UNDARK 2026 – “THE MICROPHYSICS OF NON-THERMAL PROCESSES’

SUGGESTED EXERCISES

EXERCISE 1 - EXTENDED AIR SHOWER

An ultra-high energy cosmic ray (UHECR) is impinging vertically on the Earth’s atmosphere and initiates an air shower. We want to estimate the typical height of the interaction above ground.

For simplicity, let’s assume that the atmosphere is in thermal equilibrium with temperature $T = 273$ K (0°C). The air pressure on ground is typically $p \simeq 1\,\mathrm{atm}$.

> **1.1** Use the ideal gas law to estimate the air density $n(0)$ at sea level in units of particles per volume.

Now, assume that the atmosphere is also in diffusive equilibrium. From your thermodynamics course you know that this implies equal chemical potential at every atmospheric layer at height h. We can approximate the total chemical potential as the sum of the internal chemical potential of an ideal (monatomic) gas, $\mu_{\rm int} = k_BT\ln(n(h)/n_Q)$ (n_Q the “quantum density”) and the external chemical potential of gravity, $\mu_{\rm ext} = Mgh$ ($g \simeq 9.81\mathrm{m/s}^2$ and M is the (average) mass of the gas atom).

> **1.2** Show that the gas density scales with height as $n(h) = n(0)\exp(-Mgh/k_BT)$.

The inelastic interaction cross section of UHECRs impinging on the Earth’s atmosphere is of the order of $\sigma_{\rm CR} \simeq 0.5\mathrm{barn} \equiv 5\times10^{-25}\mathrm{cm}^2$.

> **1.3** Estimate the typical interaction height $h_{\rm CR}$ of UHECR, as the solution to the equation:
>
> $$1 = \sigma_{\rm CR}\int\limits_{h_{\rm CR}}^{\infty} \mathrm{d}h\, n(h)$$

You can assume that the atmosphere is dominated by dinitrogen (N_2) with a mass $M \simeq 28u$ (mass unit Dalton is $1u \simeq 1.7\times10^{-27}$ kg $\simeq 1.5\times10^{-10}$ J$/c^2$).

EXERCISE 2 - MULTI-MESSENGER SOURCE

Ultra-high energy cosmic rays can reach energies of the order of 10^{20} eV. At these extreme energies, deflections in Galactic and extragalactic magnetic fields are not well described as a diffusive process. Instead, one can consider the CR deflections as a small perturbation on the arrival direction.

> **2.1** Estimate the maximal angular deflection of 10^{20} eV protons from a source at 10 Mpc assuming an extragalactic magnetic field with strength $B = 10^{-9}$ G.

Imagine that the same UHE CR proton was produced in a γ-ray burst, *i.e.* a transient event.

> **2.2** For the same UHE CR proton as in part 2.1, estimate the maximal time delay compared to the arrival of a γ-ray in your detector.

Some of the UHE CRs might interact in the source, producing a EeV (10^18 eV) neutrino.

> **2.3** Determine the time delay between a γ-ray and a neutrino with mass 0.1eV observed at Earth. Is it possible to measure neutrino masses this way?

Exercise 3 - Galactic PeVatrons

Recently, the H.E.S.S. Collaboration reported evidence of a cosmic ray source in the Galactic center that is capable of accelerating cosmic rays up to 1 PeV ($= 10^{15}$ eV) – see arXiv:1603.07730 for more information. The H.E.S.S. observatory is an Imaging Atmospheric Cherenkov Telescope (IACT) located in Namibia. They studied diffuse γ-ray emission in different locations along the Galactic disk in the vicinity of Sagittarius A*. The search regions are indicated in figure 1. Under the assumption that the diffuse γ-ray emission in each search region is due to the decay of neutral pions, $\pi^0 \to \gamma + \gamma$, that are produced via cosmic ray interactions with molecular gas, one can infer the cosmic ray density in the Galactic center. This is shown in figure 2 in terms of the distance r from Sagittarius A*.

In the lecture we derived the density of CRs for a source at distance r that emits a burst of $N_{\rm CR}$ cosmic rays. The solution can be extended to an emission spectrum $N(E)$ (number of CRs per energy) to give the local spectral energy density:

$$n(t,E,r) = \frac{N(E)}{(4\pi K(E)t)^{3/2}} \exp\left(-\frac{r^2}{4K(E)t}\right) .$$

Consider now the emission of this source over a long period $T \to \infty$ with a CR emission rate $Q(E) = dN(E)/dt$ (number of CRs per energy and time).

3.1 Show that the radial distribution of the CR spectral density scales like $1/r$.

Hint: The result can be related to a Gamma function:

$$\int_0^\infty \mathrm{d}x x^{\alpha-1} e^{-x} = \Gamma(\alpha) .$$

3.2 Use the best-fit $1/r$ result of the local CR energy density $w(r) = \int_{10\ \rm TeV}^{\infty} \mathrm{d}E E n(E,r)$ (red dashed line in figure 2) to estimate the source luminosity in erg/s:

$$L(\geq 10\ \mathrm{TeV}) = \int_{10\ \mathrm{TeV}}^{\infty} \mathrm{d}E E Q(E) ,$$

assuming that the diffusion coefficient is constant in energy with $K \simeq 10^{30}\mathrm{cm}^2/\mathrm{s}$.

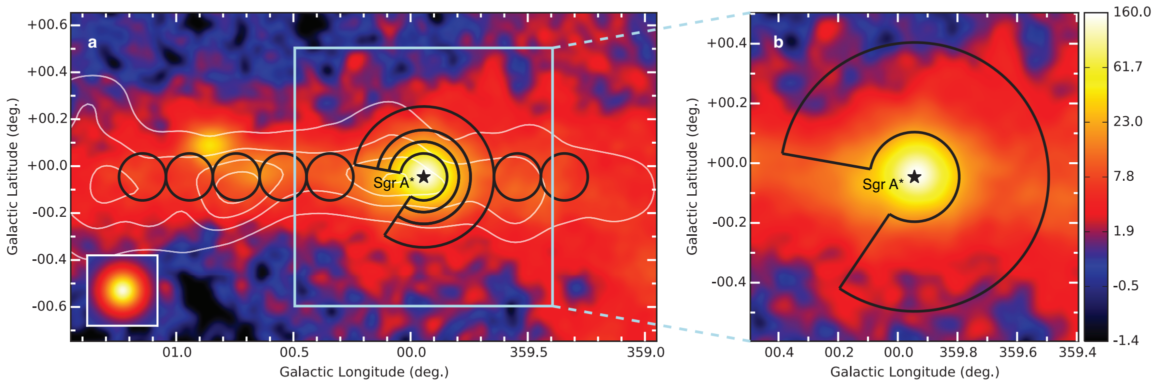

Figure 1: **VHE γ-ray image of the Galactic Centre region.** The colour scale indicates counts per $0.02° \times 0.02°$ pixel. *Left panel:* The black lines outline the regions used to calculate the CR energy density throughout the central molecular zone. A section of 66° is excluded from the annuli (see Methods). White contour lines indicate the density distribution of molecular gas, as traced by its CS line emission[30]. The inset shows the simulation of a point-like source. *Right panel:* Zoomed view of the inner ~ 70 pc and the contour of the region used to extract the spectrum of the diffuse emission.

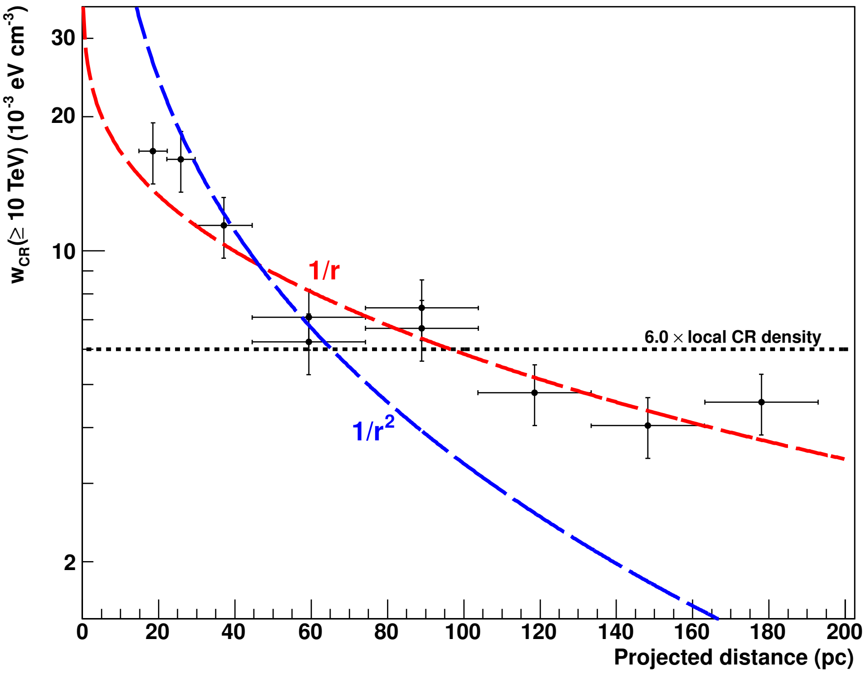

Figure 2: **Spatial distribution of the CR density versus projected distance from Sgr A*.** The vertical and horizontal error bars show the 1σ statistical plus systematical errors and the bin size, respectively. A fit to the data of a $1/r$ (red line, $\chi^2/\text{d.o.f.}$ = 11.8/9), $1/r^2$ (blue line, $\chi^2/\text{d.o.f.}$ = 73.2/9) and an homogeneous (black line, $\chi^2/\text{d.o.f.}$ = 61.2/9) CR density radial profiles integrated along the line of sight are shown. The best fit of a $1/r^\alpha$ profile to the data is found for $\alpha = 1.10 \pm 0.12$ (1σ).The $1/r$ radial profile is clearly preferred by the H.E.S.S. data.

Exercise 4 - Fermi Acceleration

In the lecture we discussed particle acceleration via diffusive shock acceleration in astrophysical shocks. A simplified version of this process is as follows: Charged particles scatter back-and-forth between the two sides of the shock due to diffusion in turbulent magnetic fields. Starting with initial energy E_0, in each cycle the particle gains an energy which can be expressed via an acceleration time-scale τ_{acc} and $\mathrm{d}E/\mathrm{d}t = E/\tau_{\text{acc}}$. At the same time, diffusions also allows the escape of the particle. Their number at the shock vicinity N_{in} decreases following $\mathrm{d}N_{in}/\mathrm{d}t = -N_{in}/\tau_{\text{esc}}$ with escape time τ_{esc}.

4.1 Show that this mechanism predicts that the spectrum of cosmic rays escaping the shock region follows a power-law distribution, $\mathrm{d}N_{out}/\mathrm{d}E = C(E/E_0)^{-\alpha}$ with some constant C, and determine the expression for the spectral index α.

In diffuse shock acceleration, the acceleration and escape time scales are related to the velocities u_1 (upstream) and u_2 (downstream) on both sides of the shock:

$$\tau_{\text{acc}} = \frac{3}{4}\frac{\lambda}{u_1 - u_2} \qquad \text{and} \qquad \tau_{\text{esc}} = \frac{\lambda}{4u_2}\,, \tag{1}$$

where λ is a length scale that parametrizes the particle's scattering length. Using energy, momentum and particle number conservation across the shock, one can derive a relation between the velocities (Rankine-Hugoniot jump condition)

$$\frac{u_1}{u_2} = \frac{(\gamma+1)\,\mathcal{M}^2}{(\gamma+1)+(\gamma-1)(\mathcal{M}^2-1)}\,, \tag{2}$$

where $\mathcal{M}$ is the shock Mach number and γ the adiabatic index.

4.2 Use relations (1) and (2) to determine the value of the spectral index α for the case of a strong shock (Mach number $\mathcal{M} \to \infty$) and monatomic gas (adiabatic index $\gamma = 5/3$).