

Analytical Studies of the Pion Vector Form Factor

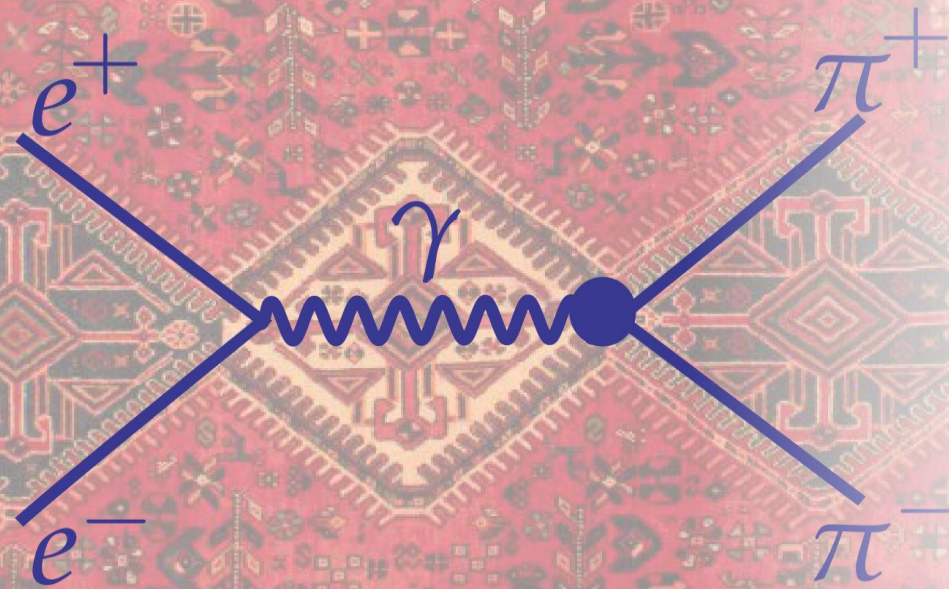
GDR-InF Annual
Workshop 2025

INTENSITY
frontier GDR-InF

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Motivation

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- Non-perturbative QCD interactions are often a major source of theory uncertainty.

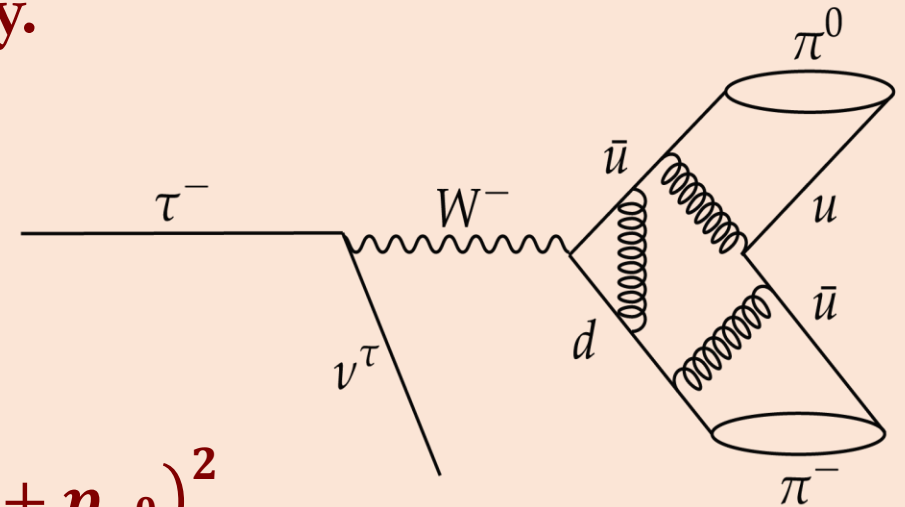
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- Non-perturbative QCD interactions are often a major source of theory uncertainty.
- In some processes, these uncertainties are encoded in form factors.

$$\langle \pi^0 \pi^- | J_W^{V,\mu} | 0 \rangle = (p_{\pi^-} - p_{\pi^0})^\mu F_\pi^V(s); \quad s = (p_{\pi^-} + p_{\pi^0})^2$$

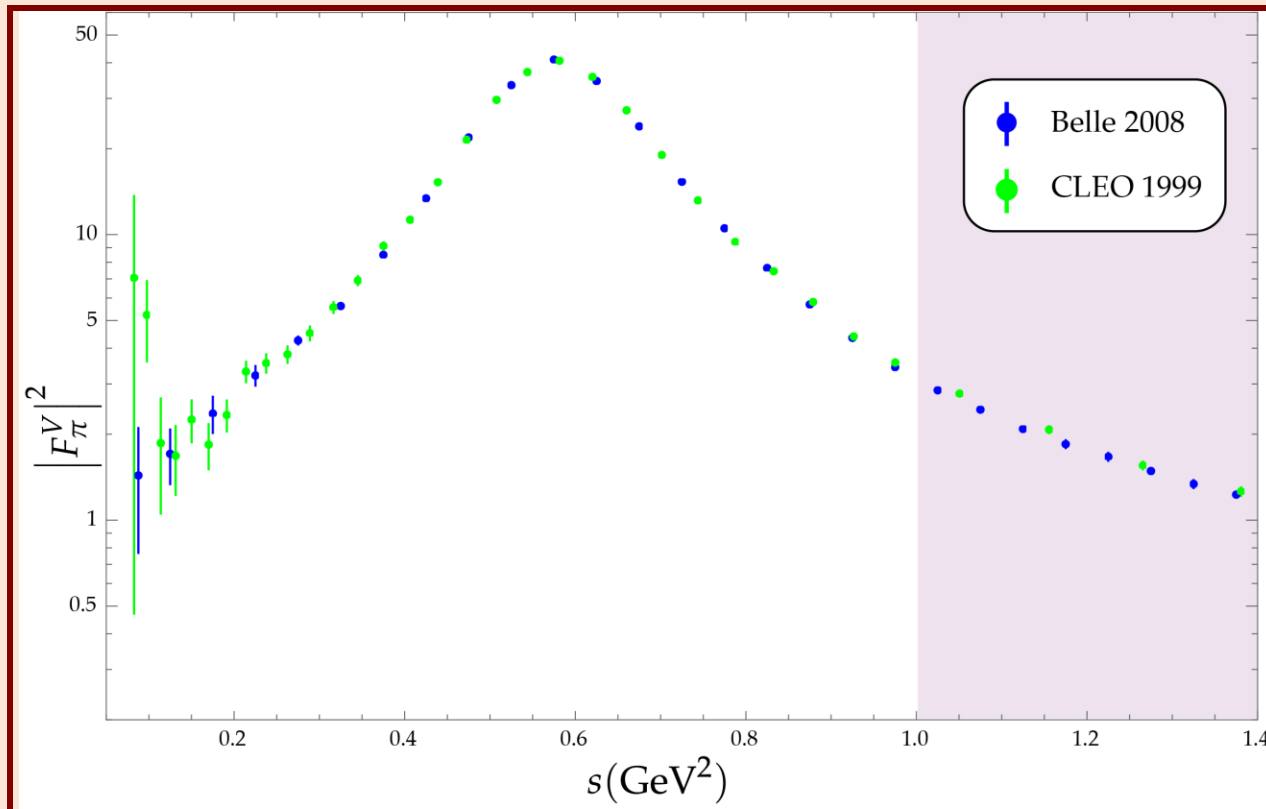
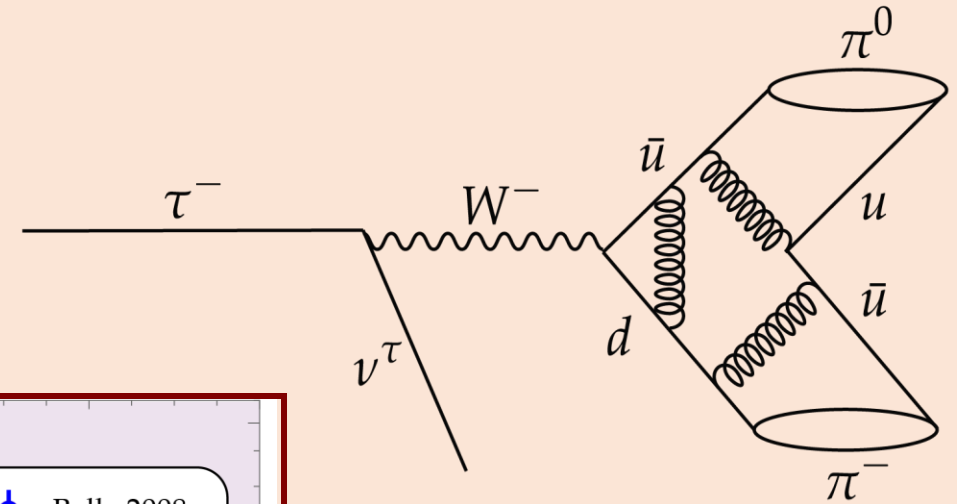


Pion vector form factor



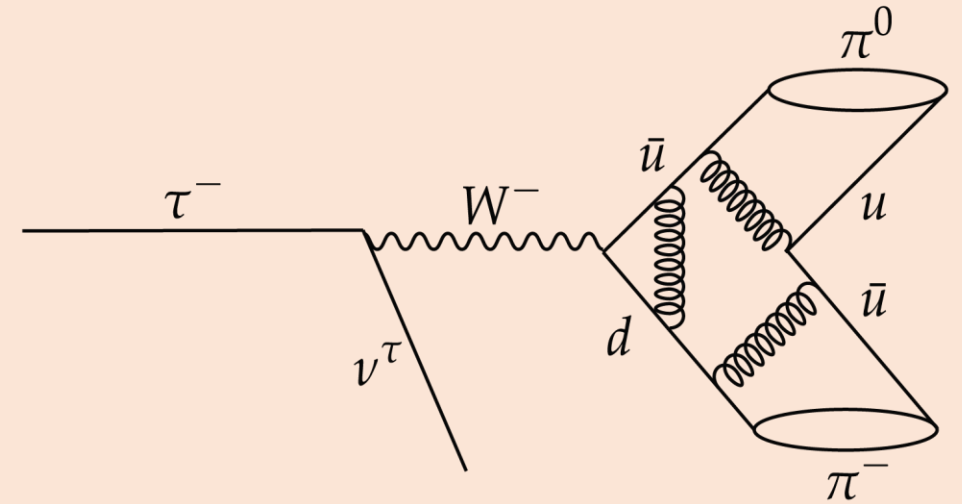
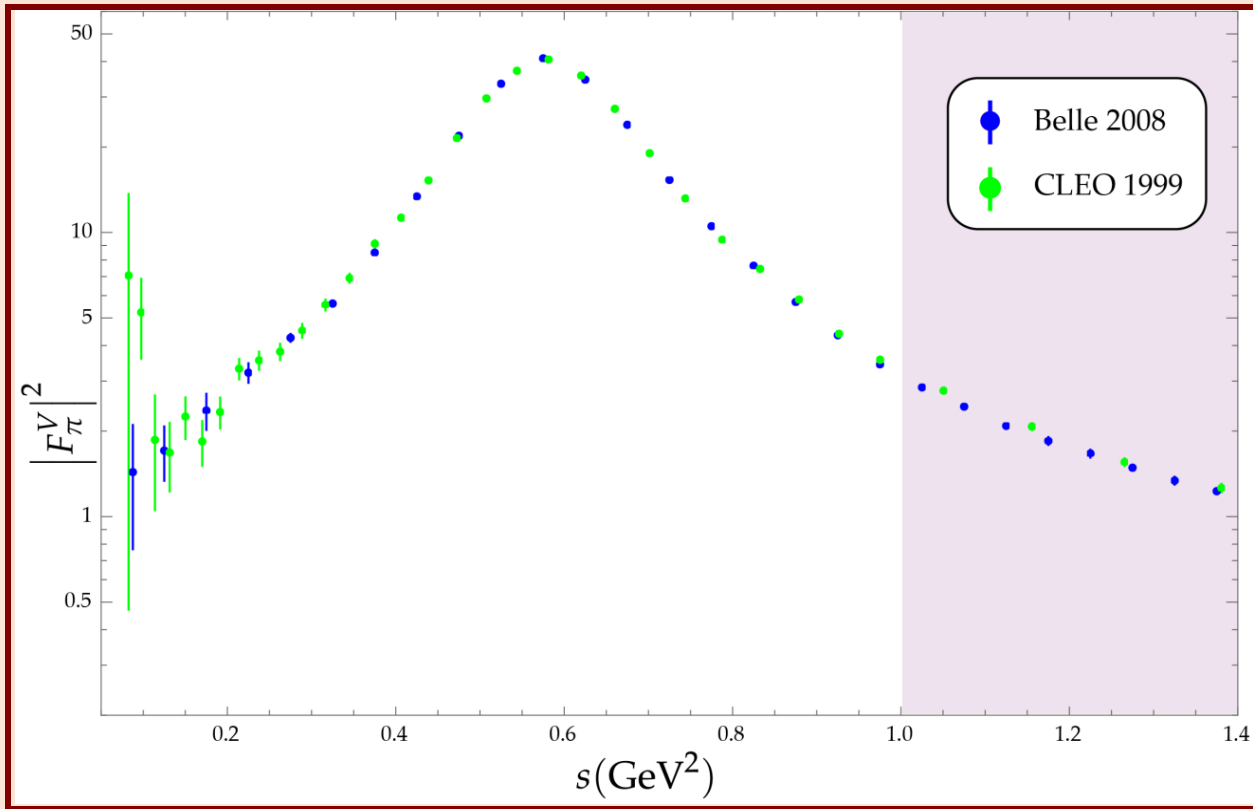
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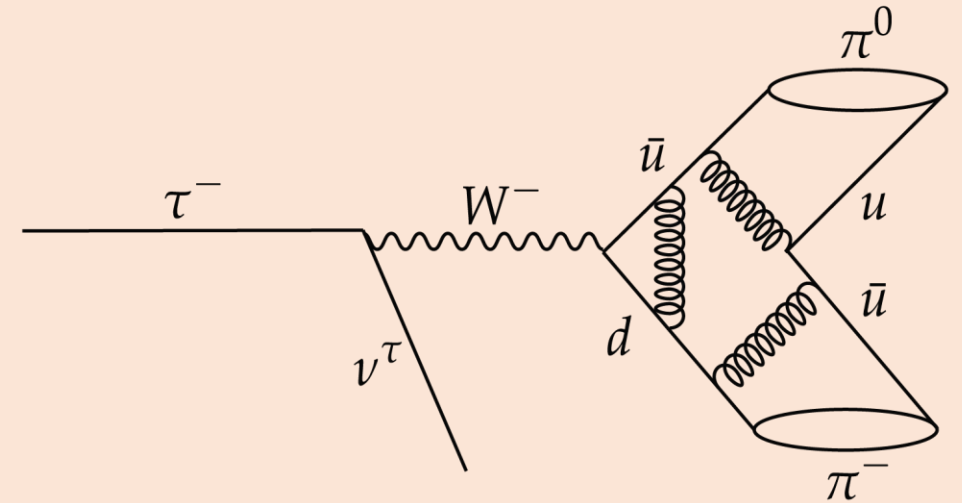
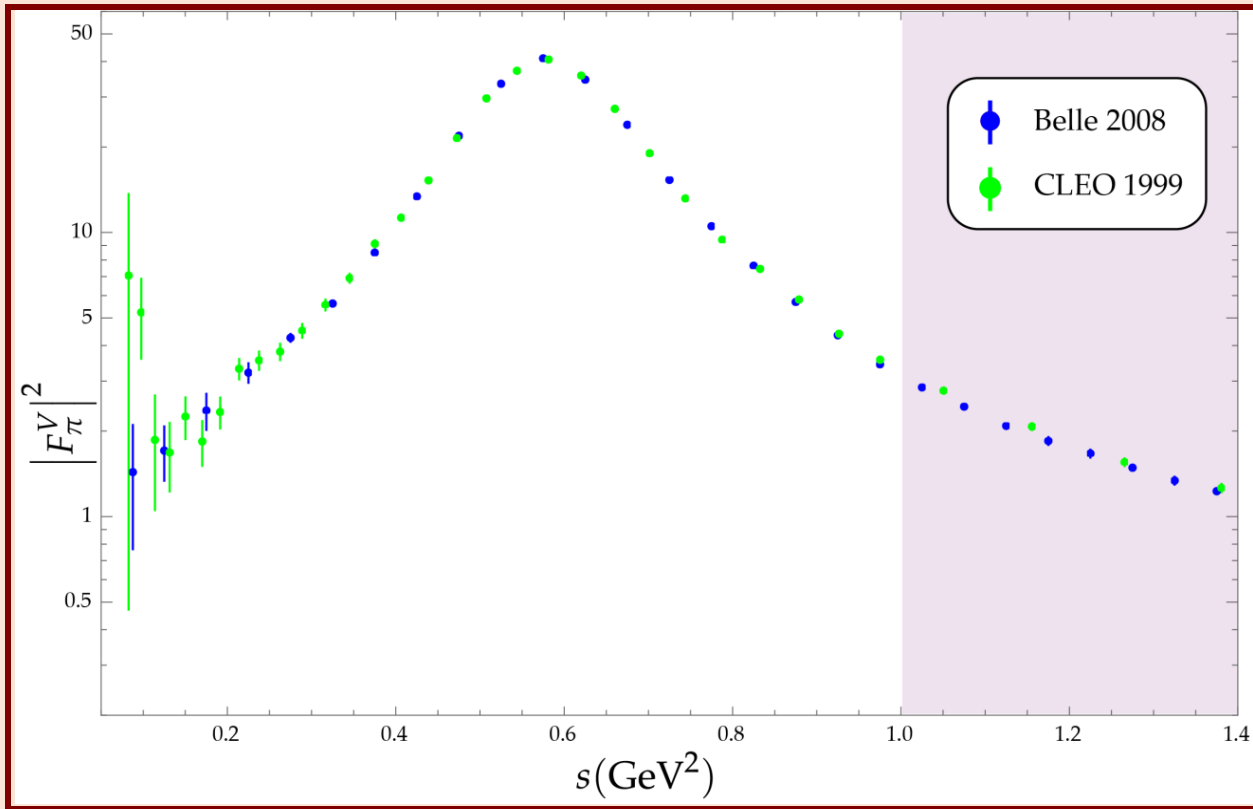
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- The peak corresponds to the isovector ρ (770) resonance.
- How to describe the peak ?

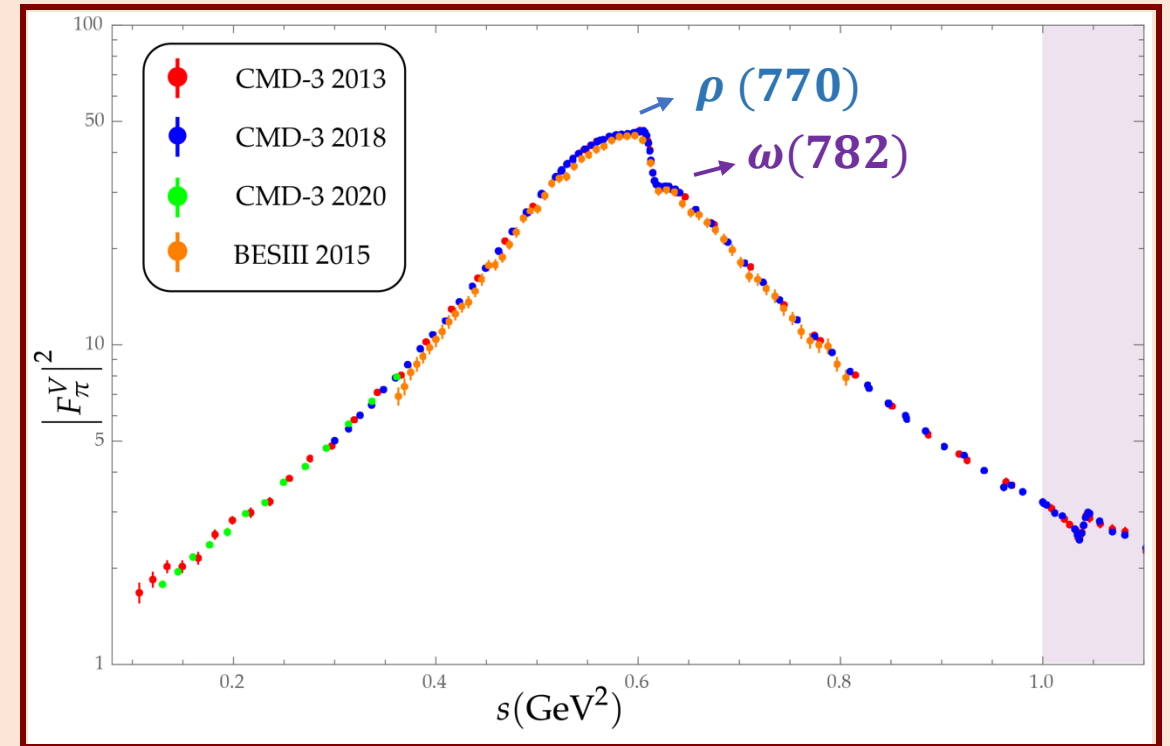
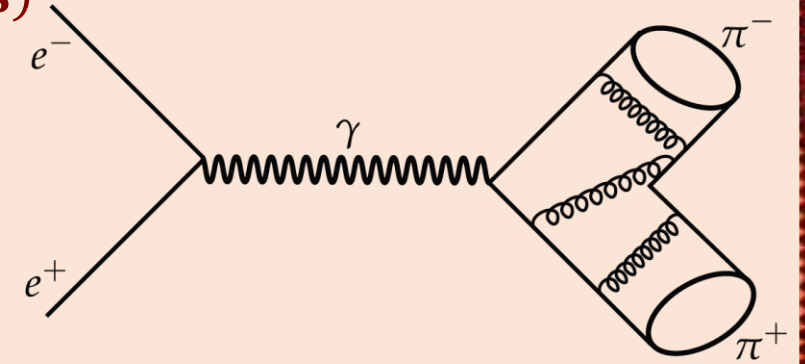
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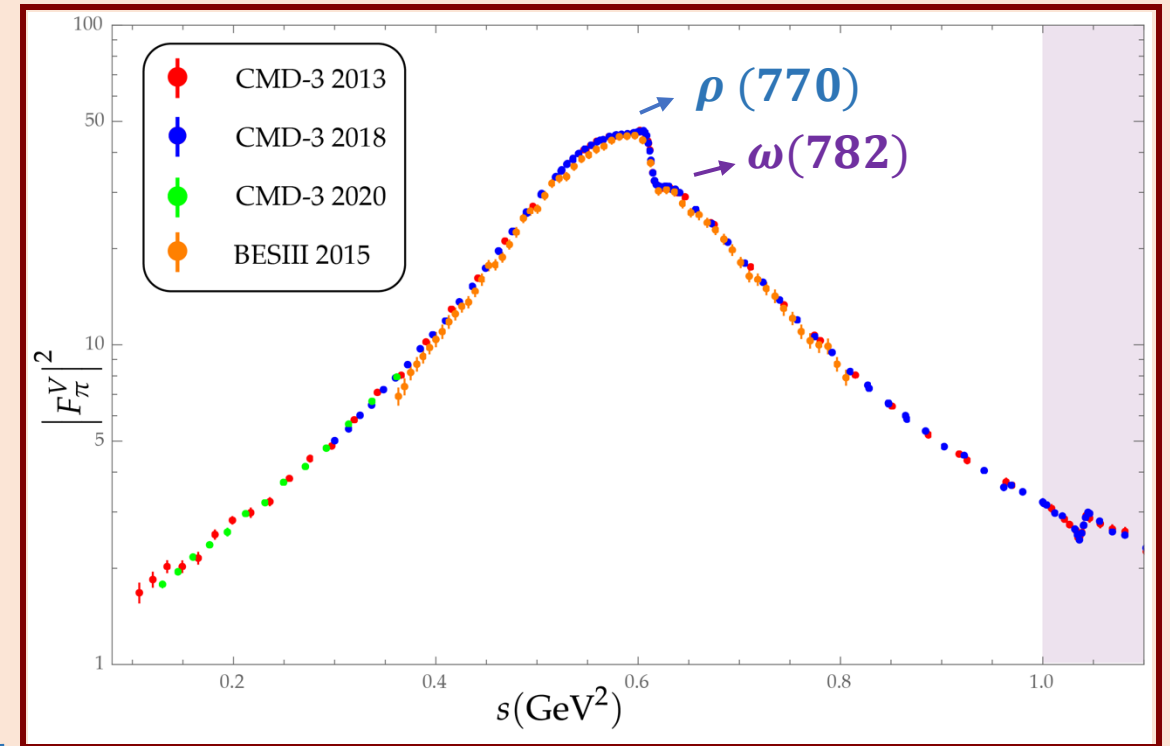
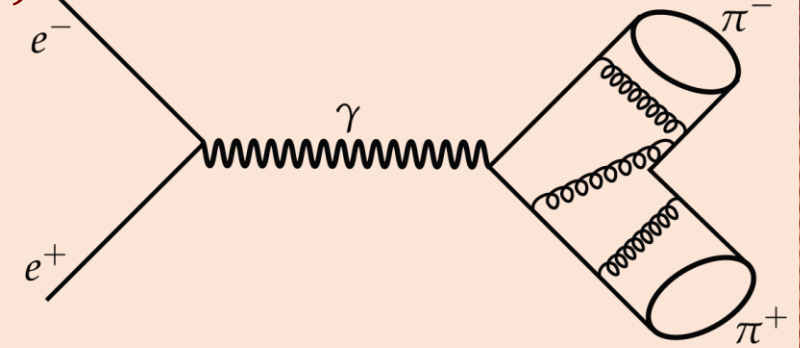
$$\langle \pi^+ \pi^- | J_{EM}^\mu | 0 \rangle = (p_{\pi^+} - p_{\pi^-})^\mu F_\pi^V(s)$$

- Breit-Wigner ? Can not describe the interference, can not produce thresholds, etc.
- It is common to use Omnès representation:

[see e.g. Colangelo, Hoferichter, Stoffer, 1810.00007, '19]

$$F_\pi^V(s) \propto \Omega_1^1(s) = \exp \left\{ \frac{s}{\pi} \int_{4m_\pi^2}^{\infty} ds' \frac{\delta_1^1}{s'(s' - s)} \right\}$$

Phase of the scattering amplitude, an external input

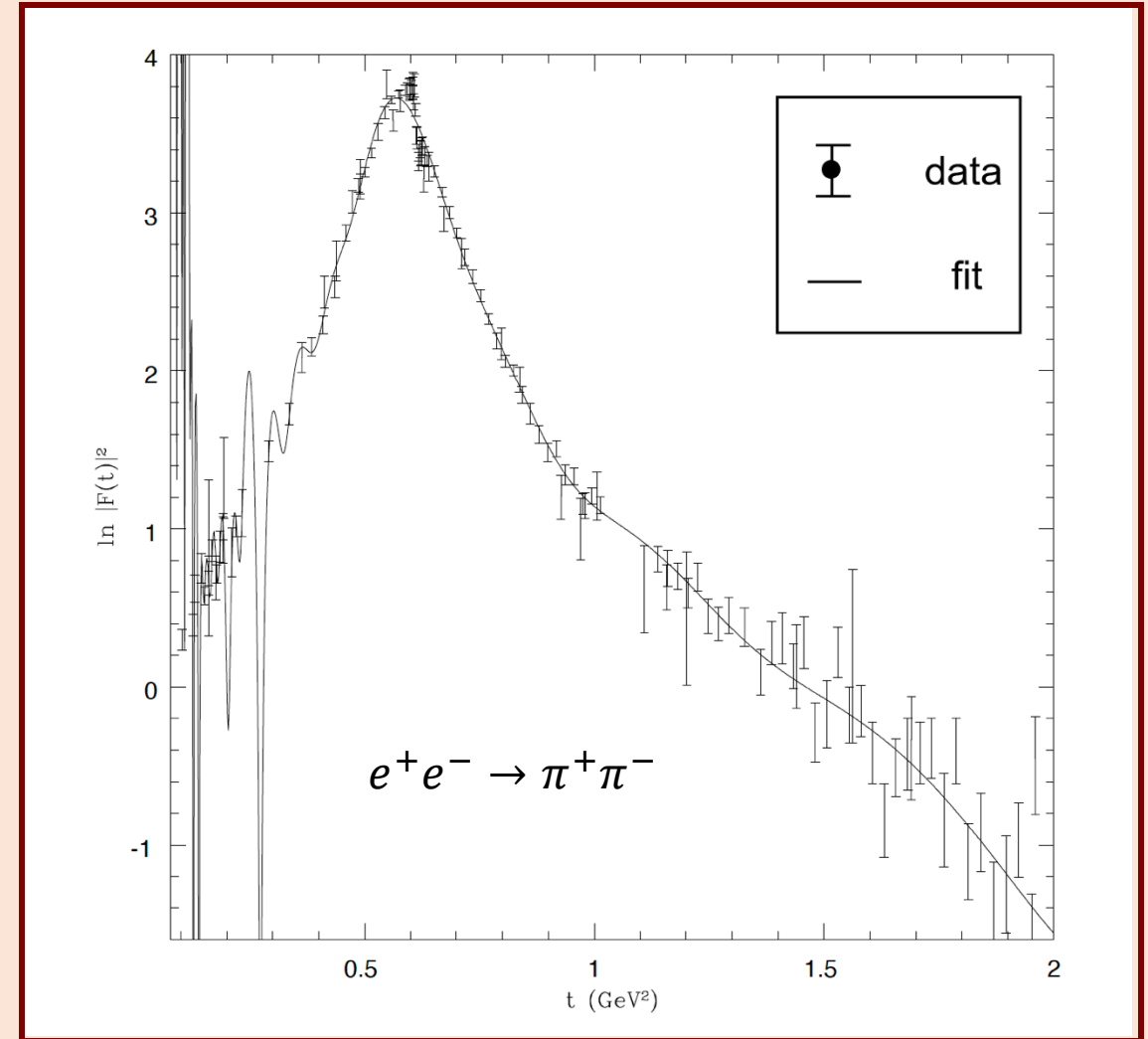


Motivation

- Another approach: a Taylor expansion in a conformal variable.
[Grinstein, Boyd, Lebed, 90's]

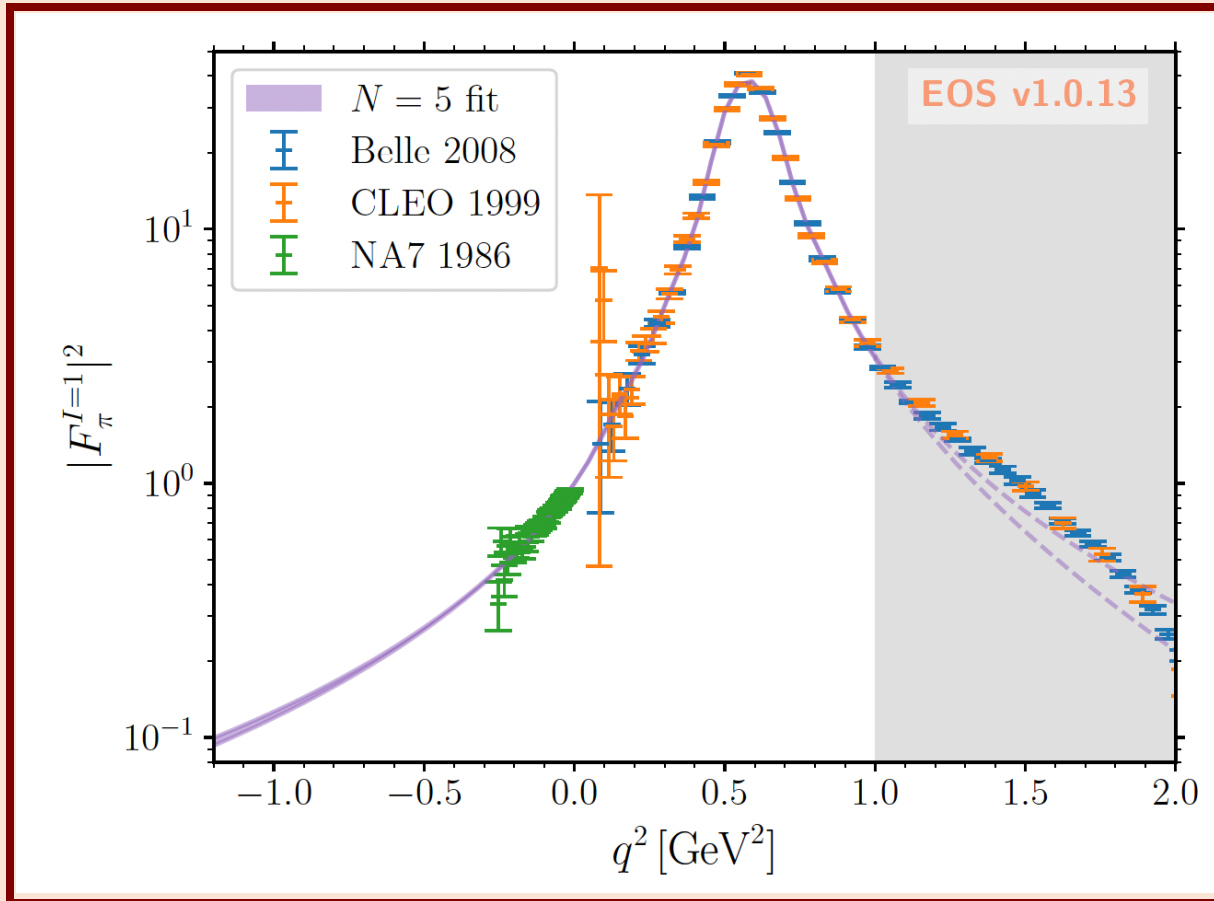
Motivation

- Another approach: a Taylor expansion in a conformal variable.
[Grinstein, Boyd, Lebed, 90's]
- Convergent Taylor series but unable to capture the rapid oscillations even after keeping 60 terms (Runge's phenomenon).
- We need a better description of form factors.



[Buck, Lebed, hep-ph/9802369, '98]

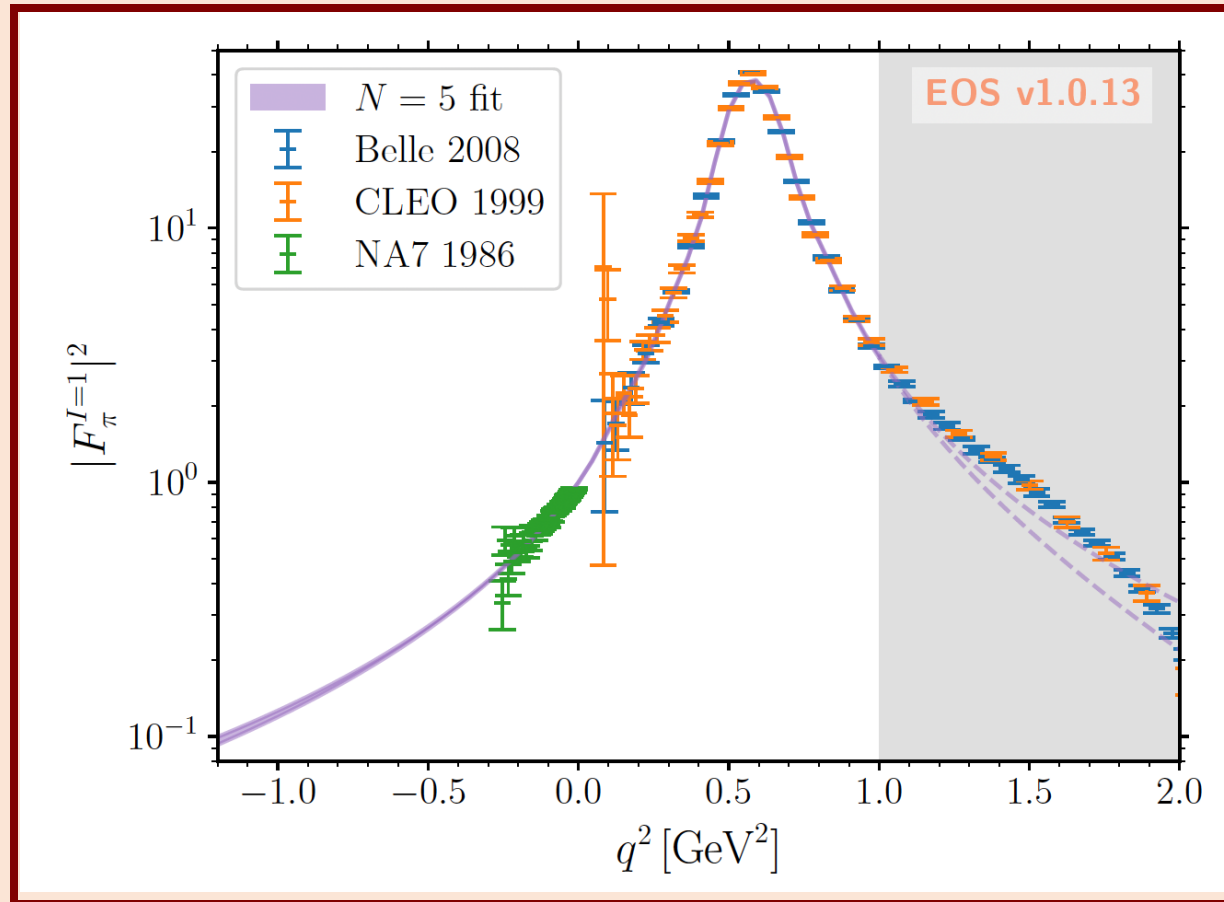
Improved Parametrization [Kirk, Kubis, Reboud, van Dyk, '25]



- Successful to describe $\tau^- \rightarrow \pi^- \pi^0 \nu_\tau$ data with only 5 terms of expansion.

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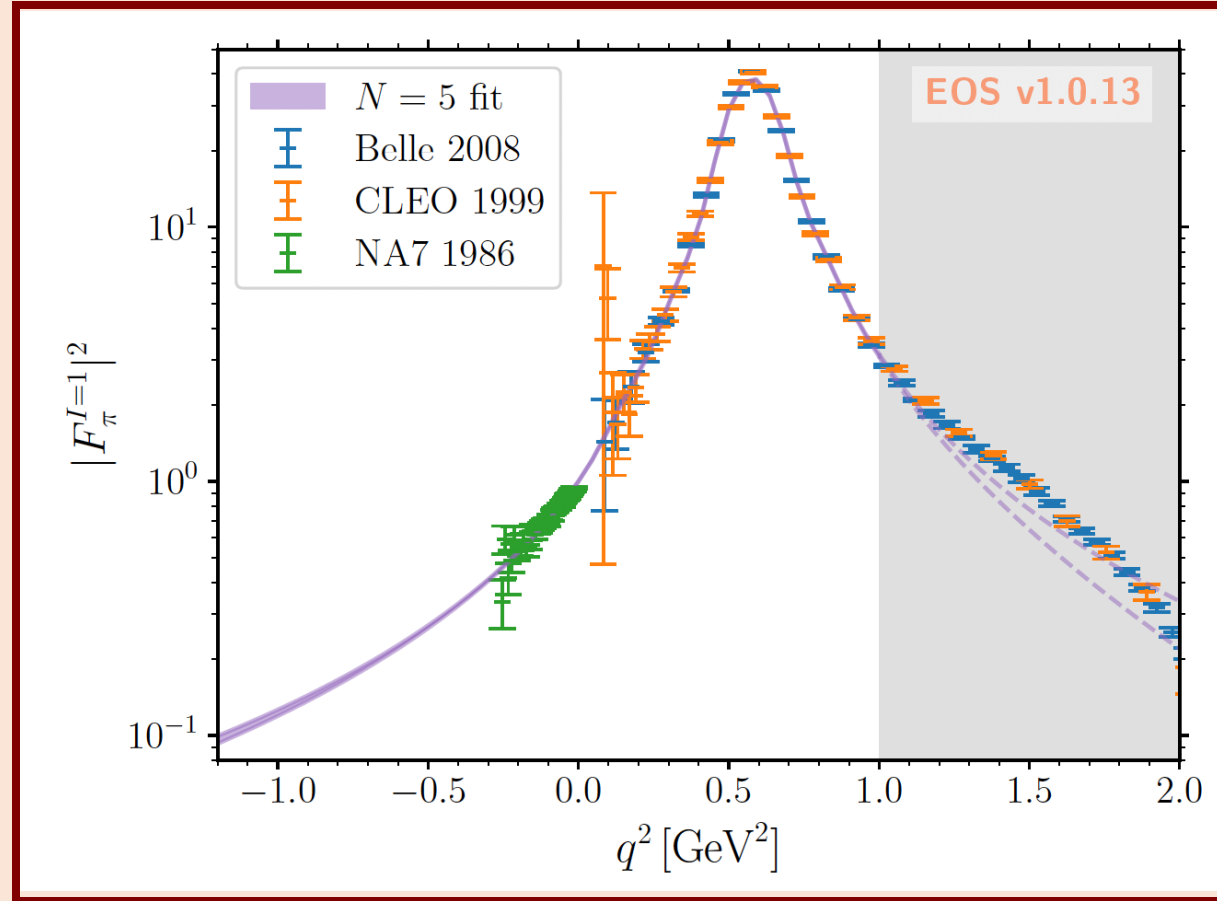


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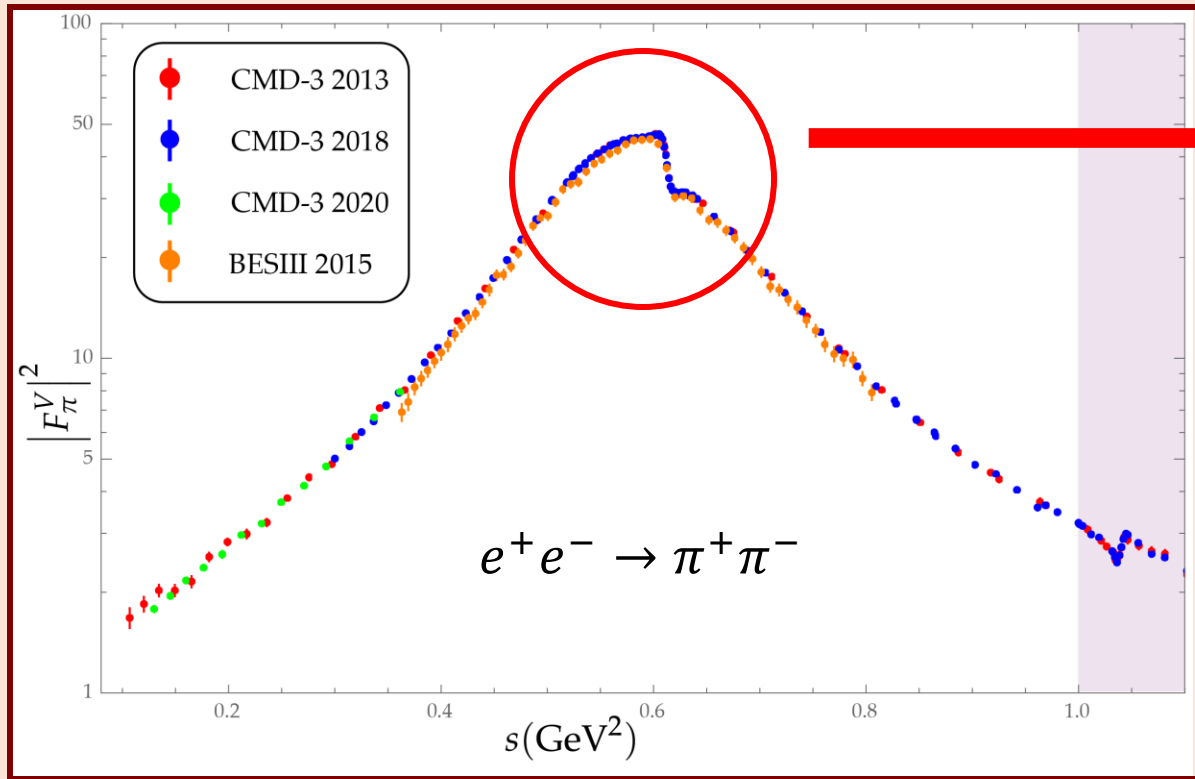
$$|I = 1, I_3 = -1\rangle$$

Very good parametrization for isospin-1 projection of the pion vector form factor $F_\pi^{I=1}$



The aim of this project ...

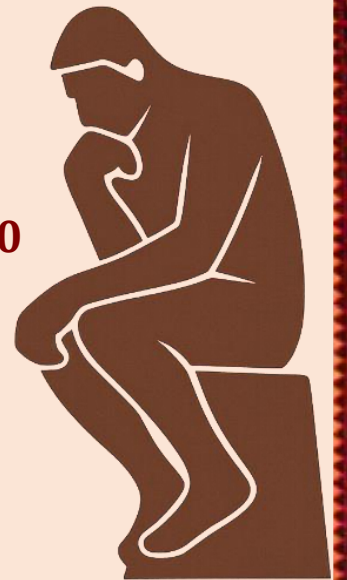
How to generalize this parametrization to account for the $\rho - \omega$ mixing ?



The dominant peak is due to isospin preserving $\rho^0 \rightarrow \pi^+ \pi^-$ decay but the small distortion is due to isospin violating $\omega^0 \rightarrow \pi^+ \pi^-$ decay.

$$I = 0 \neq I = 1$$

We need to parametrize $F_{\pi}^{I=0}$



Form Factor Properties [Caprini, Functional Analysis and Optimization Methods, '00]

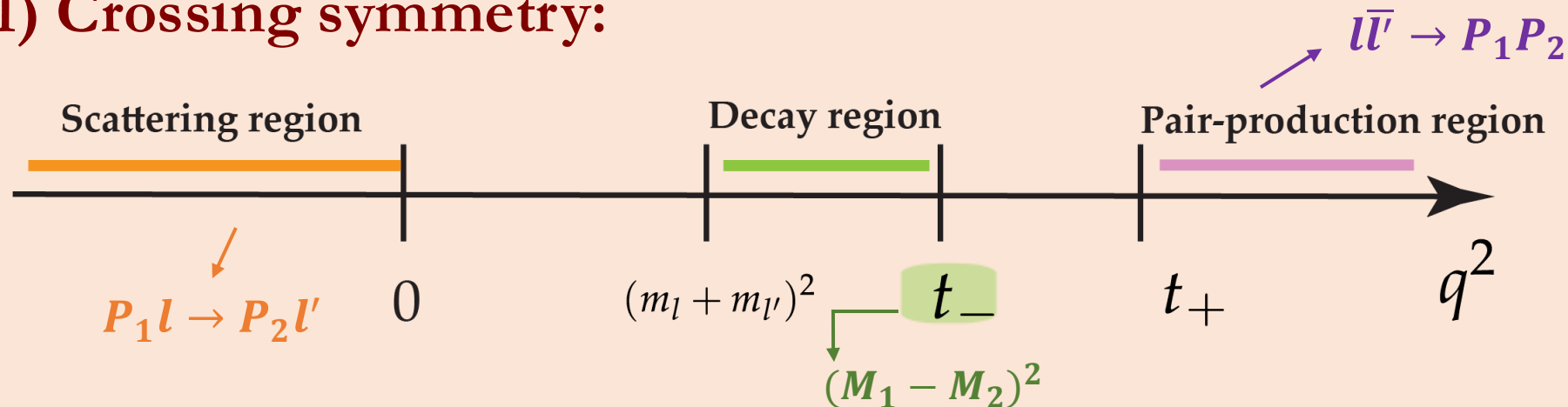
- Consider a semileptonic decay: $P_1 \rightarrow P_2 \bar{l} l'$

$$\langle P_2(p_2) | J^\mu | P_1(p_1) \rangle = \overset{\text{vector ff}}{f^+(q^2)} \left[(p_1 + p_2)^\mu - \frac{M_1^2 - M_2^2}{q^2} q^\mu \right] + \overset{\text{scalar ff}}{f^0(q^2)} \frac{M_1^2 - M_2^2}{q^2} \overset{\text{momentum transfer}}{q^\mu}$$

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1) Crossing symmetry:



2) Branch cut starts at pair-production threshold $t_+ = (M_1 + M_2)^2$.

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3) Unitarity and optical theorem requires that :

$$\frac{1}{\pi \chi_T(q^2)} \int_{t_+}^{\infty} dt \frac{\overset{\text{Phase factor}}{\omega(t)} |f_+(t)|^2}{(t - q^2)^3} \leq 1 \quad \text{Dispersive bound}$$

Known function, from theory or
lattice computations

[Bharucha, Feldmann, Wick, 1004.3249, '11]

Resonances

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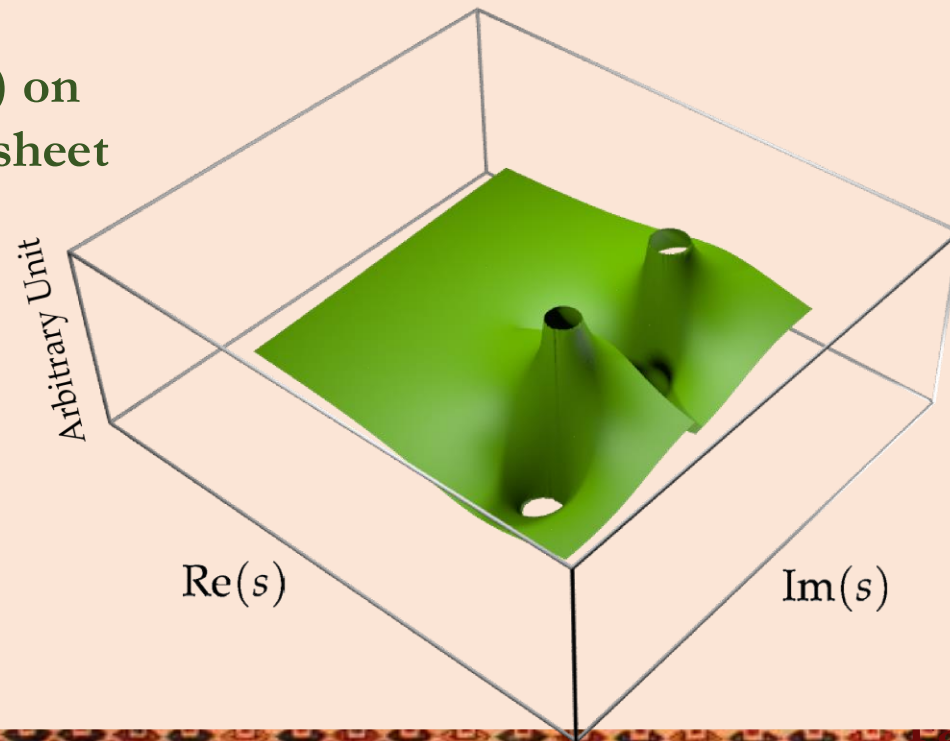
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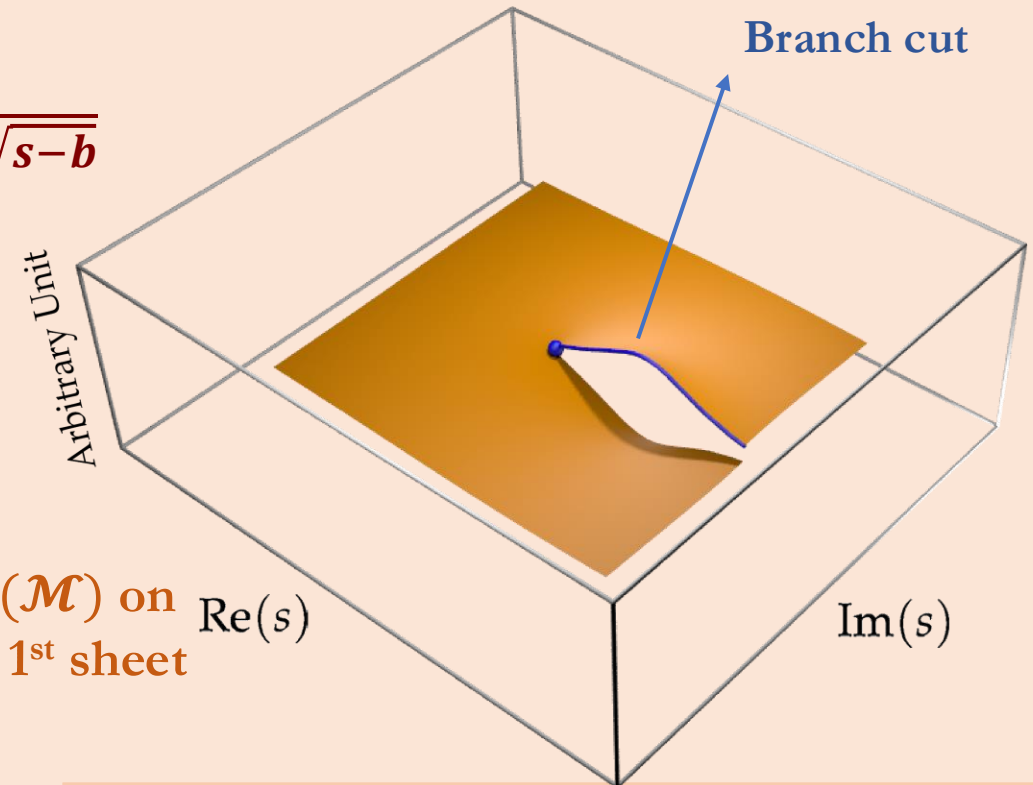
- Pair-production threshold opens another Riemann sheet.
- **Resonance:** Pole on the 2nd (unphysical) sheet, leaves an imprint on the 1st (physical) sheet.

Example: consider an amplitude $\mathcal{M} \sim \frac{1}{\sqrt{s-a}\sqrt{s-b}}$

Im ($\mathcal{M}$) on
the 2nd sheet



Im ($\mathcal{M}$) on
the 1st sheet



A Conformal Map

- In q^2 plane two Riemann sheets are connected non-trivially.

A Conformal Map

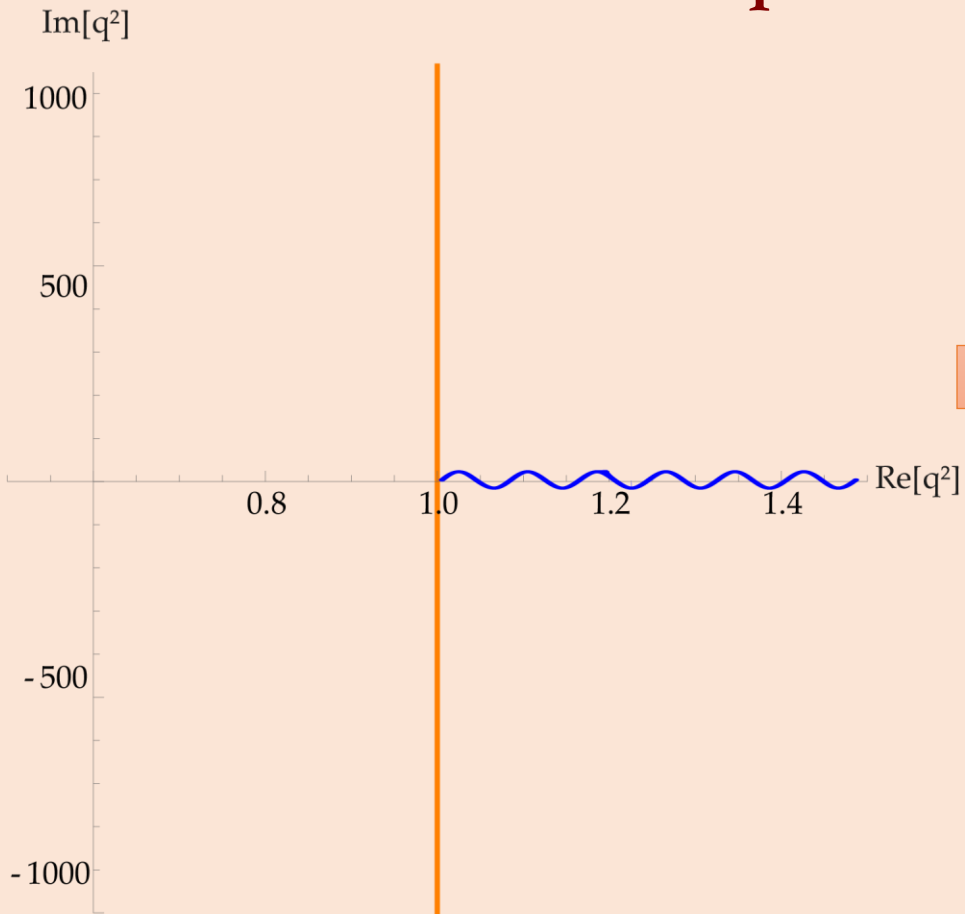
- In q^2 plane two Riemann sheets are connected non-trivially.
- A conformal mapping allows us to “unfold” the branch cut :

$$z(q^2; t_+, t_0) = \frac{\sqrt{t_+ - q^2} - \sqrt{t_+ - t_0}}{\sqrt{t_+ - q^2} + \sqrt{t_+ - t_0}}$$

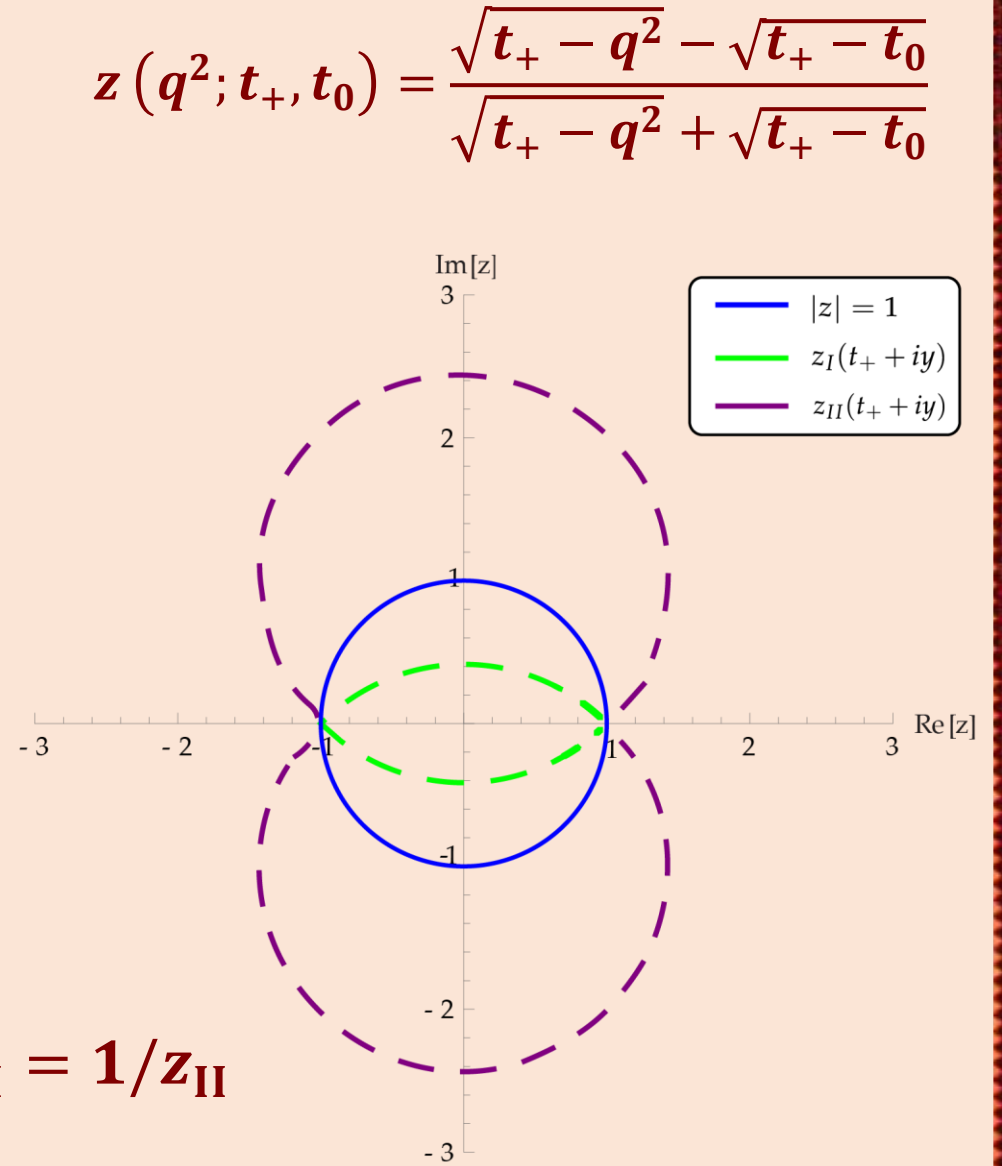
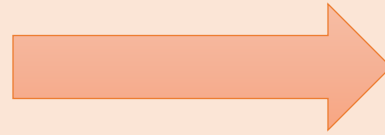
Free parameter t_0 [Grinstein, Boyd, Lebed, 90's]

A Conformal Map

- Branch cut is mapped on the unit circle and two sheets are separated:



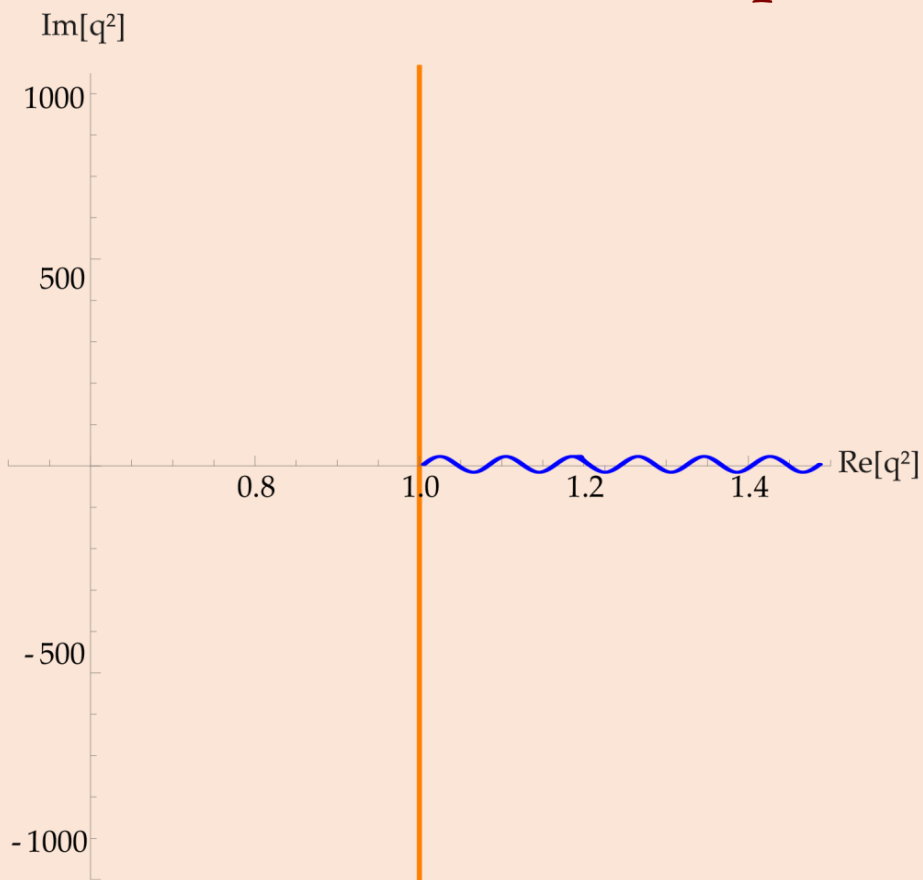
Conformal map



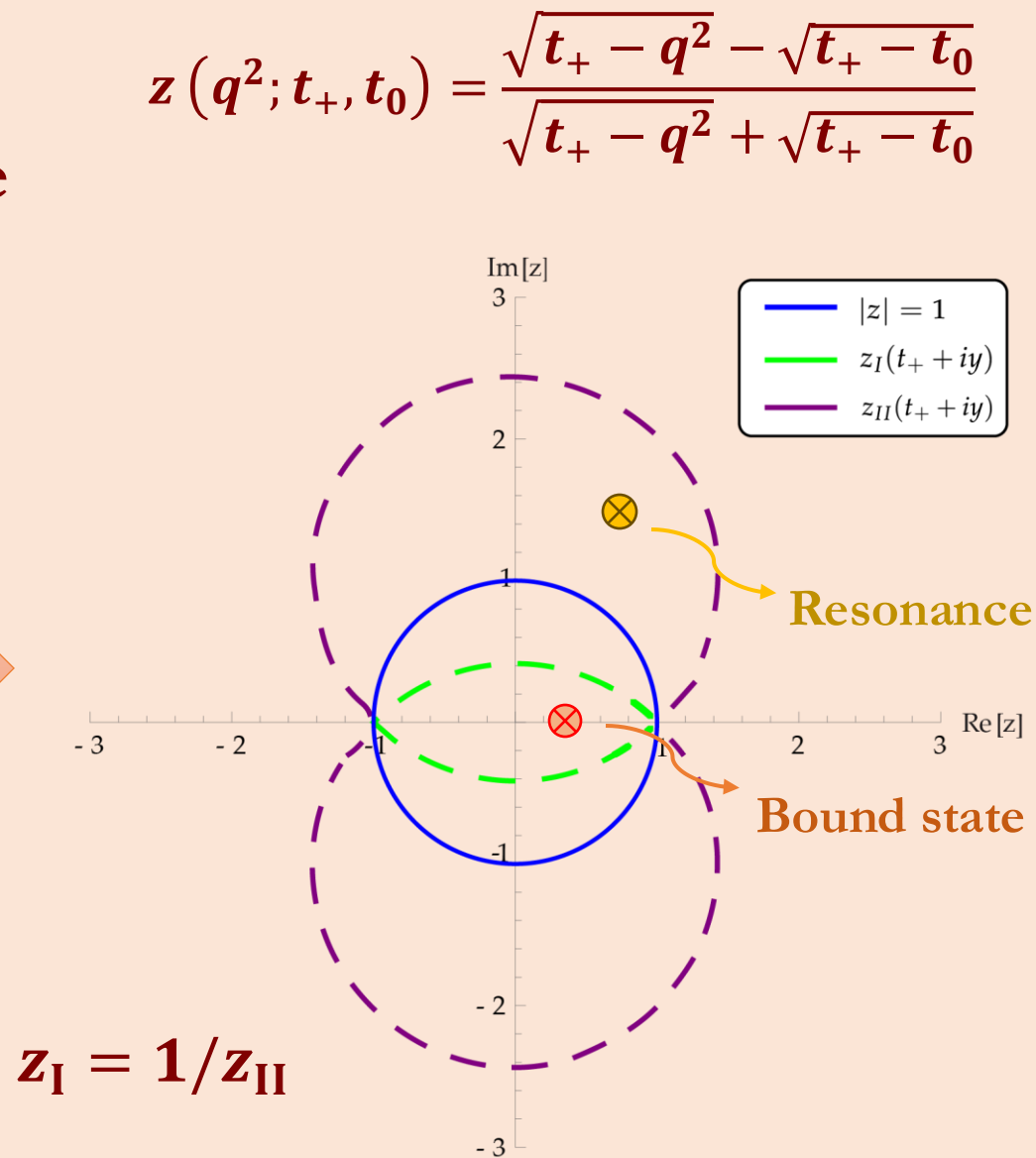
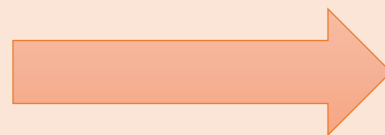
$$z_I = 1/z_{II}$$

A Conformal Map

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Conformal map



An Expansion for f_+

- Dispersive bound in terms of z :

$$\frac{1}{2\pi i} \oint_{|z|=1} \frac{dz}{z} |\phi(z)f_+(z)|^2 \leq 1$$

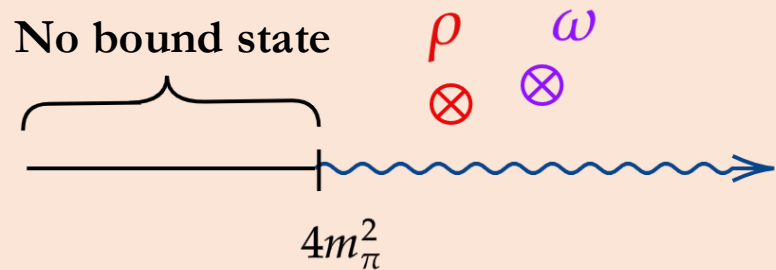
- Outer function $\phi(z)$ is a known function which is analytic and non-vanishing inside the unit disk by construction.

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$F_\pi^V(z)\phi(z)$ is analytic inside the unit disk



[Grinstein, Boyd, Lebed, 90's]

$$F_\pi^V(z) = \frac{1}{\phi(z)} \sum_{n=0}^{\infty} a_n z^n$$

Constraints on f_+ [Buck, Lebed, hep-ph/9802369, '98]

- Angular momentum conservation require that :

$$\text{Im } f_+(t) \sim (t - t_+)^{\frac{3}{2}} \quad \text{as } t \rightarrow t_+,$$

equivalently in terms of z it will be :

$$\left. \frac{df_+}{dz} \right|_{z=-1} = 0 \quad \Rightarrow \quad \text{P-wave constraint}$$

- Perturbative calculations at high energy require :

$$f_+(q^2) \sim \frac{1}{q^2} \quad \text{as } q^2 \rightarrow \infty.$$

- Charge conservation implies that :

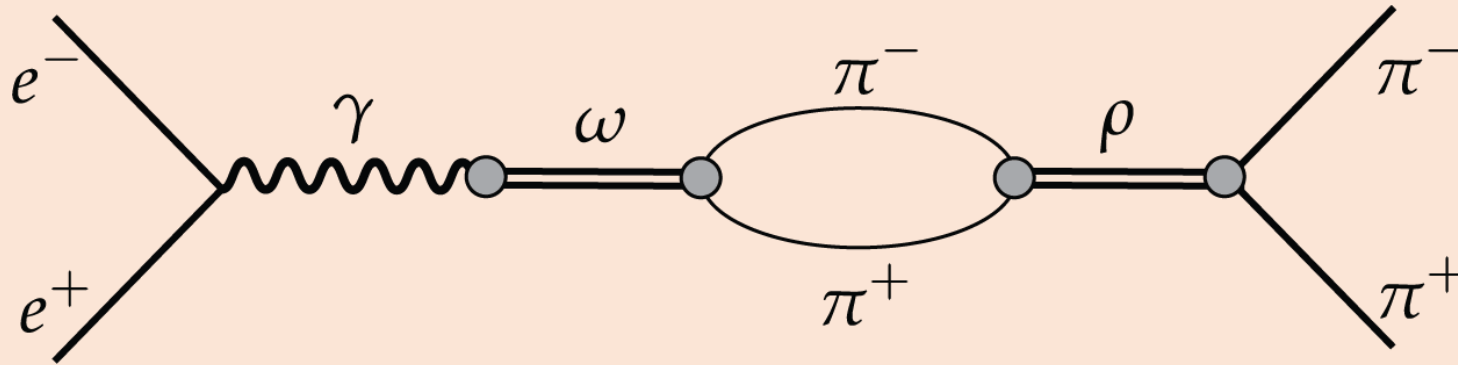
$$F_\pi^V(q^2 = 0) = 1 \quad \Rightarrow \quad \text{Normalization constraint}$$

Our Ansatz for $F_{\pi}^{I=0}$

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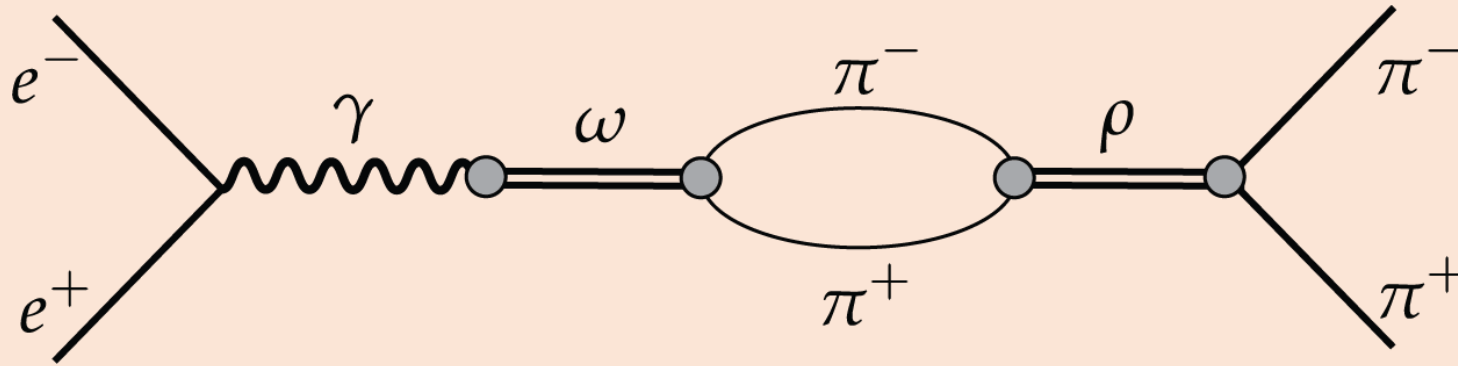


- Multiplicative ansatz is favorable, we take :

$$F_{\pi}^V(z) = F_{\pi}^{I=1}(z)[1 + F_{\pi}^{I=0}(z)]$$

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$$F_{\pi}^V(z) = F_{\pi}^{I=1}(z)[1 + \cancel{F_{\pi}^{I=0}(z)}]$$

In isospin limit $F_{\pi}^V(z) \cong F_{\pi}^{I=1}(z)$

Parametrization of $F_{\pi}^{I=1}$

$$F_{\pi}^{I=1}(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^N b_n z^n}{(z - z_{\rho})(z - \bar{z}_{\rho})}$$

Analytic inside the unit disk

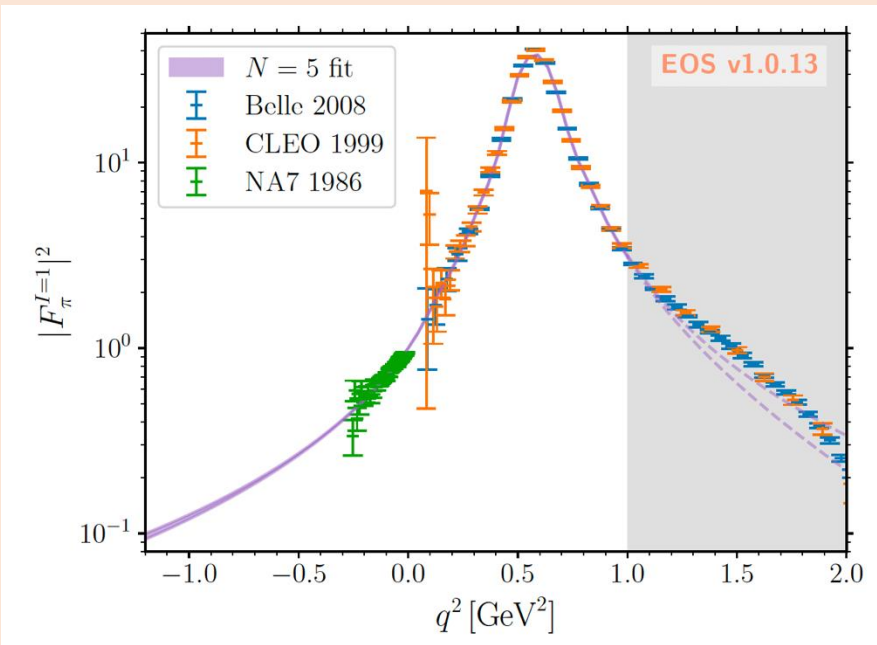
[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

- $W(z) = (1+z)^2(1-z)^{5/2}$ ensures the large q^2 behavior and cancels the kinematical singularities of the outer function $\phi(z)$ “on” the unit circle.
- $z_{\rho} = \frac{1}{z((M_{\rho} - i\Gamma_{\rho})^2)}$ is the position of ρ resonance on the second sheet, M_{ρ} and Γ_{ρ} are its mass and width.

Parametrization of $F_{\pi}^{I=1}$

$$F_{\pi}^{I=1}(z) = \frac{W(z)}{\phi(z)} \underbrace{\frac{\sum_{n=0}^N b_n z^n}{(z - z_{\rho})(z - \bar{z}_{\rho})}}$$

Factorizing the ρ pole avoid the Runge problem that [Buck, Lebed, hep-ph/9802369, '98] faced



[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

truncation N	χ^2	d.o.f.	p value [%]	$\langle r_{\pi}^2 \rangle$ [fm ²]	M_{ρ} [MeV]	Γ_{ρ} [MeV]	bound saturation
1	≈ 3300	92	$< 10^{-10}$	—	—	—	0.46
2	≈ 1500	91	$< 10^{-10}$	—	—	—	0.44
3	117.4	90	2.8	0.474 ± 0.0022	760.4 ± 0.4	143.0 ± 0.6	0.46
4	98.17	89	23.8	0.457 ± 0.0045	760.2 ± 0.4	145.9 ± 0.9	0.46
5	97.9	88	22.1	0.460 ± 0.0061	760.0 ± 0.6	146.1 ± 0.9	0.46

Dispersive bound check

[Kirk, Kubis, Reboud, van Dyk, 2410.13764, '25]

Adding $F_{\pi}^{I=0}$ to the picture

$$F_{\pi}^{I=0}(z) = \frac{\sum_{m=0}^{N'} c_m z^m}{(z - z_{\omega})(z - \bar{z}_{\omega})}$$

Adding $F_{\pi}^{I=0}$ to the picture

$$F_{\pi}^V(z) = \frac{W(z)}{\phi(z)} \frac{\sum_{n=0}^N b_n z^n}{(z - z_{\rho})(z - \bar{z}_{\rho})} \left[1 + \frac{\sum_{m=0}^{N'} c_m z^m}{(z - z_{\omega})(z - \bar{z}_{\omega})} \right]$$

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- $N' = 2$; Small ω effect does not need many terms to be described.

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- $N = 4$ was the best result for $F_{\pi}^{I=1}$.
- $N' = 2$; Small ω effect does not need many terms to be described.
- P-wave and normalization constraints need to be implemented.

Fit Results

(N, N')	χ^2	d.o.f.	P-value [%]
(4, 2)	413.8	318	0.0234

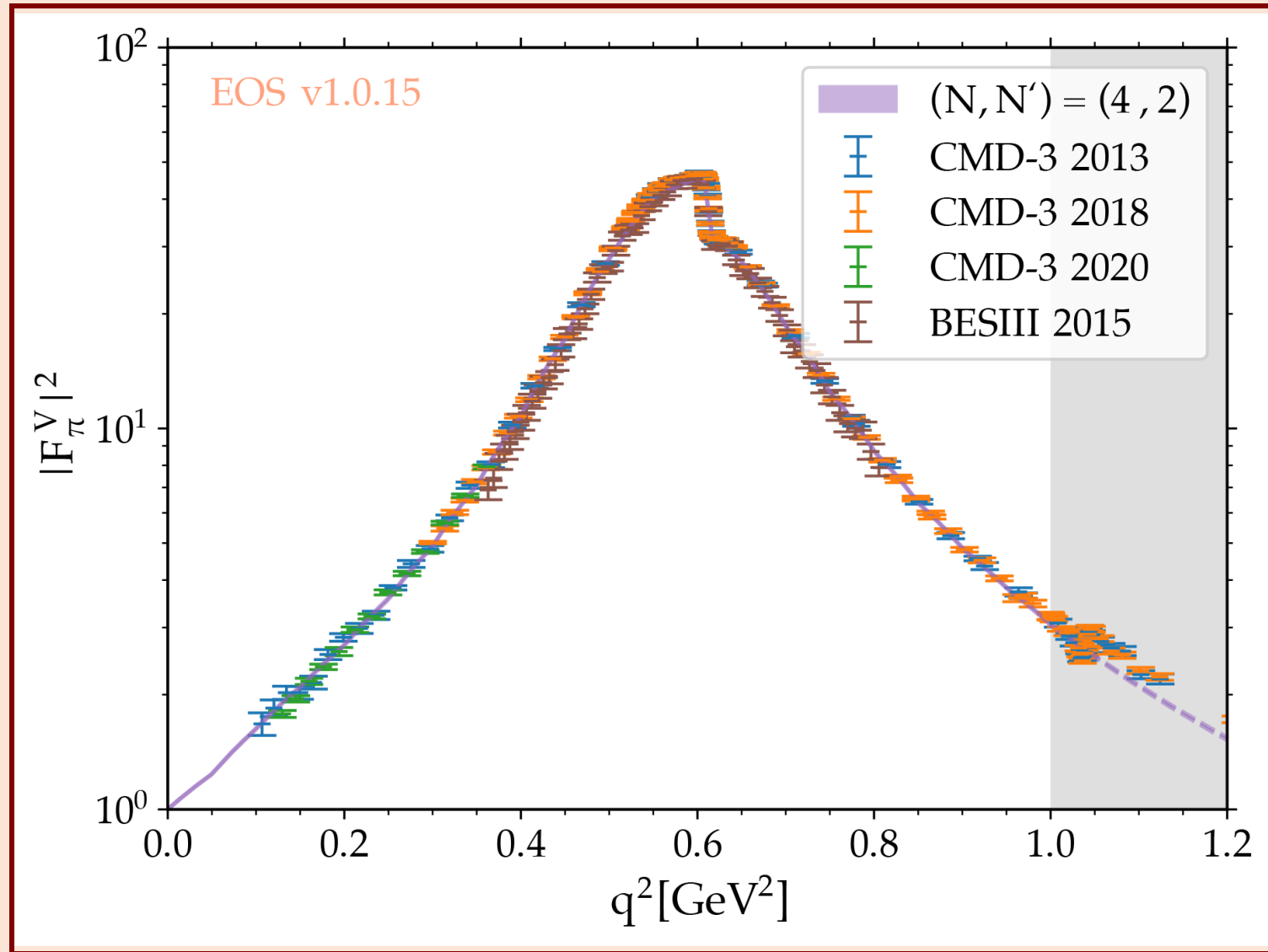


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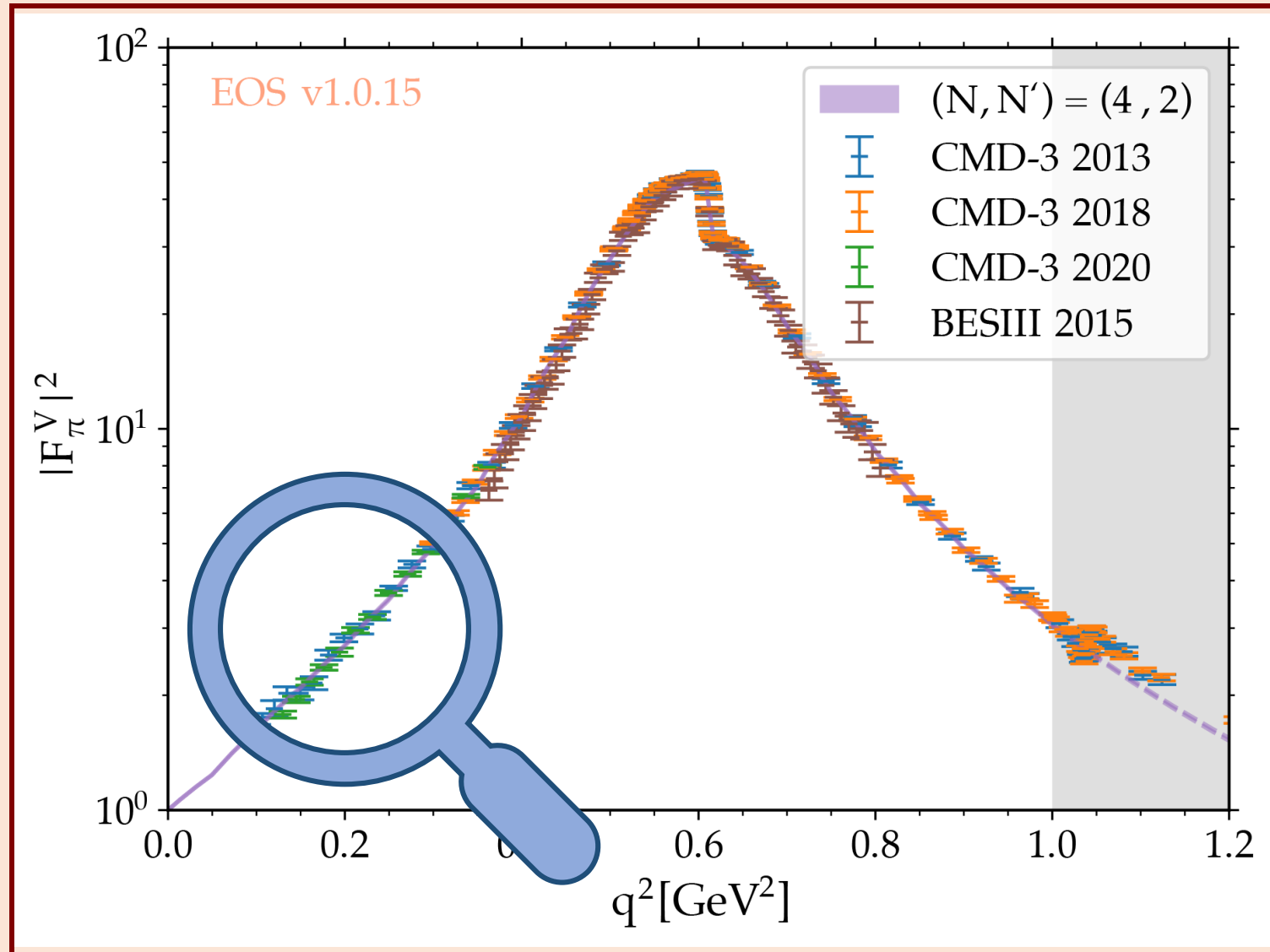
Dataset	χ^2	d.o.f	Local p-value [%]
BESIII 2015	61.05	60	43.81
Belle 2008	23.60	19	21.19
CLEO 1999	43.33	29	4.42
CMD-3 2013	76.67	66	17.36
CMD-3 2018	93.74	93	45.89
CMD-3 2020	69.36	13	0.00
NA7 1986	45.80	45	43.87
Total	413.79	318	0.0234



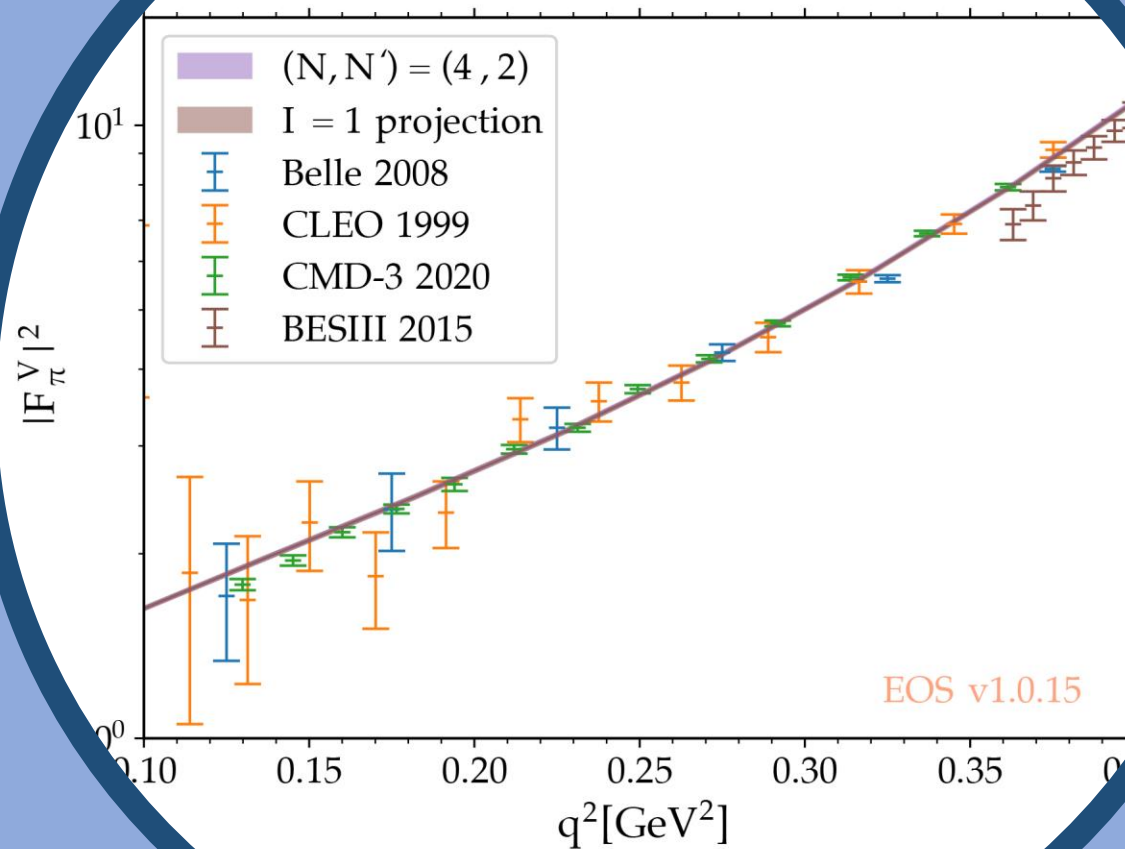
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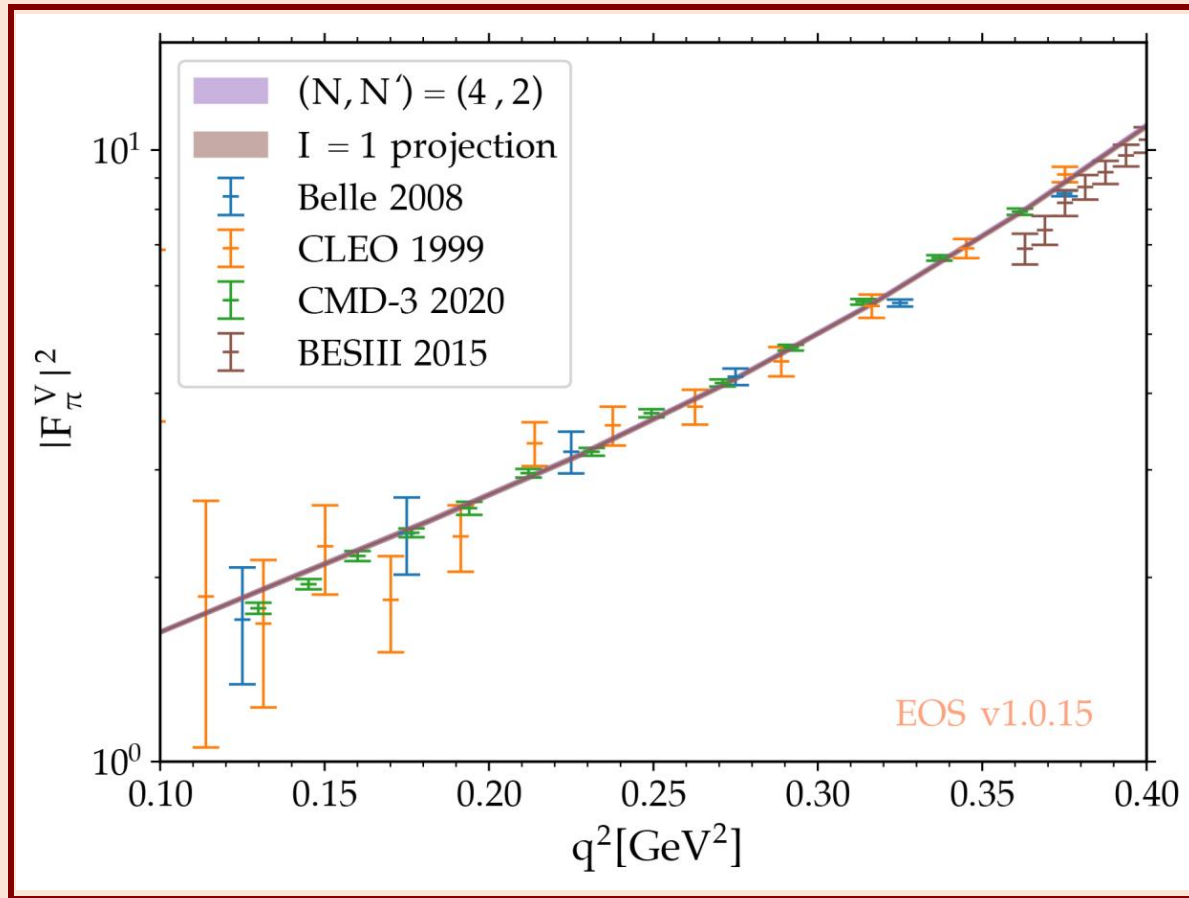
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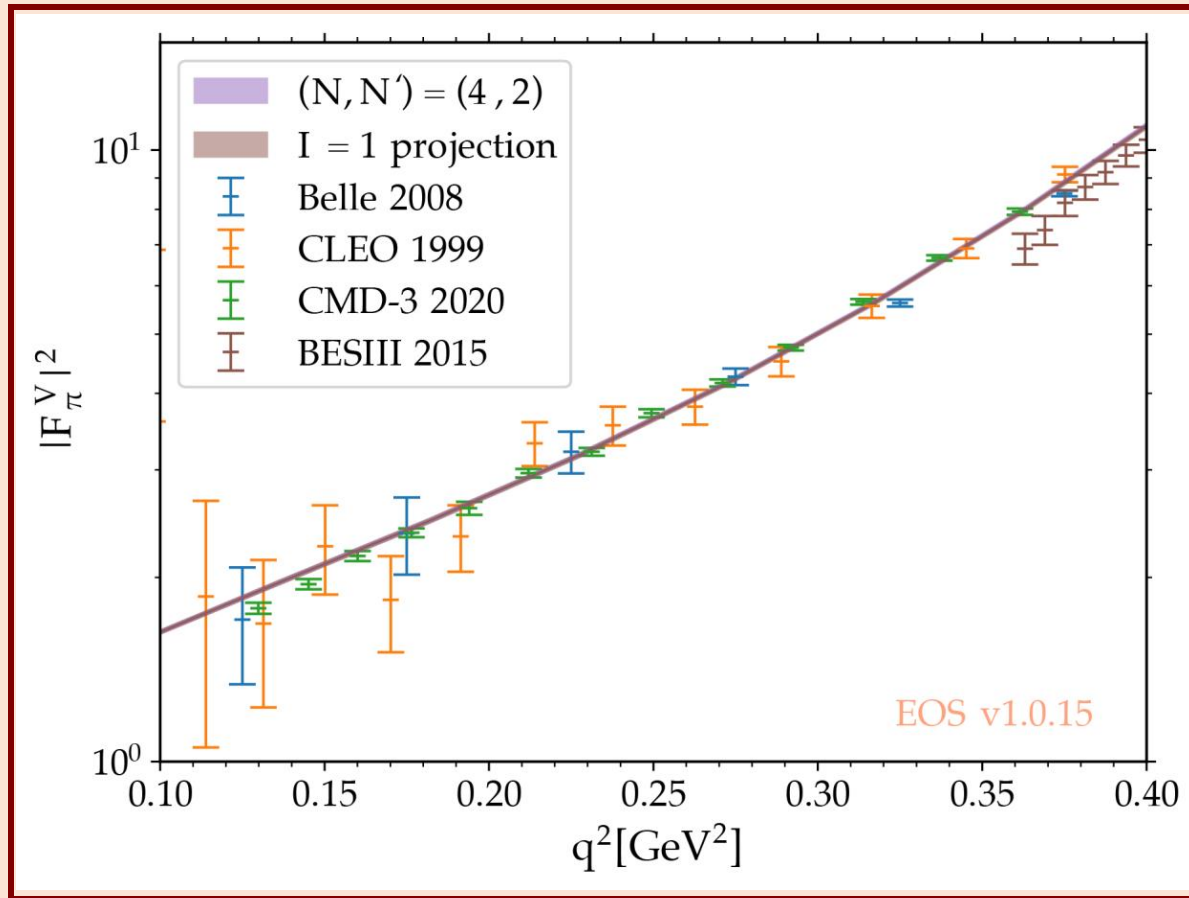
Fit Results



- The best-fit curve is closer to the CMD-3 2020 data than what we expected !



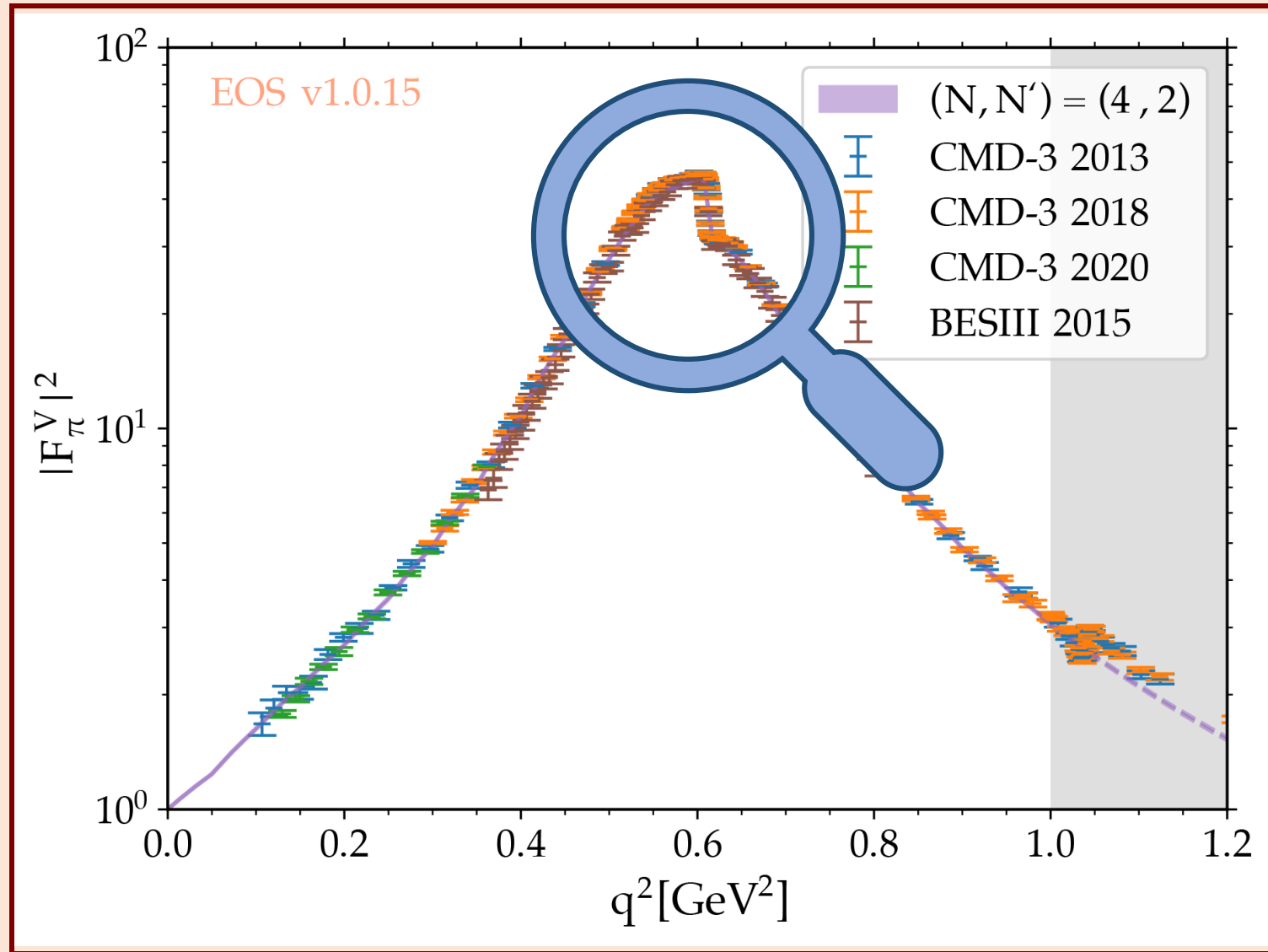
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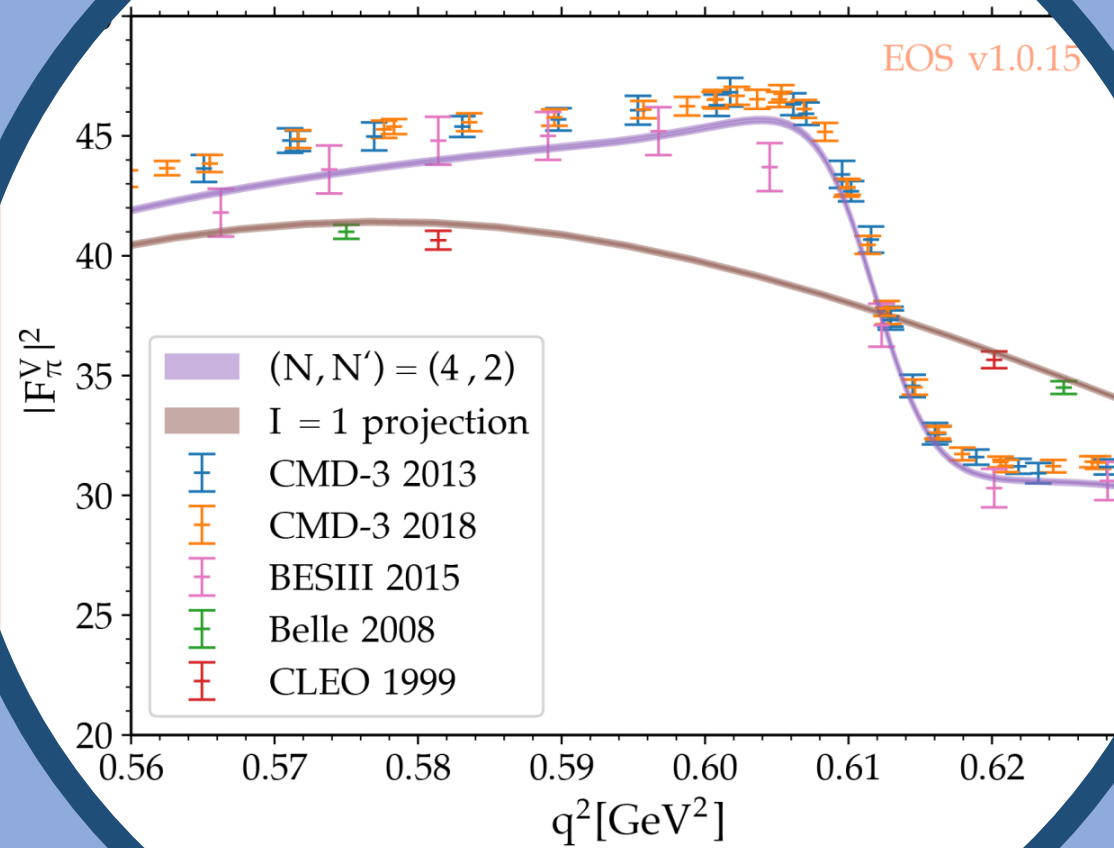
- The best-fit curve is closer to the CMD-3 2020 data than what we expected !
- Compare F_π^V with $F_\pi^{I=1}$: The two curves coincide perfectly in this region, confirming that, as expected, the ω contribution is negligible at low momentum transfer.



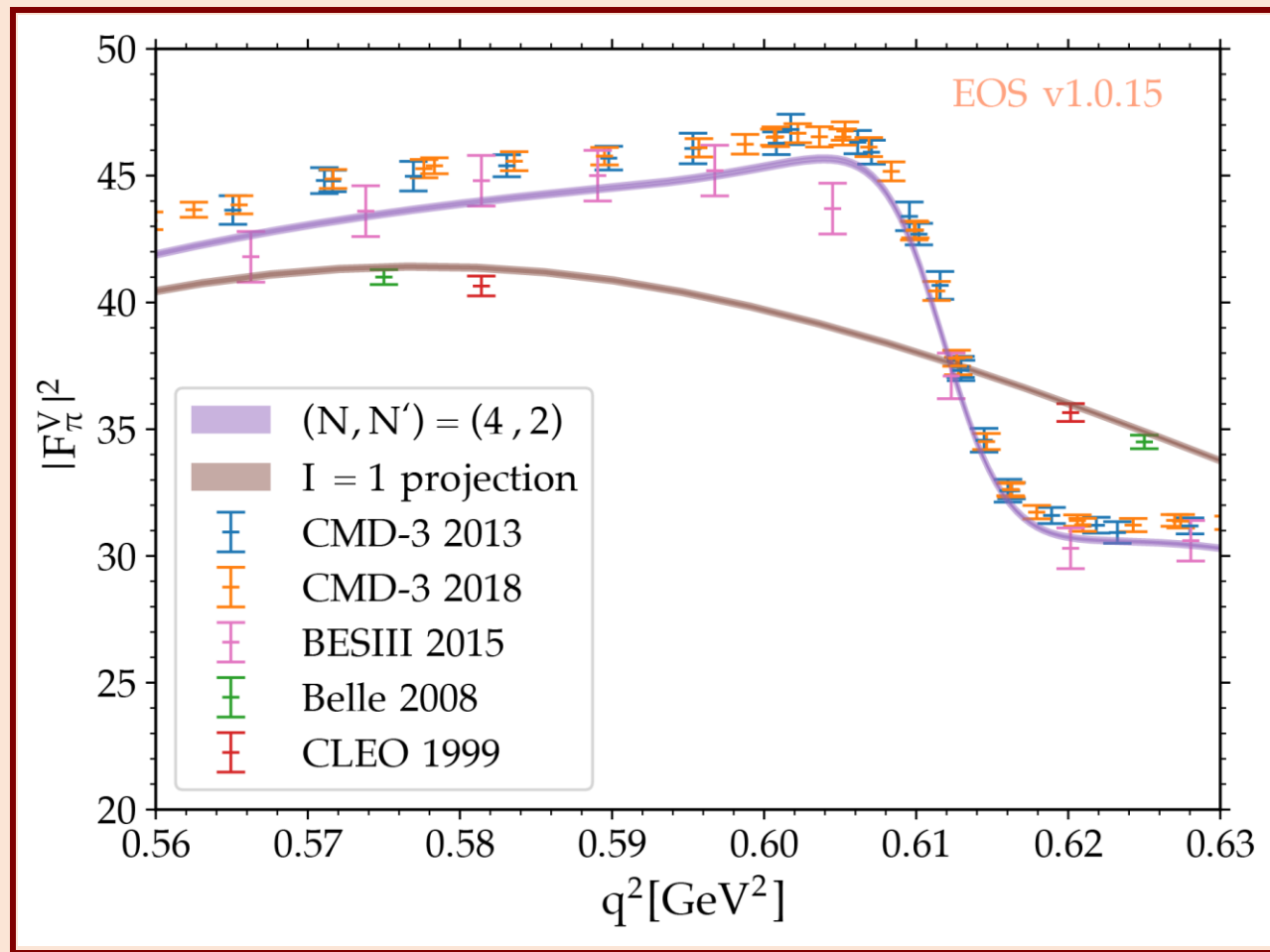
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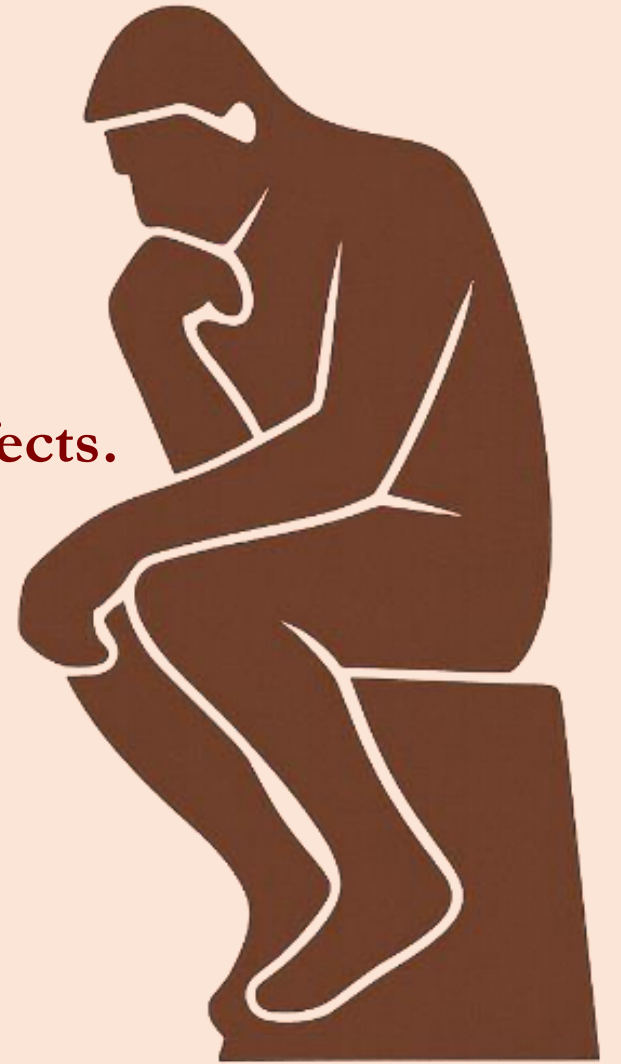


- Best-fit curve deviates from the experimental data in this region.
- Tension between different datasets.
- The tension originates from the parametrization, from inconsistencies in the experimental inputs, or both ?



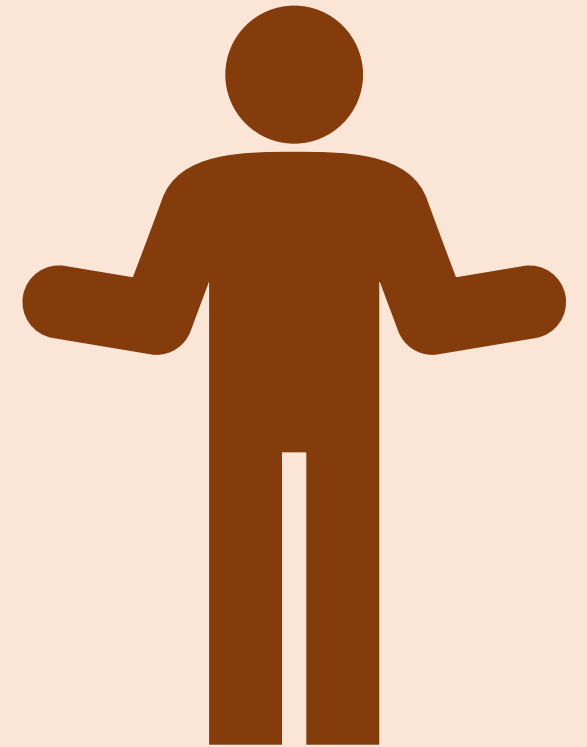
Outlook

- The CMD-3 2020 dataset was extremely precise so maybe it is sensitive to the QED corrections or other isospin-breaking effects.



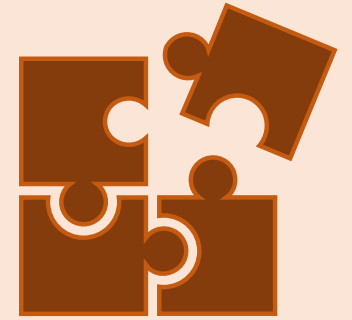
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Outlook

- Why does this matter at all?
- Pion form factor is the playground for understanding more complicated form factors.
- It is the dominant contribution to the vacuum polarization and can help reducing theory uncertainty on muon $g-2$.
- It can help constraint rare decays like B decays.





Thank you for your attention