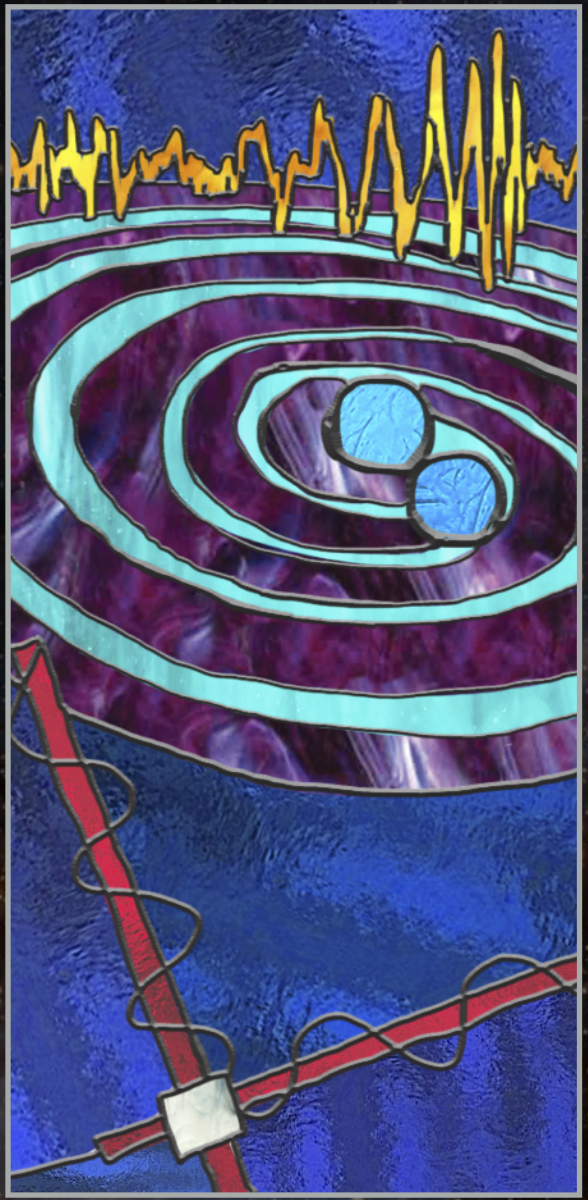
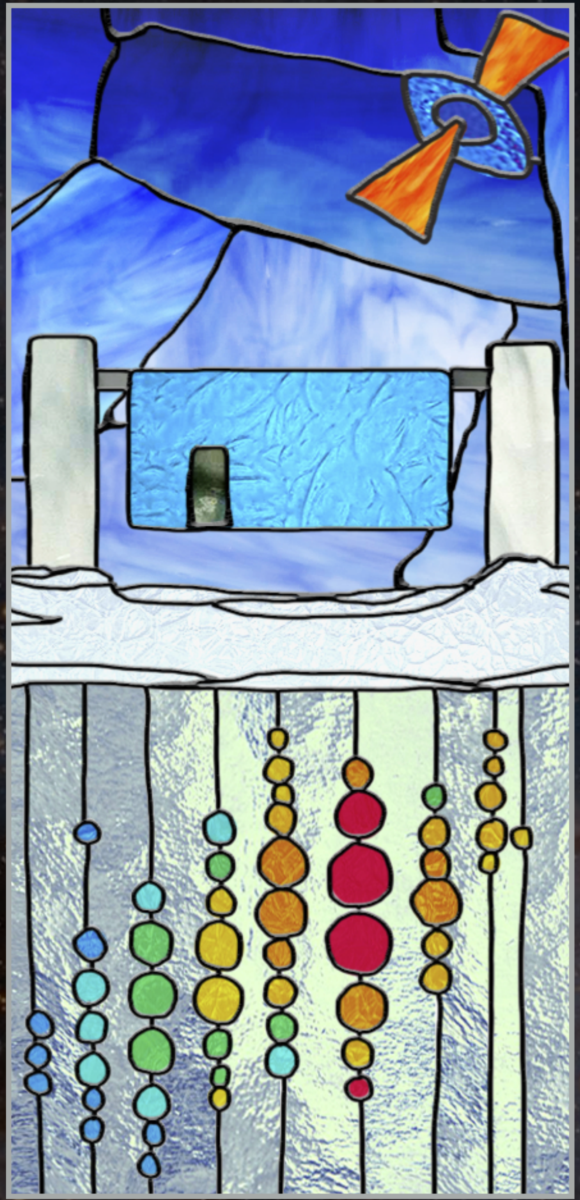
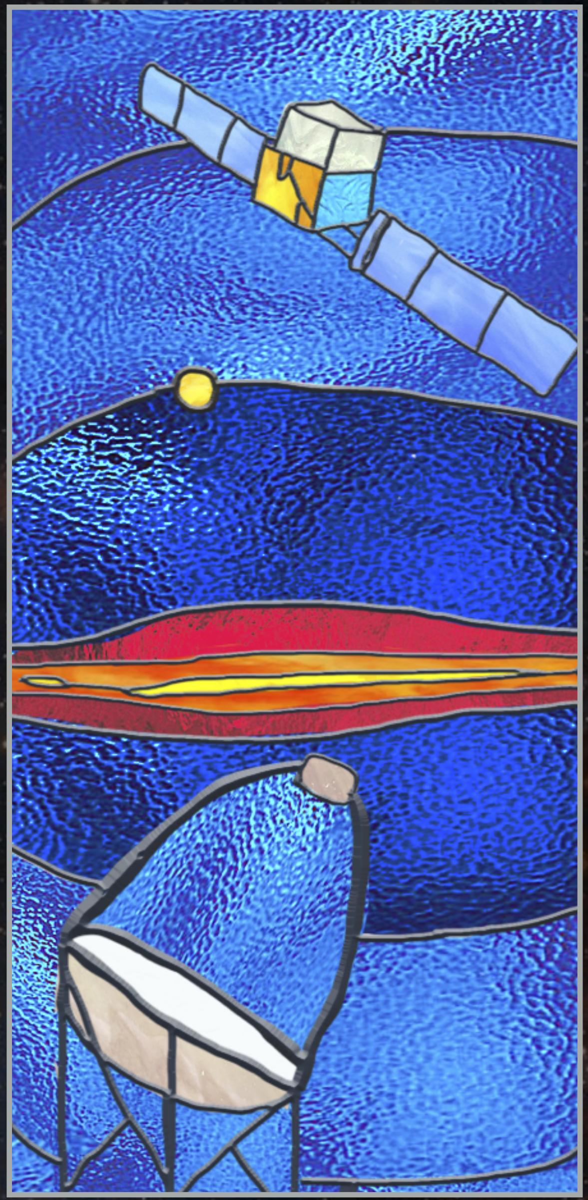






Radiative Processes in Astroparticle Physics

Markus Ahlers
Niels Bohr Institute

2nd UNDARK School 2026

CARLSBERG FOUNDATION

KØBENHAVNS UNIVERSITET

Lecture Outline

Lecture 1: Leptonic Processes

preliminaries, synchrotron, Compton & inverse-Compton, bremsstrahlung, pair production

Lecture 2: Hadronic Processes

cosmic rays, diffusive shock acceleration, transport & interaction, pion production & decay

Lecture 3: Multi-Messenger Astronomy

MM paradigm, neutrino astronomy

Suggested Literature

Radiative Processes in Astrophysics

by George B. Rybicki & Alan P. Lightman

Radiative Processes in High Energy Astrophysics

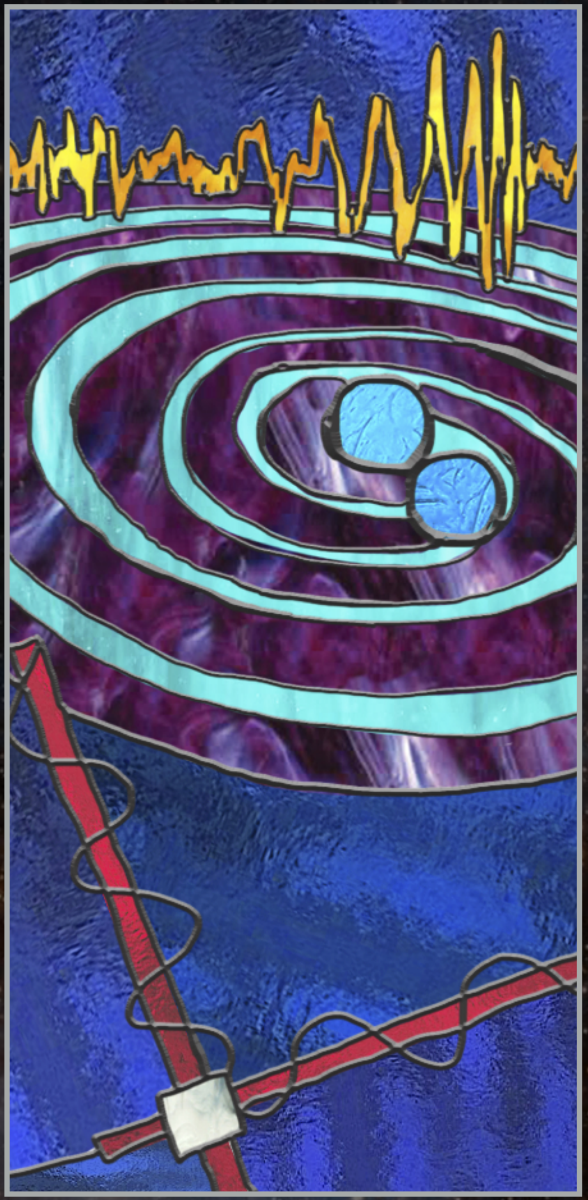
by Gabriele Ghisellini

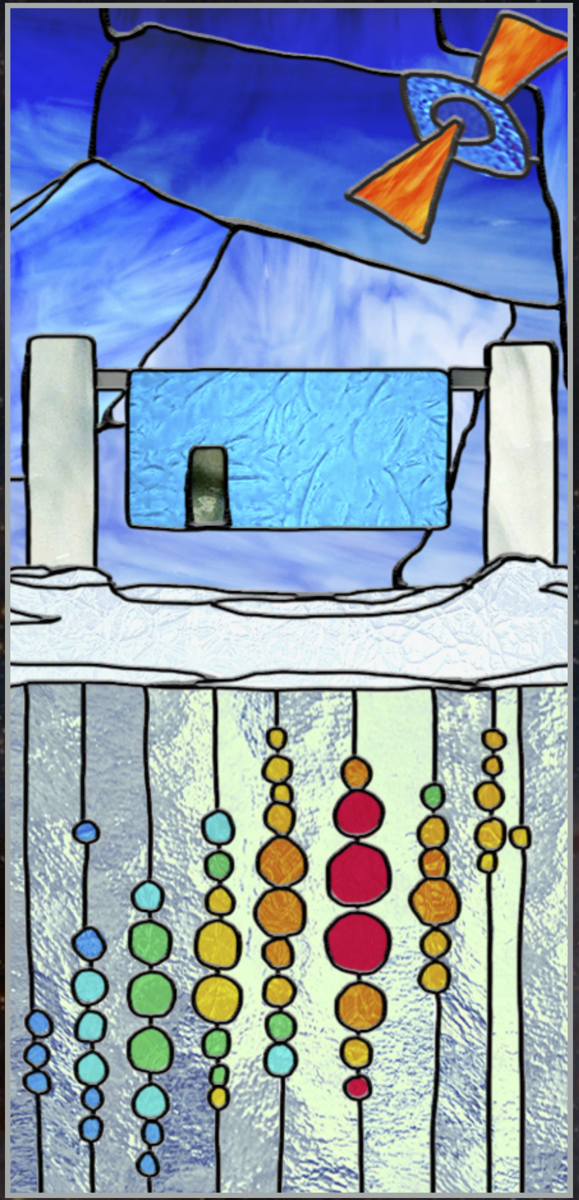
Cosmic Rays and Particle Physics

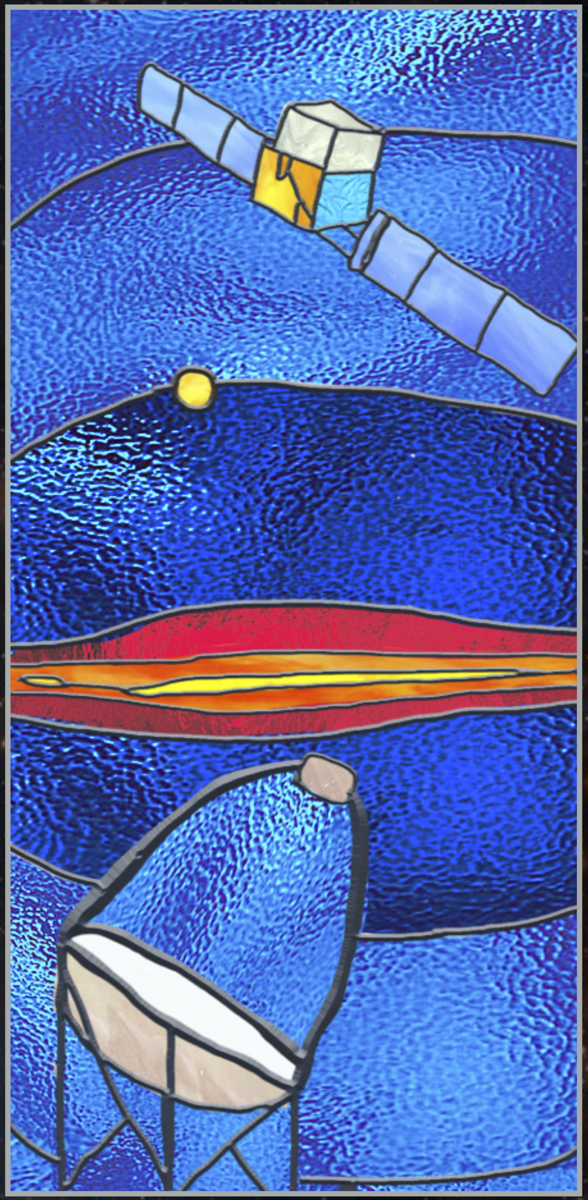
by Thomas K. Gaisser, Ralph Engel & Elisa Resconi

Probes of Multimessenger Astrophysics

by Maurizio Spurio







Lecture 1: *Leptonic Processes*

Markus Ahlers

Niels Bohr Institute

2nd UNDARK School 2026

CARLSBERG FOUNDATION

KØBENHAVNS UNIVERSITET

Preliminaries

Revision: Maxwell's Equations

- We will work in **Gaussian units** (*EM fields have same units*).
- **Maxwell's equations**:

$$\underbrace{\nabla \cdot \mathbf{B} = 0}_{\text{Gauss's law}} \qquad \underbrace{\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial}{\partial t}\mathbf{B}}_{\text{Faraday's law}}$$

$$\underbrace{\nabla \cdot \mathbf{E} = 4\pi\rho}_{\text{Gauss's law}} \qquad \underbrace{\nabla \times \mathbf{B} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial}{\partial t}\mathbf{E}}_{\text{Ampere's law}}$$

- **Lorentz force**: $$\mathbf{F} = q\left(\mathbf{E} + \frac{\mathbf{v}}{c} \times \mathbf{B}\right)$$
- **Elementary charge unit**: $$\frac{e^2}{\hbar c} = \alpha \simeq \frac{1}{137}$$

Conversion between Metric Systems

- **Gaussian (G)** to **SI units** (*recall relation* $\epsilon_0\mu_0 = c^{-2}$):

$$\rho_{\rm SI} = \sqrt{4\pi\epsilon_0}\,\rho_{\rm G} \quad \mathbf{j}_{\rm SI} = \sqrt{4\pi\epsilon_0}\,\mathbf{j}_{\rm G} \quad \mathbf{E}_{\rm SI} = \frac{\mathbf{E}_{\rm G}}{\sqrt{4\pi\epsilon_0}} \quad \mathbf{B}_{\rm SI} = \sqrt{\frac{\mu_0}{4\pi}}\mathbf{B}_{\rm G}$$

- Gaussian to **Heaviside-Lorentz (HL) units** (*common in particle physics*):

$$\rho_{\rm HL} = \sqrt{4\pi}\,\rho_{\rm G} \qquad \mathbf{j}_{\rm HL} = \sqrt{4\pi}\,\mathbf{j}_{\rm G} \qquad \mathbf{E}_{\rm HL} = \frac{\mathbf{E}_{\rm G}}{\sqrt{4\pi}} \qquad \mathbf{B}_{\rm HL} = \frac{\mathbf{B}_{\rm G}}{\sqrt{4\pi}}$$

- **Elementary charge unit**:

$$e_{\rm G}^2 = \frac{e_{\rm SI}^2}{4\pi\epsilon_0} = \frac{e_{\rm HL}^2}{4\pi} \simeq \frac{\hbar c}{137}$$

- In addition, **natural HL units** further use $\hbar = c = 1$.

Natural Units

- In **natural units**, the speed of light c and Planck's reduced constant $\hbar$ are:

$$c = \hbar = 1$$

- Time, length, mass and momentum are now expressed in **units of energy**:

$$\tilde{x} \equiv \frac{x}{\hbar c} \qquad \tilde{t} \equiv \frac{t}{\hbar} \qquad \tilde{p} \equiv pc \qquad \tilde{m} \equiv mc^2$$

- Conversion back to standard (SI) units can be done via the identities:

$$c \simeq 3 \times 10^8\,\mathrm{m/s} \qquad \hbar c \simeq 200\,\mathrm{MeV\,fm}$$

- It is also convenient to set **Boltzmann's constant** $k_B = 1$ with conversion:

$$\frac{k_B T}{300\,\mathrm{K}} \simeq \frac{1}{40}\mathrm{eV}$$

Electromagnetic Potentials

- Introduce **scalar potential** ϕ and **vector potential** $\mathbf{A}$ such that:

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\nabla\phi - \frac{1}{c}\frac{\partial}{\partial t}\mathbf{A}$$

- Invariance of $\mathbf{E}$ and $\mathbf{B}$ under *gauge transformation*:

$$\phi \to \phi - \frac{1}{c}\frac{\partial\Lambda}{\partial t} \qquad \mathbf{A} \to \mathbf{A} + \nabla\Lambda$$

- We can choose the **Lorentz gauge** ($\partial_\mu A^\mu = 0$): $$\frac{1}{c}\frac{\partial}{\partial t}\phi + \nabla\mathbf{A} = 0$$
- This gives us:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi = 4\pi\rho \qquad \left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\mathbf{A} = \frac{4\pi}{c}\mathbf{j}$$

Covariant Form of Maxwell

- Combine **scalar potential ϕ** and **vector potential $\mathbf{A}$** to $A^\mu \equiv (\phi, \mathbf{A})$
- Lorentz gauge becomes $\partial_\mu A^\mu = 0$ with $\partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$.
- Field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

- **Maxwell equations** in covariant form:

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} j^\mu \qquad j^\mu \equiv (c\rho, \mathbf{j})$$

Revision: Special Relativity

- *Postulate*: **Vacuum speed of light c is the same in all inertial frames.**

$$c = \frac{|\Delta \mathbf{x}|}{|\Delta t|} = \frac{|\Delta \mathbf{x}'|}{|\Delta t'|}$$

- **Contravariant** vector: $\Delta x^\mu = (c\Delta t, \Delta \mathbf{x})$

- Minkowski space with **metric tensor**:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- **Covariant** vector: $\Delta x_\mu = \eta_{\mu\nu}\Delta x^\nu = (c\Delta t, -\Delta \mathbf{x})$

- Invariance of c can be expressed as: $\Delta x'^\mu \Delta x'_\mu = \Delta x^\mu \Delta x_\mu$

Revision: Lorentz Boosts

- **Lorentz transformations** defined via: $\Delta x'^{\mu} \equiv \Lambda^{\mu}{}_{\nu} \Delta x^{\nu}$
- Set of $\Lambda^{\mu}{}_{\nu}$ of **rotations and boosts**.
- *For instance*, boost to coordinate system with velocity $\mathbf{v} = v\mathbf{e}_x = c\beta\mathbf{e}_x$:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

velocity:
(units of c)

$$\beta = \frac{v}{c}$$

Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

- **Contravariant four-momentum** (any system):

$$p^{\mu} = (E/c, \mathbf{p}) = (E/c, p_x, p_y, p_z)$$

- Lorentz-transformed four-momentum: $p'^{\mu} = \Lambda^{\mu}{}_{\nu} p^{\nu}$

Revision: Relativistic Relations

- **Time dilation** (for clocks at rest in original system):

$$\Delta x = 0 \rightarrow \Delta t' = \gamma \Delta t$$

- **Length contraction** (for rulers evaluated in transformed system):

$$\Delta t' = 0 \rightarrow \Delta x' = \Delta x / \gamma$$

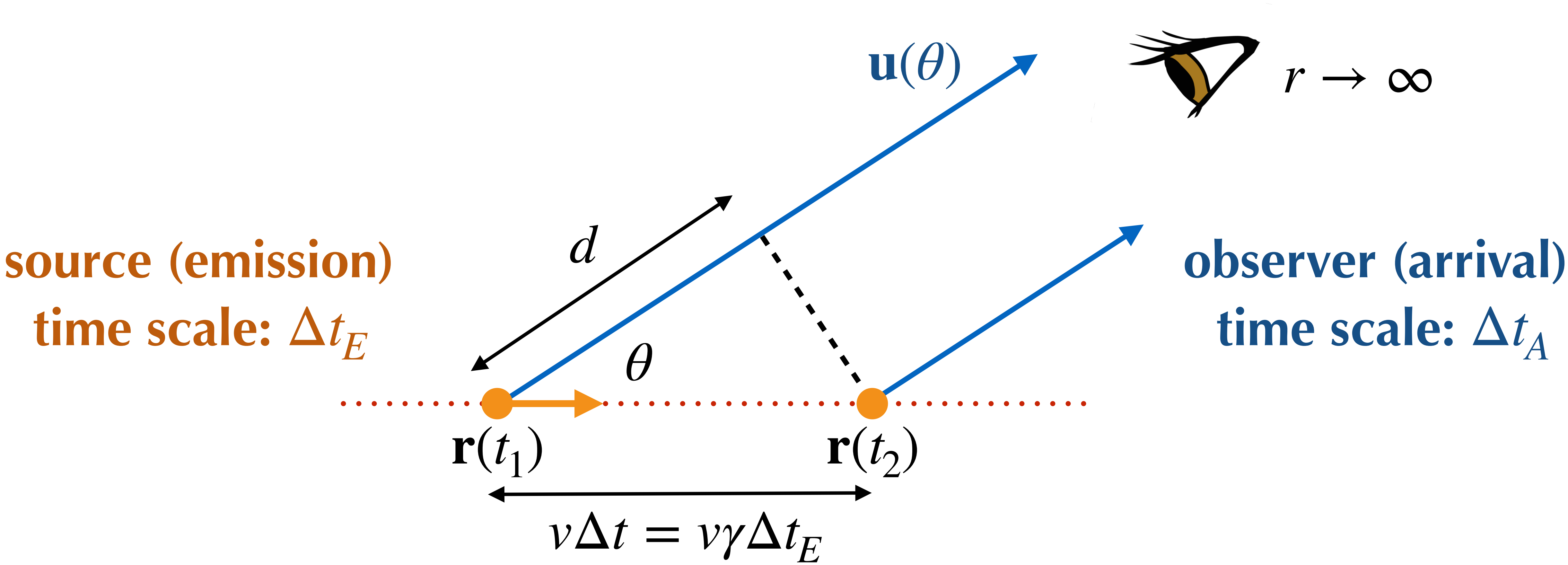
- Invariance of $p^2 \equiv p^\mu p_\mu$ yields the **energy-momentum relation**:

$$(mc^2)^2 = E^2 - (\mathbf{p}c)^2$$

- Lorentz boost of a **particle at rest**:

$$E' = mc^2 \quad \& \quad p' = 0 \quad \rightarrow \quad E = \gamma mc^2 \quad \& \quad p = \gamma m \beta c$$

Relativistic Doppler Effect



$$\Delta t_A = \Delta t - \frac{d}{c} = \Delta t - \frac{v}{c}\cos\theta\Delta t = \frac{\Delta t_E}{\delta(\beta,\theta)}$$

Doppler factor: $\delta(\beta,\theta) \equiv \dfrac{1}{\gamma(1-\beta\cos\theta)}$

Relativistic Doppler Effect

- Transformation of **(angular) frequencies and energies**:

$$\omega_A = \delta(\beta, \theta)\omega_E$$

- Transformation of **velocity u**(θ):

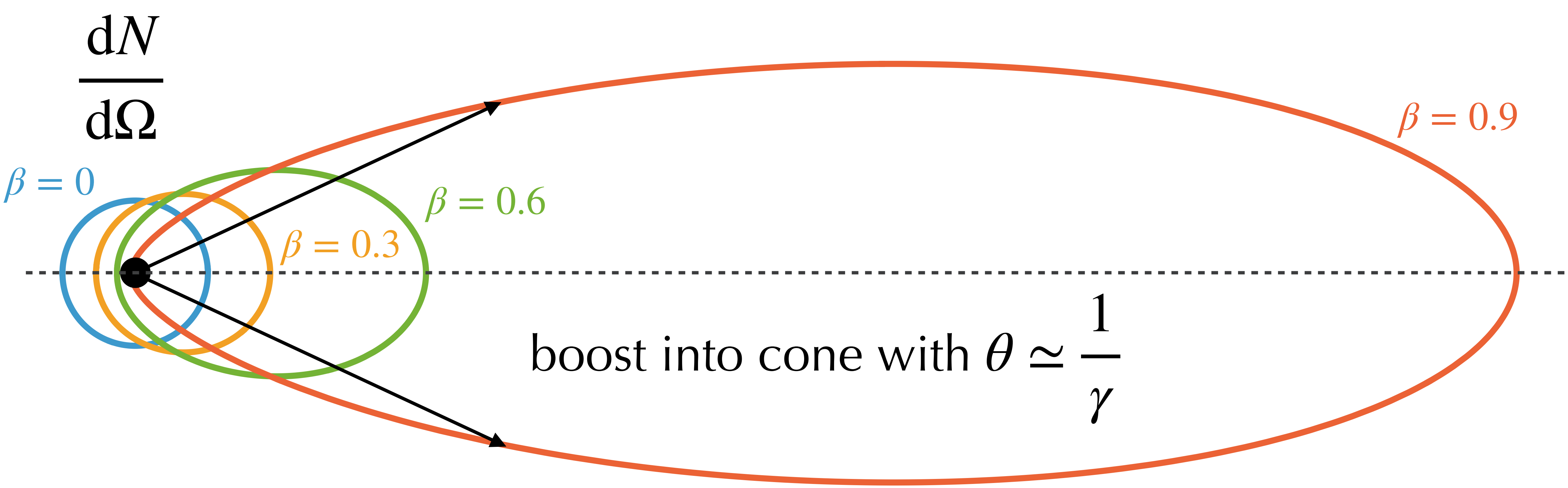
$$\Delta x_{\|} = \gamma(\Delta x'_{\|} + \beta c \Delta t') \qquad c\Delta t = \gamma(c\Delta t' + \beta \Delta x'_{\|})$$

$$u_{\|} = \frac{\Delta x_{\|}}{\Delta t} = \frac{u'_{\|} + v}{1 + v u'_{\|}/c^2} \quad \rightarrow \quad \cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

- Gives us relation between **observation and emission angles**:

$$\frac{\mathrm{d}\cos\theta}{\mathrm{d}\cos\theta'} = \delta^2(-\beta, \theta') = \frac{1}{\delta^2(\beta, \theta)}$$

Doppler-Boosted Emission



consider isotropic emission in **rest frame**: $$\frac{dN}{d\Omega'} = \frac{1}{4\pi}$$

emission in **observer frame**: $$\frac{dN}{d\Omega} = \frac{dN}{d\Omega'}\frac{d\Omega'}{d\Omega} = \frac{\delta^2(\beta,\theta)}{4\pi}$$

Larmor Formula

Lienard-Wiechart Potentials

- Solutions for EM potentials are in the form:

$$\phi(\mathbf{r},t) = \int \mathrm{d}^3\mathbf{r}' \frac{\rho(\mathbf{r}',t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|} \qquad \mathbf{A}(\mathbf{r},t) = \frac{1}{c}\int \mathrm{d}^3\mathbf{r}' \frac{\mathbf{j}(\mathbf{r}',t_{\rm ret})}{|\mathbf{r}-\mathbf{r}'|}$$

- The charge and current distributions are evaluated at the **retarded time**:

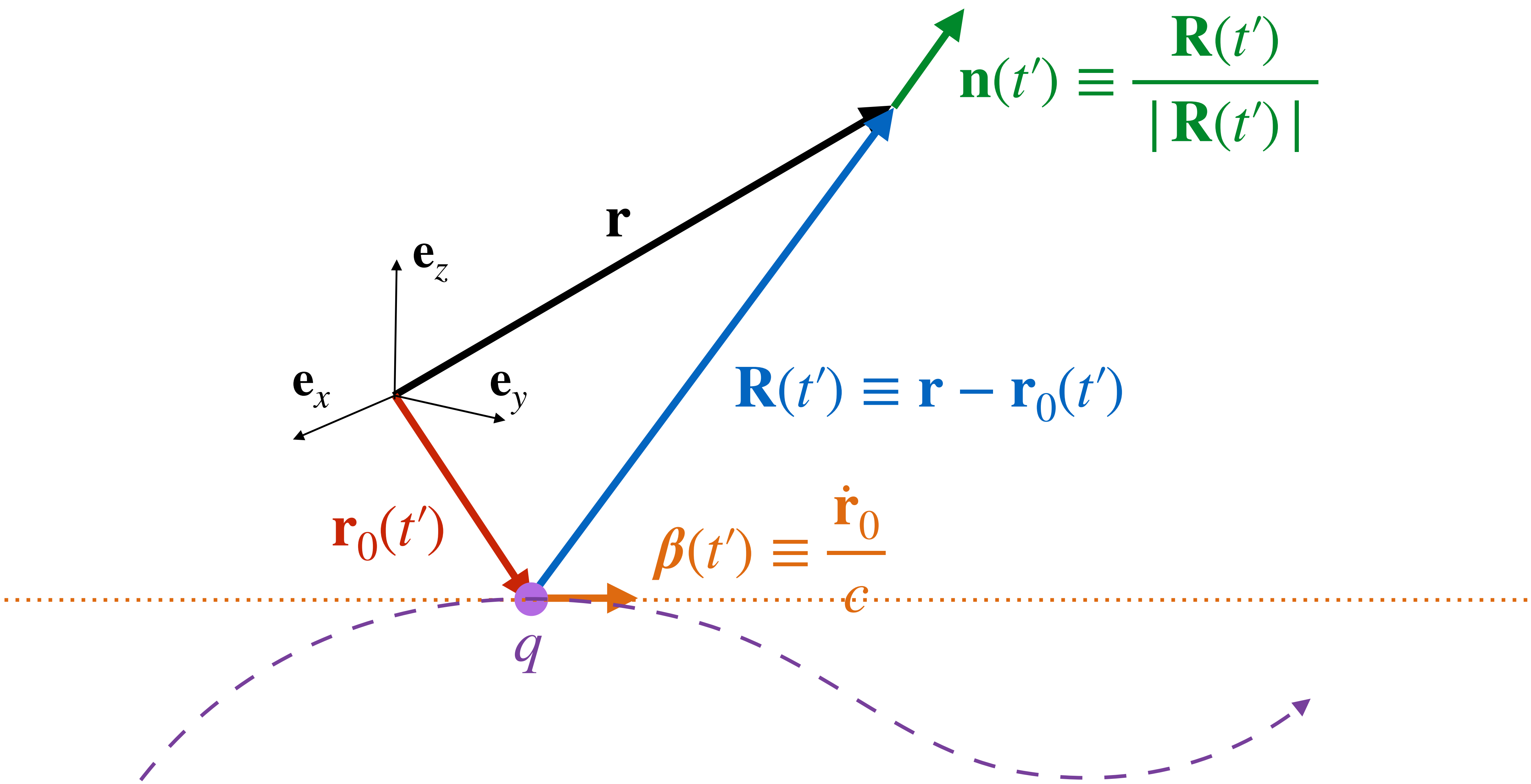
$$t_{\rm ret} = t_{\rm ret}(t,\mathbf{r},\mathbf{r}') \equiv t - \frac{|\mathbf{r}-\mathbf{r}'|}{c}$$

- Consider **point charge moving along path $\mathbf{r}_0(t)$**:

$$\rho(\mathbf{r},t) = q\delta^{(3)}(\mathbf{r}-\mathbf{r}_0(t))$$

$$\mathbf{j}(\mathbf{r},t) = q\dot{\mathbf{r}}_0(t)\delta^{(3)}(\mathbf{r}-\mathbf{r}_0(t))$$

Moving Point Charge



$$\mathbf{R}(t') \equiv \mathbf{r} - \mathbf{r}_0(t') \qquad \mathbf{n}(t') \equiv \frac{\mathbf{R}(t')}{|\mathbf{R}(t')|} \qquad \kappa(t') \equiv 1 - \mathbf{n}(t') \cdot \boldsymbol{\beta}(t')$$

Liénard–Wiechert Potentials

- Solutions are of the form:

$$\phi(\mathbf{r},t) = \frac{q}{\kappa(t_{\rm ret})R(t_{\rm ret})} \qquad \mathbf{A}(\mathbf{r},t) = \frac{q\dot{\mathbf{r}}_0(t_{\rm ret})}{\kappa(t_{\rm ret})R(t_{\rm ret})}$$

- **Retarded time** is solution of:

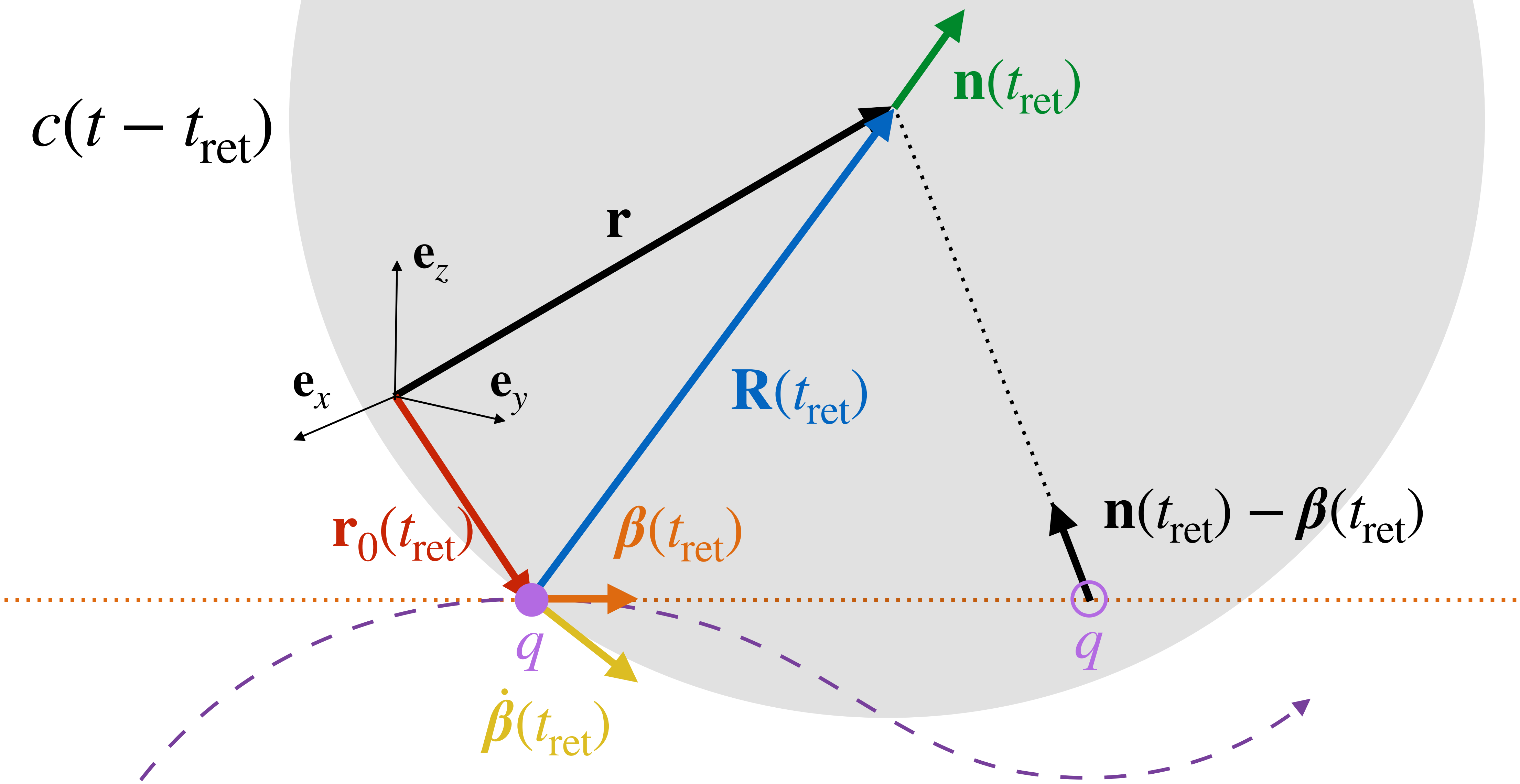
$$c(t - t_{\rm ret}) = R(t_{\rm ret})$$

- Electromagnetic fields become:

$$\mathbf{E}(\mathbf{r},t) = q\left[\frac{(\mathbf{n}-\boldsymbol{\beta})(1-\beta^2)}{\kappa^3R^2}\right]_{t_{\rm ret}} + \frac{q}{c}\left[\frac{\mathbf{n}\times\left[(\mathbf{n}-\boldsymbol{\beta})\times\dot{\boldsymbol{\beta}}\right]}{\kappa^3R}\right]_{t_{\rm ret}}$$

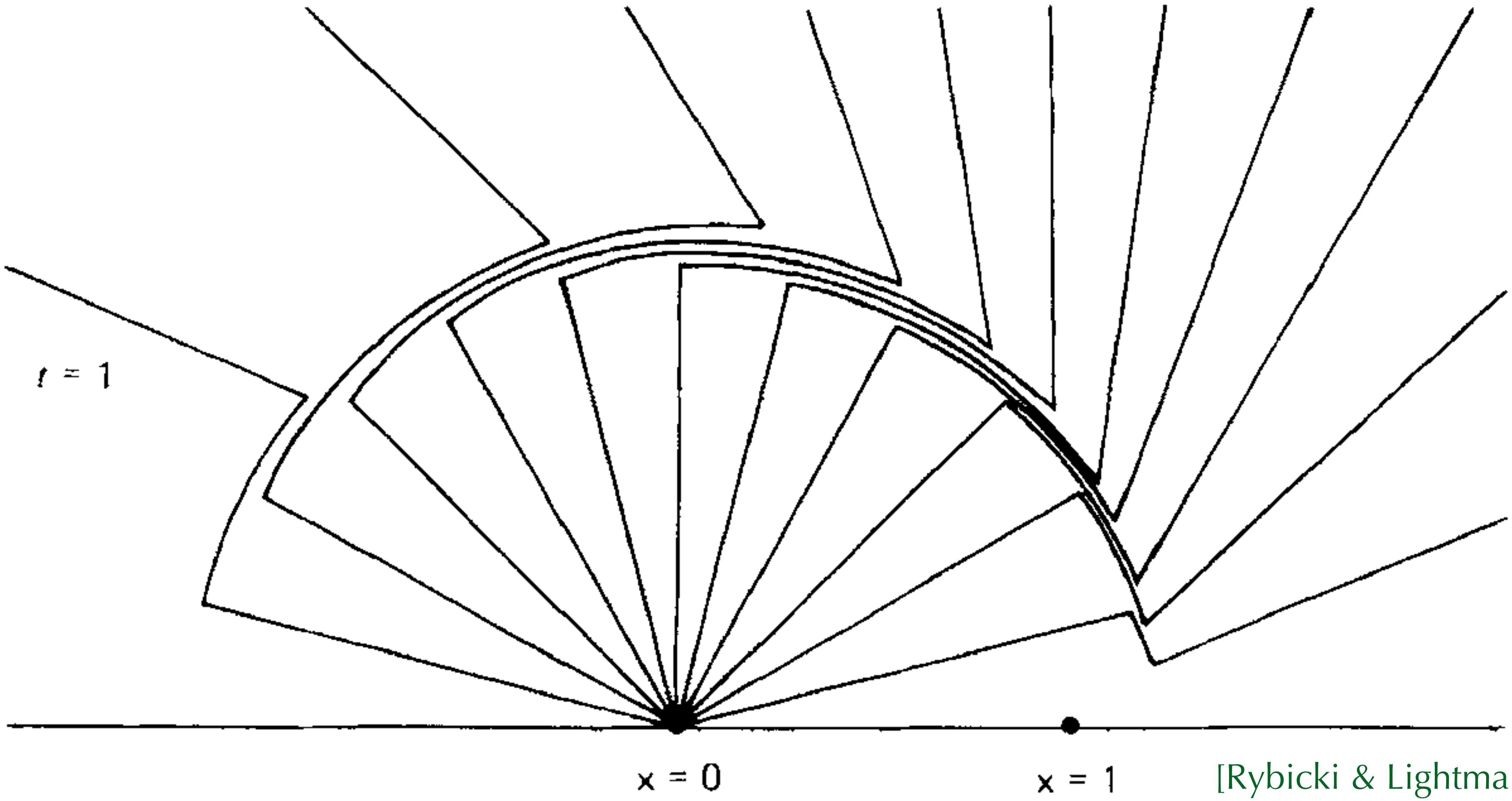
$$\mathbf{B}(\mathbf{r},t) = \mathbf{n}\times\mathbf{E}(\mathbf{r},t)$$

Moving Point Charge



$$c(t - t_{\mathrm{ret}}) = R(t_{\mathrm{ret}})$$

Example: Moving Point Charge



[Rybicki & Lightman'79]

Electric field observed at $t = 1$ from a point charge that suddenly comes to rest at $t = 0$ and $x = 0$

Poynting Vector

- **Poynting vector** describes the energy flux per area and time:

$$\mathbf{S} \equiv \frac{c}{4\pi}\mathbf{E}\times\mathbf{B}$$

- Poynting's theorem:

$$\underbrace{\int_V \mathrm{d}^3 r \left[\mathbf{j}\cdot\mathbf{E} + \frac{\partial}{\partial t}\left(\frac{\mathbf{E}^2+\mathbf{B}^2}{8\pi}\right)\right]}_{\dot{U}_{\rm mech}+\dot{U}_{\rm field}} = -\underbrace{\int_{\partial V} \mathrm{d}\mathbf{n}\cdot\mathbf{S}}_{\text{energy flux}}$$

- **Total energy flux** per area:

$$\frac{\mathrm{d}W}{\mathrm{d}A} = \int \mathrm{d}t\,\mathbf{n}\cdot\mathbf{S} = \int\limits_{-\infty}^{\infty}\mathrm{d}t\underbrace{\left[\frac{c}{4\pi}|\mathbf{E}(t)|^2\right]}_{=\frac{\mathrm{d}W}{\mathrm{d}A\mathrm{d}t}} = \int\limits_{0}^{\infty}\mathrm{d}\omega\underbrace{\left[c\,|\tilde{\mathbf{E}}(\omega)|^2\right]}_{=\frac{\mathrm{d}W}{\mathrm{d}A\mathrm{d}\omega}}$$

Larmor's Formula

- Consider radiation generated by matter confined to a small region.
- At large distance, $|\mathbf{r}| \gg |\mathbf{r_0}|$, and $v \ll c$ gives **dipole radiation**:

$$\mathbf{E}_{\rm rad}(\mathbf{r},t) \simeq \frac{1}{c^2}\frac{\mathbf{n}\times\left[\mathbf{n}\times\ddot{\mathbf{d}}\right]}{|\mathbf{r}|} \qquad \underbrace{\mathbf{d} \equiv q\mathbf{r}_0}_{\text{electric dipole}}$$

- Power per solid angle Ω (Θ is angle between $\mathbf{n}$ and $\ddot{\mathbf{d}}$):

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \mathbf{r}^2\frac{\mathrm{d}W}{\mathrm{d}t\mathrm{d}A} = \mathbf{r}^2\frac{c}{4\pi}|\mathbf{E}_{\rm rad}|^2 \simeq \frac{c}{4\pi}\frac{\ddot{\mathbf{d}}^2}{c^4}\sin^2\Theta$$

- **Larmor's formula** (total power):

$$P = \int \mathrm{d}\Omega\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \frac{2}{3}\frac{\ddot{\mathbf{d}}^2}{c^3}$$

Thomson Scattering

- Consider a **plane wave** in direction $\mathbf{e}_x$ with polarization $\boldsymbol{\varepsilon} \perp \mathbf{e}_x$:

$$\mathbf{E} = E_0\boldsymbol{\varepsilon}\sin(\omega_0 t) \qquad \mathbf{B} = \mathbf{e}_x \times \mathbf{E}$$

- Incoming Poynting flux:

$$\mathbf{S} = \frac{c}{4\pi}\mathbf{e}_x E_0^2 \sin^2\omega_0 t \qquad \langle\,|\mathbf{S}|\,\rangle_t = \frac{c}{8\pi}E_0^2$$

- Electron at origin feels force:

$$\mathbf{F} = -\,eE_0\boldsymbol{\varepsilon}\sin\omega_0 t = m\ddot{\mathbf{r}}$$

- Induced **electric dipole** follows:

$$\ddot{\mathbf{d}} = \frac{e^2E_0}{m}\boldsymbol{\varepsilon}\sin\omega_0 t$$

Thomson Scattering

- *Time-averaged* radiation power per solid angle $d\Omega = \mathrm{d}A/\mathbf{r}^2$:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} \simeq \frac{c}{4\pi}\frac{\langle \ddot{\mathbf{d}}^2\rangle_t}{c^4}\sin^2\Theta = \frac{c}{8\pi}\frac{e^4E_0^2}{m^2c^4}\sin^2\Theta$$

- Ratio of radiated power and incoming flux defines **cross section**:

$$\frac{\mathrm{d}P}{\mathrm{d}\Omega} = \langle\,|\mathbf{S}|\,\rangle_t\frac{\mathrm{d}\sigma_\mathrm{T}}{\mathrm{d}\Omega} \qquad \langle\,|\mathbf{S}|\,\rangle_t = \frac{c}{8\pi}E_0^2$$

- **Thomson cross section** scales with electron radius $r_0 \equiv e^2/mc^2 \simeq 2.8\,\mathrm{fm}$:

$$\frac{\mathrm{d}\sigma_\mathrm{T}}{\mathrm{d}\Omega} = r_0^2\sin^2\Theta \qquad \sigma_\mathrm{T} = \frac{8\pi}{3}r_0^2 \simeq 66\,\mathrm{fm}^2 \simeq 0.66\,\mathrm{b}$$

- *Unpolarized* plane waves: $\frac{\mathrm{d}\sigma_\mathrm{T}}{\mathrm{d}\Omega} = \frac{r_0^2}{2}\left[1+\cos^2\theta\right]$ (θ : scattering angle)

Synchrotron Emission

Cosmic Ray Gyration

- **Particle momentum and energy**:

$$\mathbf{p} = \gamma m \boldsymbol{\beta} c \qquad p_0 = \gamma m c$$

- Particle motion under **Lorentz force**:

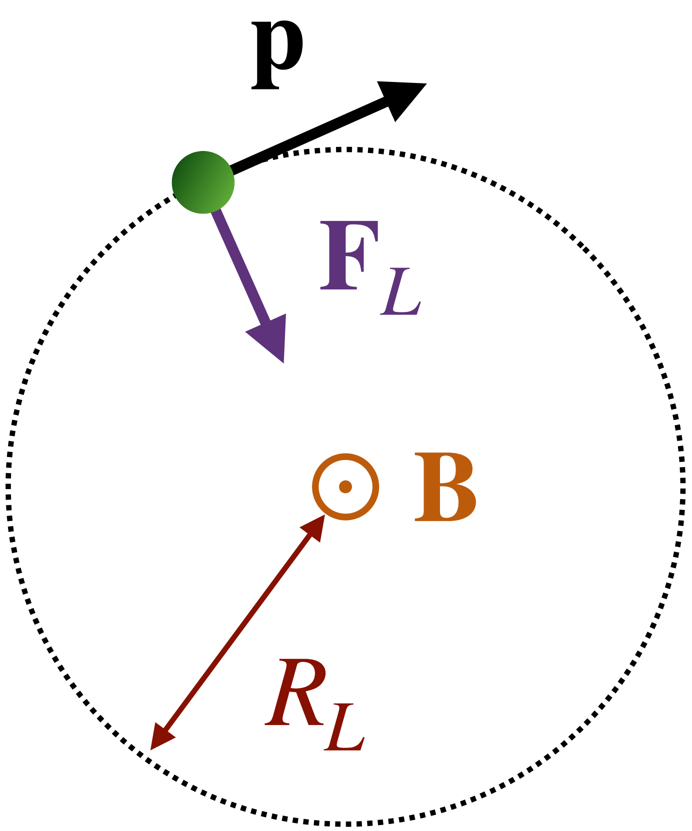
$$\dot{\mathbf{p}} = q\boldsymbol{\beta} \times \mathbf{B} = \mathbf{p} \times \left(\frac{q\mathbf{B}}{\gamma m c}\right) = \mathbf{p} \times \boldsymbol{\Omega}$$

- **Larmor frequency**:

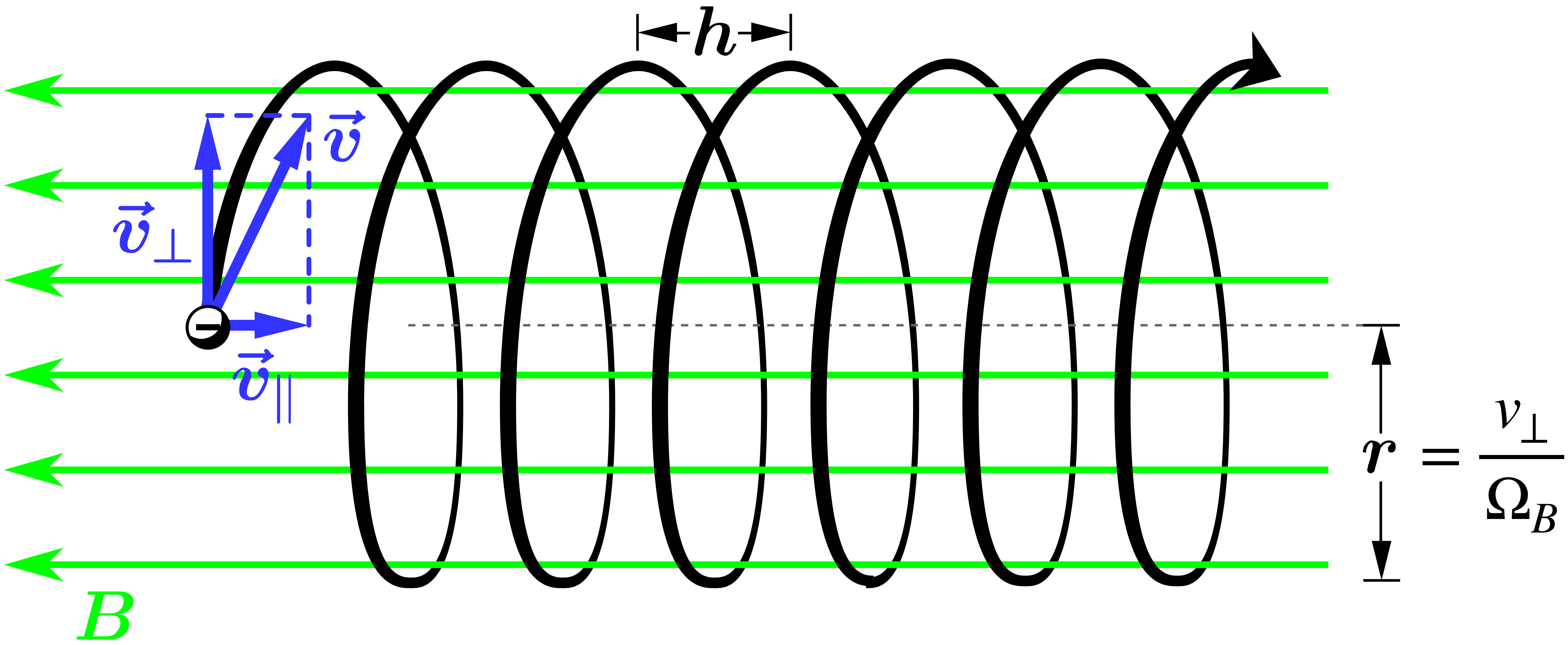
$$\Omega_B = \frac{qB}{\gamma m c} = \frac{2\pi}{T}$$

- **Larmor radius**:

$$R_L = \frac{\beta c}{\Omega_B} = \frac{|\mathbf{p}|c}{qB} = \frac{\mathcal{R}}{B}$$



Electron Gyration



Pitch angle α between $\mathbf{v}(t)$ and $\mathbf{B}_0$ remains constant in time.

Path is a superposition of circular motion in the plane perpendicular to $\mathbf{B}_0$ and linear motion along $\mathbf{B}_0$ with velocity:

$$v_\parallel = \cos\alpha v \equiv \mu v.$$

Synchrotron Radiation

- Use Larmor's formula for power in **instantaneous rest frame of electron**:

$$P' = \frac{\mathrm{d}W'}{\mathrm{d}t'} = \frac{2}{3}\frac{\ddot{\mathbf{d}}'^2}{c^3} = \frac{2}{3}\frac{e^2}{c^3}\mathbf{a}'^2 = \frac{2}{3}\frac{e^2}{c^3}\left(\mathbf{a}'^2_{\parallel} + \mathbf{a}'^2_{\perp}\right)$$

- Acceleration in observer frame:

$$\mathbf{a}_{\parallel} = \frac{\mathrm{d}\mathbf{v}_{\parallel}}{\mathrm{d}t} = 0 \qquad \mathbf{a}_{\perp} = \frac{\mathrm{d}\mathbf{v}_{\perp}}{\mathrm{d}t} = \mathbf{v}_{\perp} \times \boldsymbol{\Omega}$$

- In particle's **rest frame**: $\mathbf{a}'_{\parallel} = \gamma^3 \mathbf{a}_{\parallel} \qquad \mathbf{a}'_{\perp} = \gamma^2 \mathbf{a}_{\perp}$

- Both, energy and time are dilated and hence $P = P'$:

$$P = \frac{2}{3}\frac{e^2}{c^3}\gamma^4 \Omega_B^2 \mathbf{v}_{\perp}^2 = \frac{2}{3} r_0^2 c \boldsymbol{\beta}_{\perp}^2 \gamma^2 B^2$$

Synchrotron Radiation

- Averaging over pitch-angle distribution:

$$\langle P \rangle_\Omega = \frac{2}{3} r_0^2 c \beta^2 \gamma^2 B^2 \langle \sin^2 \alpha \rangle_\Omega = \frac{4}{9} r_0^2 c \beta^2 \gamma^2 B^2$$

- **Averaged synchrotron power** depends on **magnetic energy density**:

$$\langle P \rangle_\Omega = \frac{4}{3} c \sigma_\mathrm{T} \beta^2 \gamma^2 U_B \qquad U_B = \frac{B^2}{8\pi}$$

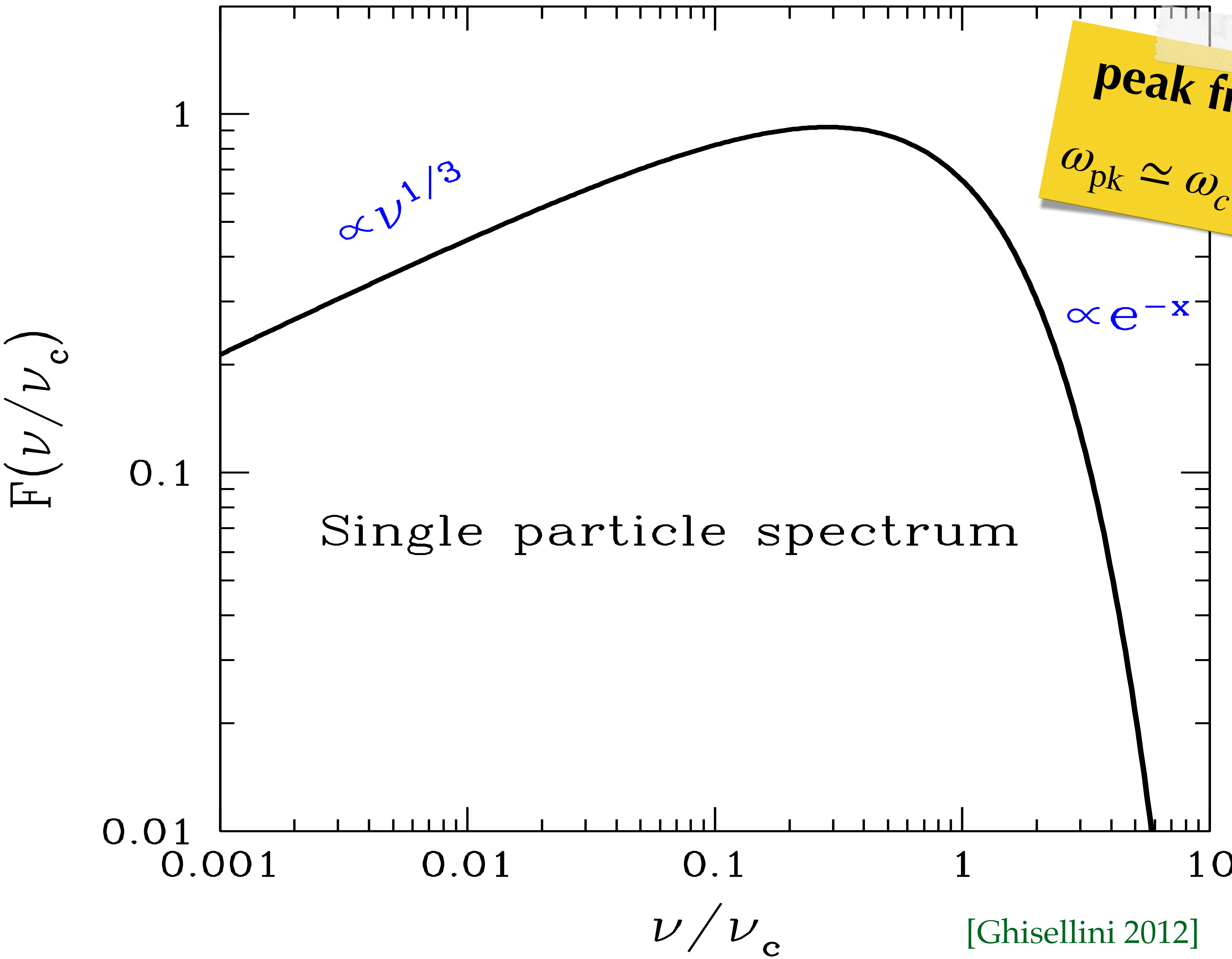
- Power spectrum trickier, but can be extracted from Liénard–Wiechert:

$$\frac{\mathrm{d}W}{\mathrm{d}\omega \mathrm{d}\Omega} = \frac{q^2 \omega^2}{4\pi^2 c} \left| \int \mathrm{d}t' \mathbf{n} \times [\mathbf{n} \times \boldsymbol{\beta}(t')] \exp\left(i\omega \left(t' - \frac{\mathbf{n} \cdot \mathbf{r}_0(t')}{c} \right) \right) \right|^2$$

result:

$$\frac{\mathrm{d}P}{\mathrm{d}\omega} \simeq \frac{\sqrt{3} e^3 B \sin\alpha}{2\pi m c^2} F\left(\frac{\omega}{\omega_c} \right) \qquad \omega_c = \frac{3}{2} \gamma^3 \Omega_B \sin\alpha$$

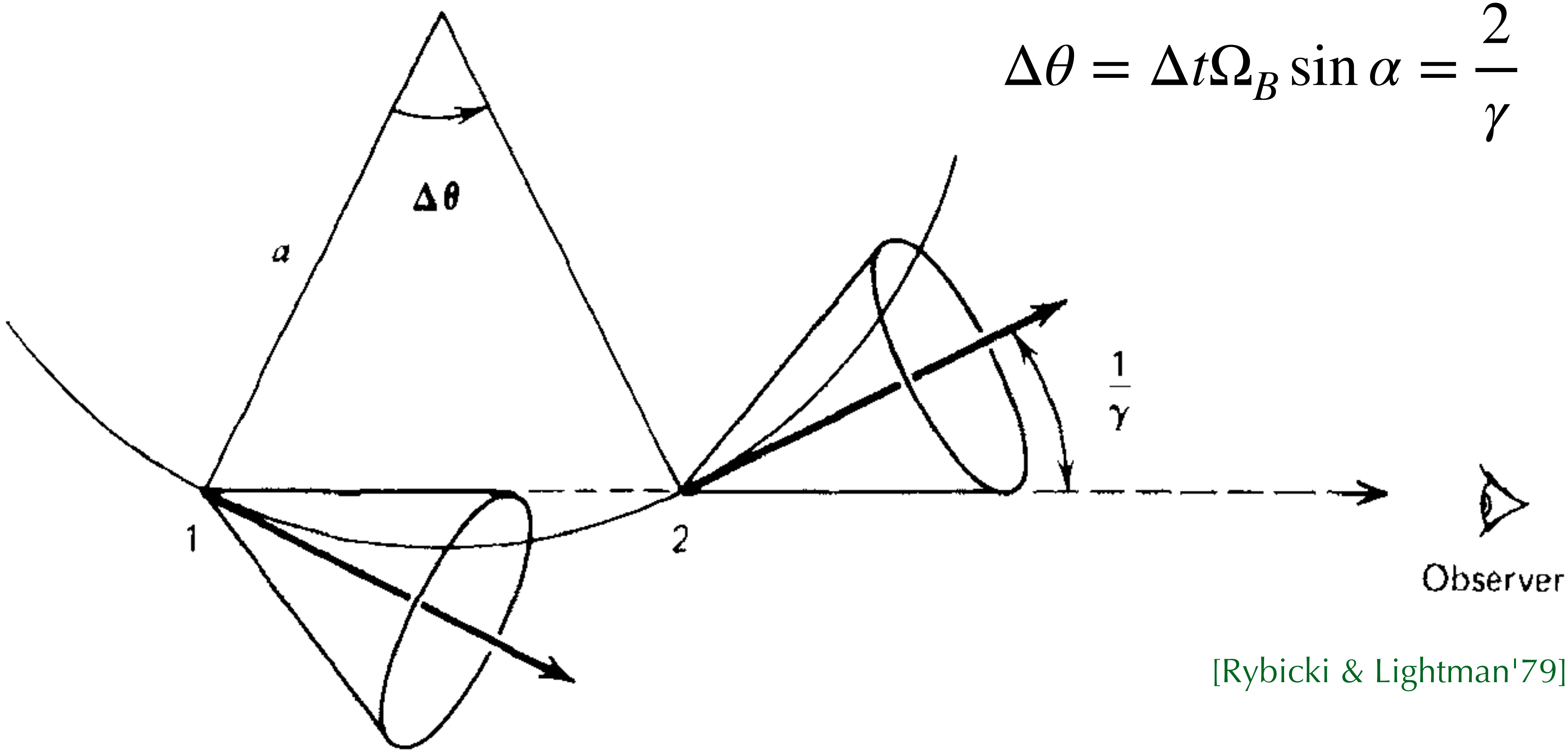
Synchrotron Spectrum



peak frequency:
$\omega_{pk} \simeq \omega_c \simeq \propto \gamma^2 qB$

[Ghisellini 2012]

Critical Synchrotron Frequency



$$\Delta\theta = \Delta t \Omega_B \sin\alpha = \frac{2}{\gamma}$$

[Rybicki & Lightman'79]

$$\Delta t_{\rm obs} \simeq \frac{2}{\gamma\Omega_B \sin\alpha}\left(1 - \frac{v}{c}\right) \simeq \frac{1}{\gamma^3\Omega_B \sin\alpha}$$

Synchrotron Spectrum

- Consider electrons following a power-law distribution:

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = C\gamma^{-p} \qquad \gamma_{\rm min} < \gamma < \gamma_{\rm max}$$

- Synchrotron power spectrum follows:

$$\frac{\mathrm{d}P_{\rm syn}}{\mathrm{d}\omega} = \int \mathrm{d}\gamma \frac{\mathrm{d}N}{\mathrm{d}\gamma}\frac{\mathrm{d}P}{\mathrm{d}\omega} \propto \int_{\gamma_{\rm min}}^{\gamma_{\rm max}} \mathrm{d}\gamma \gamma^{-p} F(\omega/\omega_c)$$

- Change of integration variable to $x = \omega/\omega_c \propto \gamma^{-2}$:

$$\frac{\mathrm{d}P_{\rm syn}}{\mathrm{d}\omega} \propto \omega^{-\frac{p-1}{2}} \int_{x_{\rm min}}^{x_{\rm max}} \mathrm{d}x x^{\frac{p-3}{2}} F(x) \propto \omega^{-\frac{p-1}{2}}$$

- **Combined spectrum scales with power $\frac{p-1}{2}$.**

Compton & inverse-Compton

Compton Radiation

- **Thomson scattering** of low-energy photons on *unpolarized* electron:

$$\frac{d\sigma_T}{d\Omega} = \frac{r_0^2}{2}\left(1 + \cos^2\theta\right) \qquad \sigma_T = \frac{8\pi}{3}r_0^2 \qquad r_0 = \frac{e^2}{mc^2}$$

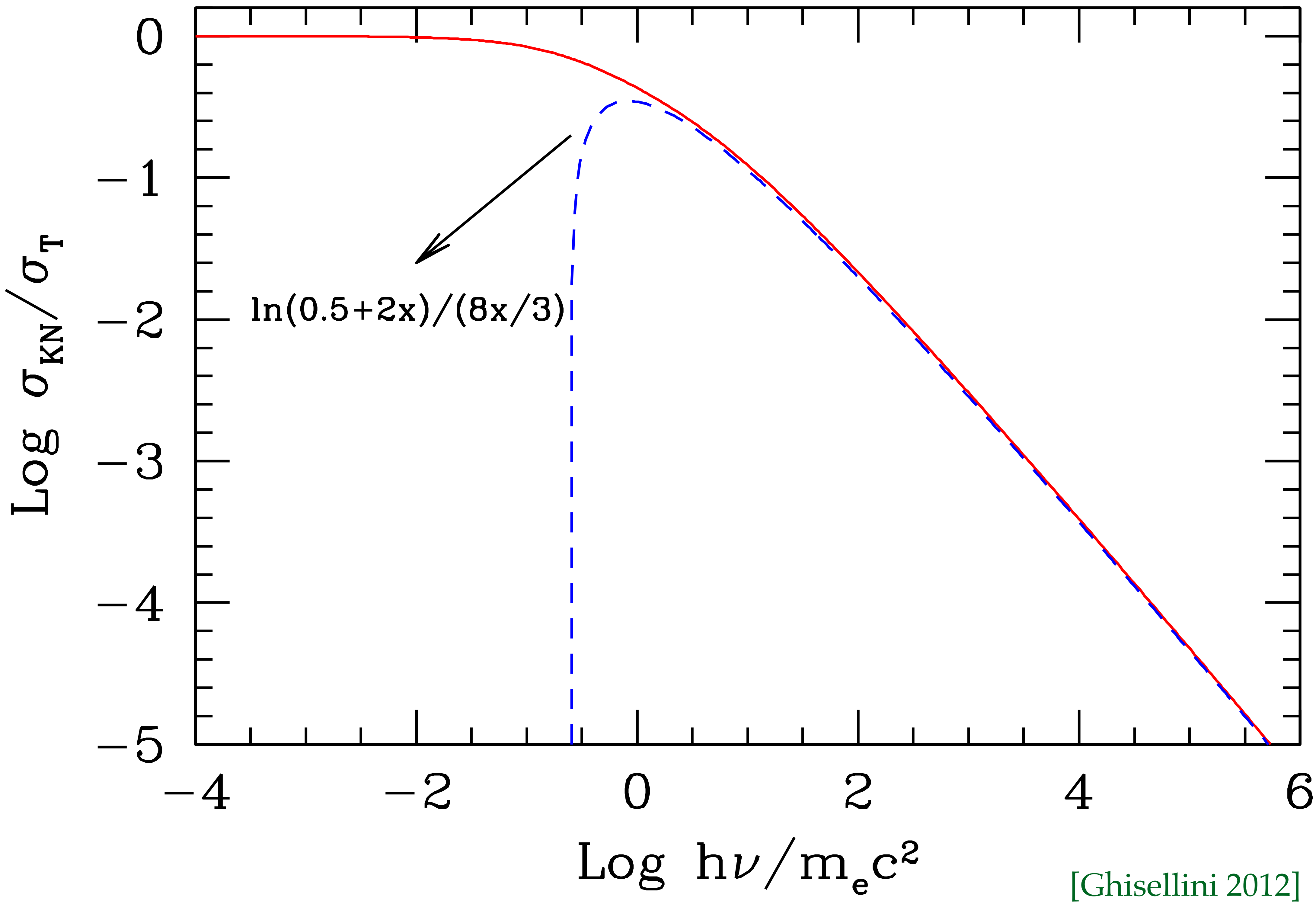
- Photons with energy ϵ approaching the **rest-mass energy** $mc^2$of electrons loose energy depending on **scattering angle** θ:

$$\epsilon_1 = \frac{\epsilon_0}{1 + \frac{\epsilon_0}{mc^2}(1 - \cos\theta)}$$

- Cross section is *reduced* for $\epsilon \gg mc^2$ (**Klein-Nishina**):

$$\frac{d\sigma_{KN}}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_1^2}{\epsilon_0^2}\left(\frac{\epsilon_0}{\epsilon_1} + \frac{\epsilon_1}{\epsilon_0} - \sin^2\theta\right)$$

Klein-Nishina Cross Section



[Ghisellini 2012]

Inverse-Compton Scattering

- Moving electron can transfer energy to incoming photon.
- For instance, head-on collisions give the **maximal energy** $\epsilon_1 = 4\gamma^2\epsilon_0$.
- Consider process in the rest frame of electron with $\epsilon' \ll mc^2$.
- **Total radiated power** of a plane wave:

$$P'_{\rm tot} = \frac{{\rm d}W'}{{\rm d}t'} = \sigma_{\rm T}\langle\,|\mathbf{S}'|\,\rangle_t = c\sigma_{\rm T}\frac{\mathbf{E}_0'^{\,2}}{8\pi} = c\sigma_{\rm T}U'_\gamma$$

- Target **photon energy density** U'_γ in terms of **phase-space density** f'_γ:

$$U'_\gamma = \int {\rm d}^3p'\epsilon' f'_\gamma$$

- Again, since ${\rm d}W = \gamma{\rm d}W'$ and ${\rm d}t = \gamma{\rm d}t'$ we get $P_{\rm tot} = P'_{\rm tot}$.

Reminder : Phase-Space Density

- **Phase-space density** (PSD) is Lorentz-invariant:

$$f(t,\mathbf{r},\mathbf{p}) \equiv \frac{\mathrm{d}N}{\mathrm{d}^3r\,\mathrm{d}^3p}$$

- Particle moving into solid angle Ω with momentum $p = \gamma\beta m$:

$$\mathrm{d}^3r \times \mathrm{d}^3p \to \beta\,\mathrm{d}t\,\mathrm{d}A_\perp \times \mathrm{d}\Omega\,p^2\mathrm{d}p$$

- **Spectral flux** ("intensity"):

$$\phi(t,\mathbf{r},E,\Omega) \equiv \frac{\mathrm{d}N}{\mathrm{d}t\,\mathrm{d}A_\perp\,\mathrm{d}\Omega\,\mathrm{d}E} = \beta p^2\frac{\mathrm{d}p}{\mathrm{d}E}f(t,\mathbf{r},\mathbf{p}) = p^2 f(t,\mathbf{r},\mathbf{p})$$

- **Spectral density**:

$$n(t,\mathbf{r},E) \equiv \frac{\mathrm{d}N}{\mathrm{d}^3r\,\mathrm{d}E} = \frac{1}{\beta}\int \mathrm{d}\Omega\,\phi(t,\mathbf{r},E,\Omega) = \frac{4\pi}{\beta}p^2\,\langle f(t,\mathbf{r},\mathbf{p})\rangle_{4\pi}$$

Inverse-Compton Scattering

- Transform to observer frame quantities:

$$U'_\gamma = \int \mathrm{d}^3p \frac{\mathrm{d}p'_x}{\mathrm{d}p_x}\epsilon' f_\gamma{}' = \int \mathrm{d}^3p\gamma^2(1-\beta\cos\theta)^2\epsilon f_\gamma$$

using: $$\frac{\mathrm{d}p'_x}{\mathrm{d}p_x} = \frac{1}{\delta(\beta,\theta)} \qquad \epsilon' = \frac{\epsilon}{\delta(\beta,\theta)} \qquad f'_\gamma = f_\gamma$$

- For isotropic target photons (in observer frame) we get:

$$U'_\gamma = \gamma^2 \left\langle (1-\beta\cos\theta)^2 \right\rangle_\Omega U_\gamma = \gamma^2\left(1+\frac{1}{3}\beta^2\right)U_\gamma$$

- ***Total radiation*** in observer frame is therefore:

$$P_{\rm tot} = c\sigma_T\gamma^2\left(1+\frac{1}{3}\beta^2\right)U_\gamma$$

Inverse-Compton Scattering

- To extract contribution to inverse-Compton, we have to subtract the "elastic" part in the limit $\beta \to 0$:

$$P_{\rm IC} = P_{\rm tot} - P_{\rm tot}(\beta = 0) = P_{\rm tot} - c\sigma_{\rm T}U_\gamma$$

- Finally, **inverse-Compton power** is:

$$P_{\rm IC} = \frac{4}{3}c\sigma_{\rm T}\beta^2\gamma^2U_\gamma$$

- Note the **relative contribution of IC and synchrotron power**:

$$P_{\rm syn} = \frac{4}{3}c\sigma_{\rm T}\beta^2\gamma^2U_B \quad \to \quad \frac{P_{\rm IC}}{P_{\rm syn}} = \frac{U_\gamma}{U_B}$$

Inverse-Compton Spectrum

- Consider an **isotropic mono-energetic target radiation flux**:

$$\phi(\epsilon) = F_0\delta(\epsilon - \epsilon_0)$$

- Transformed into the **rest-frame of the electron** ($\delta = \delta(-\beta, \theta')$):

$$\phi'(\epsilon', \theta') = \frac{F_0}{\delta^2}\delta\left(\frac{\epsilon'}{\delta} - \epsilon_0\right) = \frac{F_0}{\gamma\beta\epsilon_0}\frac{1}{\delta}\delta\left(\cos\theta' - \frac{\epsilon_0 - \gamma\epsilon'}{\gamma\beta\epsilon'}\right)$$

- **Emission spectrum** (averaged over initial photon directions):

$$\frac{\mathrm{d}N'}{\mathrm{d}\Omega'\mathrm{d}t'\mathrm{d}\epsilon'_1} \simeq \sigma_T\frac{1}{2}\int \mathrm{d}\cos\theta'\phi'(\epsilon'_1, \theta') = \frac{\sigma_\mathrm{T}}{2}\frac{F_0\epsilon'_1}{\gamma\beta\epsilon_0^2}$$

$$\frac{\epsilon_0}{\gamma(1+\beta)} < \epsilon'_1 < \frac{\epsilon_0}{\gamma(1-\beta)}$$

Inverse-Compton Spectrum

- **Emission spectrum per volume** from electrons with density n'_e:

$$j'(\epsilon'_1) \equiv n'_e \frac{\mathrm{d}N'}{\mathrm{d}\Omega' \mathrm{d}t' \mathrm{d}\epsilon'_1}$$

- Transforms as:

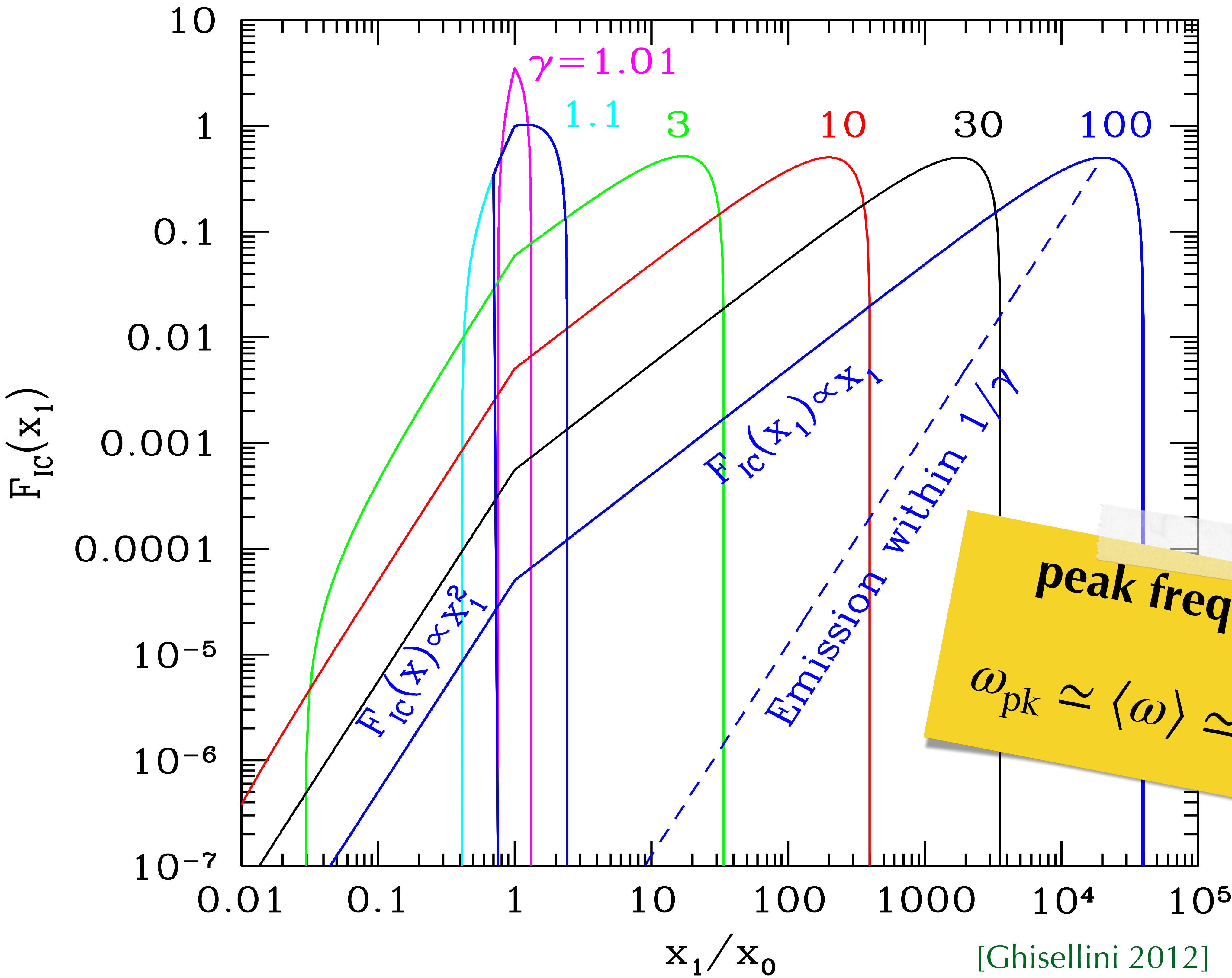
$$j(\epsilon_1, \cos\theta_1) = \delta(\beta, \theta_1) j'(\epsilon'_1) = n_e \frac{\sigma_\mathrm{T}}{2} \frac{F_0 \epsilon_1}{\gamma^2 \beta \epsilon_0^2}$$

$$\frac{\epsilon_0}{\gamma^2(1+\beta)(1-\beta\cos\theta_1)} < \epsilon_1 < \frac{\epsilon_0}{\gamma^2(1-\beta)(1-\beta\cos\theta_1)}$$

- Finally, **averaging over electron directions** yields:

$$\frac{\mathrm{d}N}{\mathrm{d}\Omega \mathrm{d}t \mathrm{d}\epsilon_1} = \sigma_\mathrm{T} F_0 \frac{1+\beta}{4\gamma^2\beta^2} \frac{mc^2}{\epsilon_0} F_\mathrm{IC}\left(\frac{\epsilon_1}{mc^2}\right)$$

Inverse-Compton Spectrum



[Ghisellini 2012]

peak frequency:

$$\omega_{pk} \simeq \langle\omega\rangle \simeq \frac{4}{3}\gamma^2\omega_0$$

Inverse-Compton Spectrum

- Consider, again, electrons following a power-law distribution in γ:

$$\frac{\mathrm{d}N}{\mathrm{d}\gamma} = C\gamma^{-p} \qquad \gamma_{\rm min} < \gamma < \gamma_{\rm max}$$

- For individual electrons, the IC power spectrum peaks at $\langle\omega\rangle = 2\gamma^2\omega_0$.
- Let us approximate this as:

$$\frac{\mathrm{d}P}{\mathrm{d}\omega} \simeq \hbar\langle\omega\rangle\delta(\omega - \langle\omega\rangle)$$

- **Combined spectrum scales with power $\frac{p-1}{2}$**, just as in synchrotron:

$$\frac{\mathrm{d}P_{\rm IC}}{\mathrm{d}\omega} = \int \mathrm{d}\gamma \frac{\mathrm{d}N}{\mathrm{d}\gamma}\frac{\mathrm{d}P}{\mathrm{d}\omega} \propto \int \mathrm{d}\gamma \gamma^{-p+1}\,\delta\left(\gamma - \sqrt{\frac{\omega}{2\omega_0}}\right) \propto \omega^{-\frac{p-1}{2}}$$

Bremsstrahlung

Bremsstrahlung

- Radiation from the acceleration of charged particles in the Coulomb field of another charged particle.
- In **dipole approximation**, need non-vanishing $\ddot{\mathbf{d}}$.

$$P = \frac{\mathrm{d}W}{\mathrm{d}t} \simeq \frac{2}{3}\frac{\ddot{\mathbf{d}}^2}{c^3} \qquad \rightarrow \qquad \frac{\mathrm{d}W}{\mathrm{d}\omega} \simeq \frac{8\pi}{3}\frac{\omega^4}{c^3}|\tilde{\mathbf{d}}(\omega)|^2$$

- For **identical particles**, $\mathbf{d}$ is proportional to center of mass with $\ddot{\mathbf{d}} = 0$.
- Let's consider the effect in **electron-proton scattering.**
- Consider electron with **impact factor** b and speed v:

$$\omega^2\tilde{\mathbf{d}}(\omega) = \frac{e}{2\pi}\int \mathrm{d}t\,\dot{\mathbf{v}}e^{i\omega t} \simeq \begin{cases} \frac{e}{2\pi}\Delta\mathbf{v} & \omega \ll v/b \\ 0 & \omega \gg v/b \end{cases}$$

Bremsstrahlung

- Bremsstrahlung power becomes:

$$\frac{\mathrm{d}W}{\mathrm{d}\omega} \simeq \begin{cases} \frac{2e^2}{3\pi c^3}\Delta\mathbf{v}^2 & \omega \ll v/b \\ 0 & \omega \gg v/b \end{cases}$$

- From Rutherford scattering and $\Delta\mathbf{v}^2 \ll \mathbf{v}^2$ we can estimate:

$$|\Delta\mathbf{v}| \simeq \frac{2e^2}{bmv}$$

- In medium with densities n_e and n_p the incoming flux of electrons is vn_e.
- Specific power spectrum from impact parameter integral:

$$\frac{\mathrm{d}P}{\mathrm{d}\omega\mathrm{d}V} \simeq vn_en_p2\pi\int \mathrm{d}bb\frac{\mathrm{d}W}{\mathrm{d}\omega}$$

Bremsstrahlung

- We arrive at:

$$\frac{\mathrm{d}P}{\mathrm{d}\omega\mathrm{d}V} \simeq \frac{16e^6}{3c^3m^2v} n_e n_p \ln\frac{b_{\min}}{b_{\max}}$$

- **Maximal impact parameter**:

$$b_{\max} \simeq \frac{v}{\omega}$$

- **Minimal impact parameter** from either $\Delta v \ll v$ or $\Delta x \Delta p \geq \hbar$:

$$b^{(1)}_{\min} \simeq \frac{e^2}{mv^2} \qquad b^{(2)}_{\min} \simeq \frac{h}{mv}$$

- Exact result uses **Gaunt factor** g_{ff} with weak dependence on v and ω:

$$\frac{\mathrm{d}P}{\mathrm{d}\omega\mathrm{d}V} \simeq \frac{16e^6}{3c^3m^2v} n_e n_p \frac{\pi}{\sqrt{3}} g_{ff}(v,\omega)$$

Pair Production

Pair Production

- All previous **emission processes** of charged particles are linked to corresponding **absorption counterparts**.
- For high-energy photons (γ-rays) we have also absorption via the process:

$$\gamma + \gamma_{\rm bg} \to e^+ + e^- \qquad\qquad \epsilon > m_e^2c^4/\epsilon_{\rm bg}$$

- Cross section in terms of electron's β in center-of-mass frame:

$$\sigma_{\gamma\gamma} \simeq \sigma_{\rm T}\frac{3}{16}(1-\beta^2)\left[2\beta(\beta^2-2)+(3-\beta^4)\ln\frac{1+\beta}{1-\beta}\right]$$

- Important absorption process for γ-rays travelling over intergalactic distances is the **cosmic microwave background (CMB)**:

$$\epsilon_{\rm th} \simeq \frac{m_e^2c^4}{k_BT_{\rm CMB}} \simeq 1\,{\rm PeV} \qquad\qquad \lambda_{\gamma\gamma} \simeq \Big[1/(\sigma_{\gamma\gamma}n_{\rm CMB})\Big]_{1\,{\rm PeV}} \simeq 10\,{\rm kpc}$$

Example: EM Cascades in CMB