# UNDARK 2026 – “THE MICROPHYSICS OF NON-THERMAL PROCESSES’

## EXAMPLE SOLUTIONS

### EXERCISE 1 - EXTENDED AIR SHOWER

**1.1** From $p = k_B T n$ we get:

$$n(0) = \frac{p}{k_B T} \simeq \frac{10^5 \mathrm{N/m}^2}{273\mathrm{K}1.4 \times 10^{-23}\mathrm{J/K}} \simeq 2.6 \times 10^{25}\mathrm{m}^{-3}\,.$$

**1.2** Equal chemical potentials across atmospheric layers imply:

$$k_B T \ln(n(0)/n_Q) = k_B T \ln(n(h)/n_Q) + Mgh\,.$$

This can be solved for $n(h) = n(0)e^{-h/\ell}$ with $\ell = k_B T/Mg$. The quantum density $n_Q$ drops out in the difference of logarithms. Note, that this is only an approximation. For instance, the rotational degrees of freedom of dinitrogen contribute to the total chemical potential as well.

**1.3** For nitrogen we have:

$$\ell = \frac{k_B T}{Mg} \simeq \frac{273\mathrm{K}1.4 \times 10^{-23}\mathrm{J/K}}{28u9.81\mathrm{m/s}^2} \simeq 8.2\mathrm{km}\,.$$

The integral gives $1 = \sigma_{\rm CR}\ell n(0)\exp(-h_{\rm CR}/\ell)$ which can be solved as:

$$h_{\rm CR} = \ell \ln(\sigma_{\rm CR}\ell n(0)) \simeq 19.4\mathrm{km}\,.$$

This is the right order of magnitude. A more sophisticated atmospheric model gives somewhat larger values.

### EXERCISE 2 - MULTI-MESSENGER SOURCE

**2.1** Larmor radius of $10^{20}$ eV protons ($Z = 1$) in magnetic field $B = 10^{-9}$ G is:

$$r_L = \frac{cp}{ZeB} \simeq 3.34 \times 10^{24}\mathrm{m} \simeq 108\ \mathrm{Mpc}\,.$$

The maximal deflection angle $\Delta\psi$ off the true position of a source at $d = 10$ Mpc follows from a magnetic field perpendicular to the line-of-sight:

$$\sin\Delta\psi = \frac{d}{2r_L}\,.$$

For small angles one can Taylor-expand $\sin\Delta\psi \simeq \Delta\psi + \mathcal{O}((\Delta\psi)^3)$ and arrive at

$$\Delta\psi \simeq \frac{d}{2r_L} \simeq 2.7^\circ\,.$$

**2.2** The arc length is $2\Delta\psi r_L$ and therefore the CR delay compared to a photon emitted at the same time is:

$$\Delta t = \frac{2\Delta\psi r_L - d}{c}\,.$$

Now, here we have to be a bit careful since we are subtracting two large numbers to find a small difference. Using our first order Taylor-term for $\Delta\psi$ would give us $\Delta t = 0$, so we have to go to the next order:

$$\sin\Delta\psi \simeq \Delta\psi - \frac{1}{6}(\Delta\psi)^3 + \mathcal{O}((\Delta\psi)^5)\,,$$

giving:

$$\Delta\psi \simeq \frac{d}{2r_L} + \frac{1}{6}(\Delta\psi)^3\,.$$

To leading order we get therefore:

$$\Delta t \simeq \frac{2\Delta\psi r_L - d}{c} \simeq \frac{1}{3}\frac{r_L}{c}(\Delta\psi)^3 \simeq 11600\text{yr}\,.$$

**2.3** The time delay between the photon and the neutrino is given by:

$$\begin{aligned}\Delta t &= \frac{d}{c}(\beta^{-1} - 1)\\ &\approx \frac{d}{2\,c}\frac{m_\nu^2}{E^2}\,.\end{aligned}$$

For d =10 Mpc, $m_\nu = 0.1\,\text{eV}$, and $E = 1\,\text{EeV}$ this gives a time difference of $\sim 5 \times 10^{-24}$ s (5 yoctoseconds). This time scale is unfortunately much larger than the activity time scale of transients, so the determination of the neutrino mass would not unfeasible.

## Exercise 3 - Galactic PeVatrons

**3.1** The following solution is – upon request – very explicit. As a warm-up, let's first start with an impulsive source that we discussed in lecture. Without loss of generality, we can assume that the source emits at time $t = 0$ and is located at the origin of our coordinate system. The source term is then simply $q(t, \mathbf{r}, E) = N(E)\delta(t)\delta^{(3)}(\mathbf{r})$. For isotropic diffusion we have a diagonal diffusion tensor $\mathbf{K} = K\mathbf{1}$ with unit matrix $\mathbf{1}$ and diagonal elements $K$ with inverse $\mathbf{K}^{-1} = \mathbf{1}/K$. The Green's function for times $t > t_s$ becomes:

$$G(t, \mathbf{r}; t_s, \mathbf{r}_s) = \frac{1}{(4\pi(t - t_s)K)^{3/2}} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_s)^2}{4(t - t_s)K}\right)\,.$$

If we observe the source after emission, $t > 0$, the solution is:

$$\begin{aligned}n(t, E, \mathbf{r}) &= \int d^3r_s \int_{-\infty}^{t} dt_s G(t, \mathbf{r}; t_s, \mathbf{r}_s) q(t_s, \mathbf{r}_s, E)\\ &= N(E) \int d^3r_s \int_{-\infty}^{t} dt_s \frac{1}{(4\pi(t - t_s)K)^{3/2}} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_s)^2}{4(t - t_s)K}\right) \delta(t_s)\delta^{(3)}(\mathbf{r}_s)\\ &= \frac{N(E)}{(4\pi tK)^{3/2}} \exp\left(-\frac{\mathbf{r}^2}{4tK}\right)\,.\end{aligned}$$

Now, instead of an impulsive source we consider a source that emits continuously with a emission rate $Q(E)$. The source term is then simply $q(t, \mathbf{r}, E) = Q(E)\delta^{(3)}(\mathbf{r})$. In this case, the local CR spectral density at time $t$ becomes:

$$\begin{aligned}n(t, E, \mathbf{r}) &= \int d^3r_s \int_{-\infty}^{t} dt_s G(t, \mathbf{r}; t_s, \mathbf{r}_s) q(t_s, \mathbf{r}_s, E)\\ &= \int d^3r_s \int_{-\infty}^{t} dt_s \frac{Q(E)}{(4\pi(t - t_s)K)^{3/2}} \exp\left(-\frac{(\mathbf{r} - \mathbf{r}_s)^2}{4(t - t_s)K}\right) \delta^{(3)}(\mathbf{r}_s)\\ &= \int_{-\infty}^{t} dt_s \frac{Q(E)}{(4\pi(t - t_s)K)^{3/2}} \exp\left(-\frac{\mathbf{r}^2}{4(t - t_s)K}\right) = \int_{0}^{\infty} d\tau \frac{Q(E)}{(4\pi\tau K)^{3/2}} \exp\left(-\frac{r^2}{4\tau K}\right)\end{aligned}$$

where we made the substitution $\tau \equiv t - t_s$ with $d\tau/dt_s = -1$ in the last step.

In order to be able to use the hint, we now define $x \equiv r^2/(4\tau K)$ with $dx/d\tau = -r^2/(4\tau^2 K) = -4Kx^2/r^2$. Another substitution gives us:

$$n(t, E, r) = Q(E) \int_0^\infty dx \frac{d\tau}{dx} \left(\frac{x}{\pi r^2}\right)^{3/2} e^{-x} = \frac{Q(E)}{4K\pi^{3/2}} \frac{1}{r} \int_0^\infty dx x^{1/2-1} e^{-x} = \frac{Q(E)}{4K\pi r} .$$

Note that this is independent of observation time $t$ and scales as $1/r$.

**3.2** The figure shows that for CR energy density at, say, $r = 100$ pc is $w_{\text{CR}} \simeq 6 \times 10^{-3}\ \text{eV/cm}^3$. The luminosity above 10 TeV is then:

$$\begin{aligned} L(\geq 10\ \text{TeV}) &\equiv \int_{10\ \text{TeV}}^\infty \mathrm{d}E E Q(E) = 4\pi K r \int_{10\ \text{TeV}}^\infty \mathrm{d}E E n(E, r = 100\text{pc}) \\ &= 4\pi K \times 100\text{pc} \times w_{\text{CR}} \simeq 2 \times 10^{49} \frac{\text{eV}}{\text{s}} \simeq 4 \times 10^{37} \frac{\text{erg}}{\text{s}} . \end{aligned}$$

## Exercise 4 - Fermi Acceleration

**4.1** We have at $t = 0$, $N_0$ particles with energy $E_0$.

$$N_0 = N_{in} + N_{out} \ , \tag{1}$$

so that,

$$\frac{\mathrm{d}N_{out}}{\mathrm{d}E} = \frac{\mathrm{d}(N_0 - N_{in})}{\mathrm{d}E} = -\frac{\mathrm{d}N_{in}}{\mathrm{d}t} \Big/ \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{\tau_{\text{acc}}}{\tau_{\text{esc}}} \frac{N_{in}}{E} = \frac{\tau_{\text{acc}}}{\tau_{\text{esc}}} \frac{N_0 - N_{out}}{E}$$

Integrating this differential equation gives:

$$N_{out}(E) = N_0 \left(1 - \left(\frac{E}{E_0}\right)^{-\frac{\tau_{\text{acc}}}{\tau_{\text{esc}}}}\right)$$

Therefore $dN_{out}/dE \propto E^{-\alpha}$ with $\alpha = 1 + \tau_{\text{acc}}/\tau_{\text{esc}}$.

**4.2** In the strong shock limit with $\gamma = 5/3$:

$$\lim_{\mathcal{M}\to\infty} \frac{v_1}{v_2} = \frac{\gamma + 1}{\gamma - 1} = 4$$

The spectral index is

$$\alpha = 1 + \frac{\tau_{\text{acc}}}{\tau_{\text{esc}}} = 1 + \frac{3}{v_1/v_2 - 1} = 1 + \frac{3}{3} = 2$$