied

But! the larger the pseudospin the higher the entanglement => non-Abelian symmetries

If you have any SU(N) field theory with arbitrary irreps, let me know

Generalization of the approach: O(N) models

O(3) Model: S=0 Sectors



Generalization of the approach: O(N) models

$S^\pm$ L I	o	Δ(ED)	Δ(DMRG)	Δ(CB)
1^- 0 1	σ	0.51893 (12)	0.51893 (28)	**0.518936** [17]
0^+ 0 2	ε	1.56189 (12)	1.61781 (26)	**1.59488** [17]
0^+ 0 4	ε'	3.77279 (12)	3.80177 (14)	3.76680 [20]
0^+ 2 1	$T_{\mu\nu}$	2.97749 (12)	3.01062 (20)	3^*
1^+ 1 1	j_μ	1.88253 (12)	1.97204 (26)	2^*
2^+ 0 1	$t_{(2)}$	1.25407 (12)	1.23767 (26)	1.20954 [17]
4^+ 0 1	$t_{(4)}$	3.26858 (11)	3.16459 (26)	2.99056 [17]

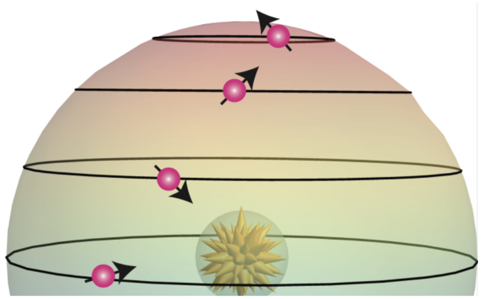
S L I	o	Δ(ED)	Δ(DMRG)
0^- 1 1	ε_μ	2.96067 (12)	3.2396 (18)
1^+ 1 3	ϕ_μ	3.70828 (12)	3.75703 (18)
1^- 0 3	σ'	3.87329 (12)	-
1^- 1 2	σ_μ	2.87055 (12)	3.04373 (26)
2^+ 0 3	$t'_{(2)}$	3.67128 (12)	3.68121 (16)
2^+ 0 4	$t''_{(2)}$	4.31261 (12)	-
2^+ 2 1	$t'_{(2)\mu\nu}$	3.02538 (12)	3.02825 (26)
2^- 1 1	$t_{(2)\mu}$	2.68970 (12)	2.78187 (26)
2^- 2 1	$t'_{(2)\mu\nu}$	3.29489 (12)	3.5341 (24)
3^+ 1 1	$\chi_{(3)\mu}$	3.68769 (12)	3.75082 (26)
3^+ 2 1	$\chi_{(3)\mu\nu}$	4.22326 (12)	-
3^- 0 1	$\sigma_{(3)}$	2.17310 (12)	2.12486 (26)
3^- 0 3		4.88895 (11)	-
3^- 2 1	$\sigma_{(3)\mu\nu}$	4.02221 (12)	3.9825 (24)
4^+ 0 3		6.25369 (11)	-
4^+ 2 1	$t_{(4)\mu\nu}$	5.17757 (11)	-
4^- 1 1		4.85139 (11)	4.87182 (26)
4^- 2 1		5.29724 (11)	-

Generalization of the approach: Yang-Mill models

Not yet published…

Just follow the recipe with a well-chosen model

Generalization of the approach: Boundary CFTs



Boundary CFTs

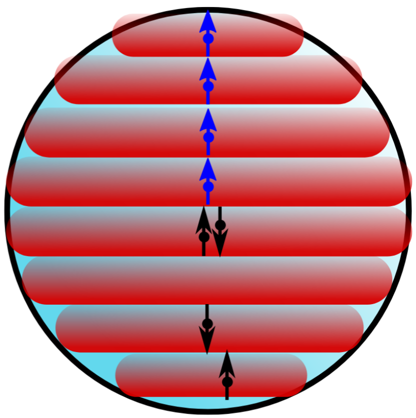
As they are gapless, CFTs have specific responses to BC

New/renormalized operators appear

Still complicated to study! => Fuzzy sphere approaches

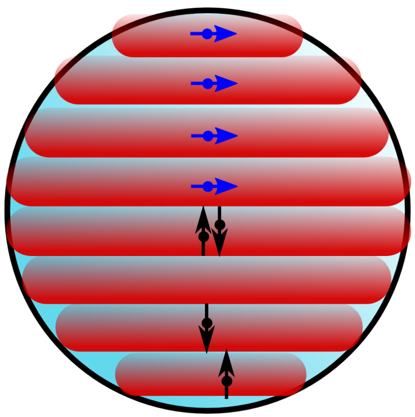
BCFT in practice (Ising)

Soft boundary conditions with a chemical potential on one hemisphere



Z2-breaking boundary

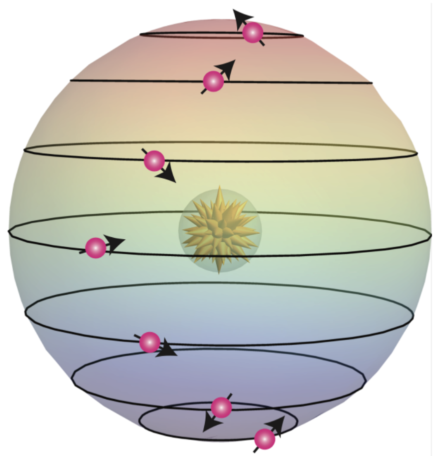
	Quantity	This work	CB	MC
Ordinary	Δ_o	1.23(4)	1.276(2)	1.2751(6)
	a_ϵ	0.74(4)	0.750(3)	
	$b_{\epsilon \mathrm{D}}$	0.92(4)		
	$b_{\sigma o}$	0.87(2)	0.755(13)	
	C_D	0.0089(2)		
	c_bd	−0.0159(5)		
Normal	a_ϵ	6.4(9)	6.607(7)	
	a_σ	2.58(16)	2.599(1)	2.60(5)
	$b_{\epsilon \mathrm{D}}$	1.74(22)	1.742(6)	
	$b_{\sigma \mathrm{D}}$	0.254(17)	0.25064(6)	0.244(8)
	C_D	0.176(2)	0.182(1)	0.193(5)
	c_bd	−1.44(6)		



Z2-preserving boundary

Zhou et al, SciPost Phys. 18, 031 (2025) ; Dedushenko, arXiv:2407.15948

Conclusions



Fuzzy sphere provides a systematic way to investigate 2+1d CFTs

Conformal data can be straightforwardly extracted from energy levels

Finite-size effects remain large even with DMRG-like methods

Future/open questions

Better treatment of non-Abelian symmetries

Combine with other approaches to further constrain conformal data

Find new models to study beyond simple zoo classifications? (SU(N), please)

What kills us is entanglement, in the end, but $S_{\mathrm{vN}} = \cancel{\alpha L} - a \log L + \ldots.$

Annoying stuff   Conformal stuff

Already not so bad, but can we do better / use hybrid approaches