Perturbative Theory Uncertainties from Theory Nuisance Parameters.

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[FT; arXiv:2411.18606]

[Thomas Cridge, Giulia Marinelli, FT; arXiv:2506.13874]

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Outline

Meaningful Theory Uncertainties.

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Comparing a measured quantity to its theory prediction to extract POI $\boldsymbol{y}$

$$\Big[f \pm \Delta f\Big]_{\text{measured}} \overset{!}{=} \Big[f(y) \pm \Delta f\Big]_{\text{theory}} \quad \Rightarrow \quad y \pm \Delta y^{\text{exp}} \pm \Delta y^{\text{th}}$$

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Theory uncertainty must be as meaningful as experimental one, so it should

- Reflect our level of knowledge
  - Quantify the intrinsic precision of the current prediction
  - It is *not* the distance to the true (or higher-order) result $\hat{f}$

$$\Delta f \neq |f - \hat{f}|$$

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- Have some form of statistical interpretation
  - We want to have some probability or level of confidence that

$$\hat{f} \in [f - \Delta f, f + \Delta f]$$

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- Have correct correlations

Theory Correlations.

Correlations can be crucial once several predictions are used in combination

- Prototype example: Ratio of two quantities f and g

$$\underbrace{f}_{\text{wanted}} = \underbrace{\big[g \pm \Delta g\big]_{\text{measured}}}_{\text{measure precisely}} \times \underbrace{\left[\frac{f \pm \Delta f}{g \pm \Delta g}\right]_{\text{theory}}}_{\text{theory uncertainties cancel}}$$

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  - For $\Delta f/f = \Delta g/g \equiv \delta$ we have $\delta_{f/g} = \delta\sqrt{2(1-\rho)}$

a correlation ρ of	99.5%	98%	95.5%	87.5%
yields ratio unc. $\delta_{f/g} =$	0.1δ	0.2δ	0.3δ	0.5δ

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- Differential spectrum $f(x) \equiv \mathrm{d}\sigma/\mathrm{d}x$
  - Uncertainty on *shape* of $f(x)$ is encoded in *point-by-point correlations* ρ_{ij} between $\Delta f(x_i)$ and $\Delta f(x_j)$

$\Rightarrow$ Critical for interpreting differential spectra when relying on shape effects

Scale Variations.

Our Standard Estimation Method.

Consider expansion of quantity f in some small parameter α

$$f(\alpha) = f_0 + f_1\,\alpha + f_2\,\alpha^2 + f_3\,\alpha^3 + \mathcal{O}(\alpha^4)$$

- To make a prediction, we calculate first few *true values* $\hat{f}_n$

NLO: $$f(\alpha) = \hat{f}_0 + \hat{f}_1\,\alpha$$

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Can also perform the expansion in a slightly different way (aka “scheme”)

$$\tilde{\alpha}(\alpha) = \alpha\big[1 + b_0\,\alpha + b_1\,\alpha^2 + b_2\,\alpha^3 + \mathcal{O}(\alpha^4)\big]$$

$$\Rightarrow \quad \tilde{f}(\tilde{\alpha}) = \tilde{f}_0 + \tilde{f}_1\,\tilde{\alpha} + \tilde{f}_2\,\tilde{\alpha}^2 + \tilde{f}_3\,\tilde{\alpha}^3 + \mathcal{O}(\tilde{\alpha}^4)$$

- To all orders: $f(\alpha) \equiv \tilde{f}(\tilde{\alpha})$
- Finite-order prediction using $\tilde{f}(\tilde{\alpha})$ differs by higher-order terms

$$\text{NLO:} \quad \tilde{f}(\tilde{\alpha}) = \hat{\tilde{f}}_0 + \hat{\tilde{f}}_1\,\tilde{\alpha} = \hat{f}_0 + \hat{f}_1\,\alpha + b_0\,\hat{f}_1\,\alpha^2 + b_1\,\hat{f}_2\,\alpha^3 + \cdots$$

Our Standard Estimation Method.

Idea: Take the difference between schemes as uncertainty

NLO: $$\Delta f(\alpha) = b_0 \hat{f}_1\, \alpha^2 + b_1 \hat{f}_1\, \alpha^3 + \mathcal{O}(\alpha^4)$$

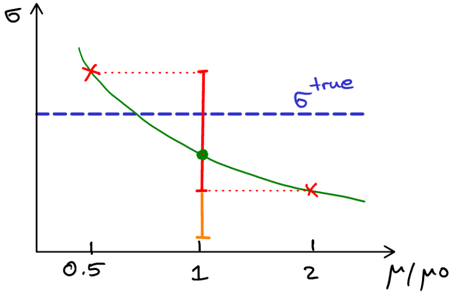
NNLO: $$\Delta f(\alpha) = \big[2 b_0 (\hat{f}_2 - b_0 \hat{f}_1) + b_1 \hat{f}_1\big] \alpha^3 + \mathcal{O}(\alpha^4)$$

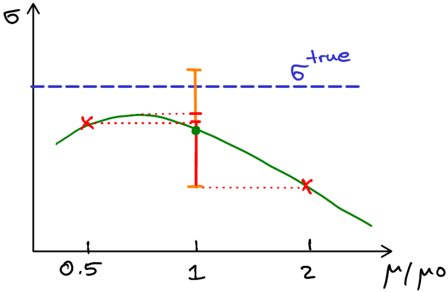
- We effectively estimate inexactness due to missing terms by approximating them by some linear combination of known lower-order terms, e.g. at NLO $f_2 \approx \hat{f}_1\, b_0$, $f_3 \approx \hat{f}_1\, b_1$

✓ Resulting estimated $\Delta f(\alpha)$ is indeed $\mathcal{O}(\alpha^{n+1})$

✗ But nothing guarantees that this is a good approximation (often it is not)
- f_{n+1} often has more complex internal structure than $f_{\leq n}$
- b_n are (rather arbitrary) numbers, have a priori nothing to do with f

Scale Variations.





$$\alpha \equiv \alpha_s(\mu_0)\,, \quad \tilde{\alpha} \equiv \alpha_s(\mu)\,, \qquad b_0 = 0.85\,L\,, \quad b_1 = 0.75\,L^2 + 0.34\,L$$

Correspond to a continuous scheme change with $L = \ln(\mu/\mu_0)/\ln 2$

- μ (or b_0) is *not* an actual parameter with a true value that f depends on
  - No value for it might ever capture the true result (happens regularly)
  - At higher order, uncertainty reduces *only because* μ (or b_0) becomes less relevant and *not because* it somehow becomes better known

What About Correlations?

$$f(\alpha) = \hat{f}_0 + \hat{f}_1\,\alpha \pm \Delta f \qquad \text{with} \qquad \Delta f = \hat{f}_1\, b_0\, \alpha^2 + \cdots$$
$$g(\alpha) = \hat{g}_0 + \hat{g}_1\,\alpha \pm \Delta g \qquad \text{with} \qquad \Delta g = g_1\, b_0\, \alpha^2 + \cdots$$

How are Δf and Δg correlated?

- We don't know – the method simply does not tell us
  - Correlations require a common uncertain parameter
    (or more generally a common source of uncertainty)
  - $b_0, b_1, \ldots$ (or μ) are not common or uncertain parameters
  - ✗ Scale variations *do not* give correct correlations or shape uncertainties
- Best we can do is *assume* some theoretically motivated but still *ad hoc* correlation model that we impose on Δf and Δg
  - Example: Correlating/uncorrelating scale variations
  - Another example: Offsetting/scanning over scale variations (→ backup)

⇒ Probably the most severe shortcoming of scale variations

Existing Alternatives to Scale Variations.

There have been precious few efforts to develop alternatives

- Bayesian models [Cacciari, Houdeau '11; Bagnaschi et al. '14; Bonvini '20; Duhr et al. '21]
- Series acceleration [David, Passarino '13]
- Using reference processes [Gosh et al. '22]
- All go in the right direction
  - ✓ Try to more directly estimate size of missing higher orders
  - ✓ Expose assumptions more explicitly
  - ✓ Try to address statistical interpretation

However

- ✗ Uncertainty estimate still based on the known lower-order terms and/or scale variations
- ✗ Do not address correlations

⇒ Unfortunately, have similar level of arbitrariness and share many limitations of scale variations

Theory Nuisance Parameters.

Parametric Theory Uncertainties.

Let's go back to our expansion of f

$$f(\alpha) = f_0 + f_1\,\alpha + f_2\,\alpha^2 + f_3\,\alpha^3 + \mathcal{O}(\alpha^4)$$

- We calculate first few true $\hat{f}_n$ to make an approximate prediction

$$\begin{aligned} \text{LO:} \quad & f(\alpha) = \hat{f}_0 \\ \text{NLO:} \quad & f(\alpha) = \hat{f}_0 + \hat{f}_1\,\alpha \\ \text{NNLO:} \quad & f(\alpha) = \hat{f}_0 + \hat{f}_1\,\alpha + \hat{f}_2\,\alpha^2 \end{aligned}$$

- Q: What is the source of uncertainty in this?
  - The missing terms $f_n\,\alpha_s^n$?
  - No, actually, it is the unknown series coefficients f_n
  - Or more precisely, their unknown true values $\hat{f}_n$

⇒ To estimate the theory uncertainty, we need to quantify our limited or lack of knowledge of f_n

Parametric Theory Uncertainties.

Consider f_n as an unknown parameter to be varied in some way

- To evaluate associated uncertainty, we need to propagate this variation into the prediction
- For that, f_n actually has to appear in the prediction, so we include it

$$\text{N}^{1+1}\text{LO:} \qquad f(\alpha, f_2) = \hat{f}_0 + \hat{f}_1\,\alpha + f_2\,\alpha^2$$

$$\text{N}^{1+2}\text{LO:} \qquad f(\alpha, f_2, f_3) = \hat{f}_0 + \hat{f}_1\,\alpha + f_2\,\alpha^2 + f_3\,\alpha^3$$

$$\text{N}^{2+1}\text{LO:} \qquad f(\alpha, f_3) = \hat{f}_0 + \hat{f}_1\,\alpha + \hat{f}_2\,\alpha^2 + f_3\,\alpha^3$$

- And that's basically it!
  - We explicitly include the (leading) sources of uncertainty in our prediction
  - f_n are well-defined parameters of our prediction with a true but unknown (or imprecisely known) value
  - We simply treat them as such

Theory Nuisance Parameters (TNPs).

In reality, $f_n \equiv f_n(x)$ has (lots of) nontrivial internal structure "x"

- Parameterize it in terms of TNPs: $f_n(x, \theta_n)$ where $\theta_n \equiv \{\theta_n^i\}$

N^{1+1}LO: $f(\alpha, x, \theta_2) = \hat{f}_0(x) + \hat{f}_1(x)\alpha + f_2(x, \theta_2)\alpha^2$

N^{1+2}LO: $f(\alpha, x, \theta_{2,3}) = \hat{f}_0(x) + \hat{f}_1(x)\alpha + f_2(x, \theta_2)\alpha^2 + f_3(x, \theta_3)\alpha^3$

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- Key condition: There must be true values $\hat{\theta}_n$ such that

$$\hat{f}_n(x) = f_n(x, \hat{\theta}_n)$$

  - This makes TNPs well-defined parameters with true but unknown value
  - Theory uncertainties become truly *parametric* with all the implied benefits

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⇒ Step 1: Derive an appropriate TNP parameterization $f_n(x, \theta_n)$
and implement it in all the predictions

Step 1: TNP Parameterization.

$f_n(x)$ are functions of various x dependences

- Discrete: partonic channels, quantum numbers, ...
- Continuous but discrete values: n_f, N_c, $E_{\rm cm}$, ...
- Fully continuous: Kinematic variables (p_T^Z, Y, q^2), particle masses, ...

- Q: Which dependences do we need to account for and parameterize?
  - All those in which we need correlations
  - Plus those that help to obtain better theory constraints

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To satisfy key condition $f_n(x, \theta_n)$ must encode correct x dependence

$$\hat{f}_n(x) = f_n(x, \hat{\theta}_n)$$

- Requires expert knowledge on underlying structure (functional form in x)
  - Deriving $f_n(x, \theta_n)$ means deriving correct theory correlation structure in x

✗ This is where scale variations, Pythia variations, etc. fail:
  - Do not yield a correct parameterization, cannot give correct correlations

Step 1: Parameterization Strategies.

General strategy

⇒ Break down internal structure until remaining unknowns $f_{n,i}$ are numbers

Specific strategy depends on what we know about functional form of $f_n(x)$

1. We know it well enough to parameterize it explicitly

   For example:

   - Dependence on partonic channels is always known exactly
   - We might know from theory that $f_n(x)$ is a polynomial in $\ln x$

$$\Rightarrow \qquad f_n(x) = \sum_{i=0}^{k} f_{n,i} \ln^i x$$

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1. We know it well enough to parameterize it explicitly
2. We know it well enough to apply 1) in some limit and can expand around that limit

$$f_n(x) = f_{n0}(x) + f_{n1}(x)\,\varepsilon + f_{n2}(x)\,\varepsilon^2 + \mathcal{O}(\varepsilon^3)$$

For example:

- Drell-Yan p_T spectrum at small p_T: $\varepsilon = p_T^2/Q^2$
- $f_{n0}(p_T)$ can be obtained from p_T resummation with strategy 1)

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Specific strategy depends on what we know about functional form of $f_n(x)$

1. We know it well enough to parameterize it explicitly
2. We know it well enough to apply 1) in some limit
3. If we don't have sufficient knowledge for either strategy 1 or 2, we can always expand in a suitable, complete functional basis $\{\phi_i\}$

$$f_n(x) = \sum_{i=0}^{\infty} f_{n,i}\, \phi_i(x) \quad \to \quad f_n(x, \theta_n) = \sum_{i=0}^{k} \theta_{n,i}\, \phi_i(x)$$

- A good basis is one that we can truncate after a few terms with high confidence that we can neglect truncated terms as a subleading source of uncertainty
- There are various ways for designing good bases

Constraining the TNPs.

Since θ_n are proper parameters, they can have proper estimates/constraints

$$\theta_n = u_n \pm \Delta u_n$$

- Can come from
  - From theory: Can be modelled as “imagined” auxiliary measurement
  - From experiment: Real auxiliary or nominal measurement

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- Could profile them as a free nuisance parameter in fit to data
  - No dependence on theory prejudice on how much to vary/assumptions
  - If the data is sensitive to a θ_n, it will constrain it, otherwise it doesn’t matter

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  - No dependence on theory prejudice on how much to vary/assumptions
  - If the data is sensitive to a θ_n, it will constrain it, otherwise it doesn’t matter
- Nevertheless, still worthwhile/useful to have a theory constraint
  - Leaving them completely free in a fit may not always be possible or practical
  - Important to know their expected “natural” size to check data constraints
  - We do like to know the uncertainty of a prediction without having to fit to data

$\Rightarrow$ Step 2: Obtain suitable constraints on θ_n

Step 2: Theory Constraints for Scalar Series.

Assume everything has been broken down to scalar series (in QCD)

$$f(\alpha_s) = 1 + \sum_{n=1} f_n \Big(\frac{\alpha_s}{4\pi}\Big)^n$$

- Parameterize coefficients f_n (numbers) as

$$f_n(\theta_n) = N_n \, \theta_n$$

  - Normalization N_n from expected "natural size" of f_n: $|\hat{f}_n| \lesssim N_n$
  - Expect θ_n to have $\mathcal{O}(1)$ natural size: $|\hat{\theta}_n| \lesssim 1$

- Impose a theory constraint

$$\theta_n = u_n \pm \Delta u_n = 0 \pm 1$$

  - Default assumption: Model u_n as normal distributed random variable

$\Rightarrow$ Defines a specific "theory estimator": Validate it on known pert. series

Step 2: Estimating Natural Size.

Let's take as an example: $q\bar{q} \to V$, $q\bar{q} \to H$, $gg \to H$ form factors

$f(\alpha_s)$	N_n	$\hat{f}_1/N_1$	$\hat{f}_2/N_2$	$\hat{f}_3/N_3$	$\hat{f}_4/N_4$
$c_{q\bar{q}V}(\alpha_s)$	1	-8.47	-48.6	-1387	-42015
$c_{q\bar{q}S}(\alpha_s)$	1	-0.47	$+87.1$	$+2309$	$+76100$
$c_{gg}(\alpha_s)$	1	$+4.93$	-24.0	-4066	-123979

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	$4^n C_F C_A^{n-1}$	-1.59	-0.76	-1.81	-4.56
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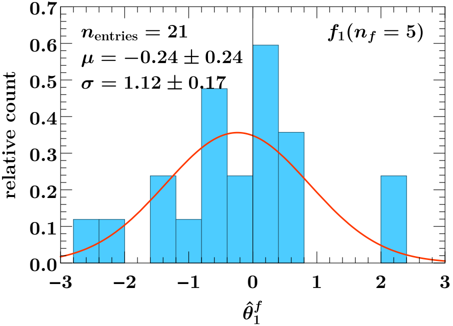
	$4^n C_A C_A^{n-1}$	$+0.41$	-0.17	-2.35	-5.98
	$4^n C_A C_A^{n-1}(n-1)!$	$+0.41$	-0.17	-1.18	-1.00

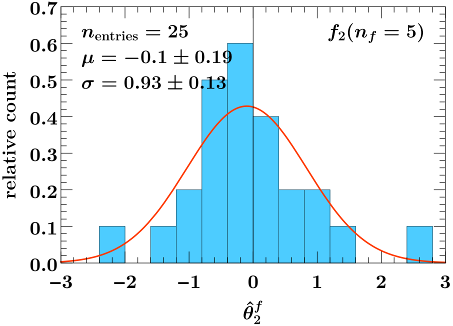
$\Rightarrow$ Let's pick: $N_n = 4C_r(4C_A)^{n-1}(n-1)!$

Step 2: Validation on Known Series.

1 loop: $\hat{\theta}_1^f$



2 loop: $\hat{\theta}_2^f$

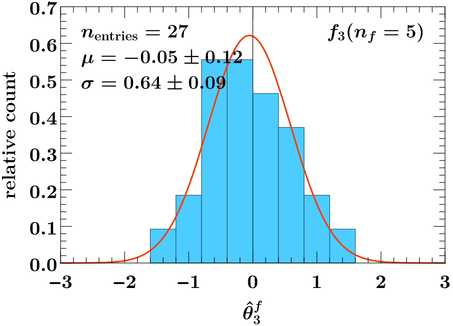


Consider set of QCD series $F = \{f\}$ of a common type/category

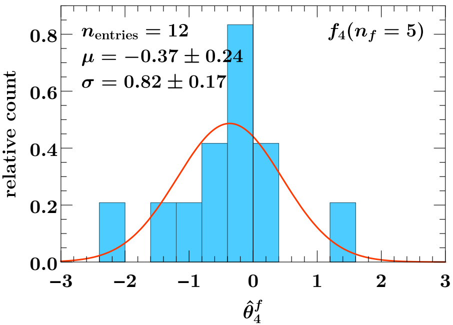
- Identify $P_{\theta_n^f}(x)$ with *population distribution* of $\hat{\theta}_n^f \in F_n$ (assumption)
  - For any given θ_n^f, its unknown $\hat{\theta}_n^f$ comes from a QCD bag of coefficients
- Estimate population distribution from a sample of known series
  - Good fit to normal distribution ("central-limit theorem of Feynman diagrams")
  - Note: Distribution is a property of this specific estimator

Step 2: Validation on Known Series.

3 loop: $\hat{\theta}_3^f$



4 loop: $\hat{\theta}_4^f$



Consider set of QCD series $F = \{f\}$ of a common type/category

- Identify $P_{\theta_n^f}(x)$ with *population distribution* of $\hat{\theta}_n^f \in F_n$ (assumption)
  - For any given θ_n^f, its unknown $\hat{\theta}_n^f$ comes from a QCD bag of coefficients
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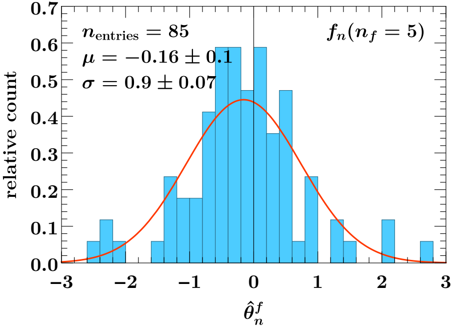
  - Note: Distribution is a property of this specific estimator

Step 2: Validation on Known Series.

1-4 loop: $\hat{\theta}_n^f$



Consider set of QCD series $F = \{f\}$ of a common type/category

- Identify $P_{\theta_n^f}(x)$ with *population distribution* of $\hat{\theta}_n^f \in F_n$ (assumption)
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Application to Drell-Yan p_T Spectrum.

TNP Parameterization for V p_T Spectrum.

Consider dependence on $x \equiv q_T = p_T^V$ (where $V = Z/\gamma, W$)

1) Apply strategy 2: Expand in $\varepsilon = q_T^2/Q^2$ (where $Q \equiv m_{\ell\ell}$)

$$\frac{d\sigma}{dq_T} = \frac{d\sigma^{(0)}}{dq_T} \times \left[1 + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)\right]$$

- $\mathcal{O}(q_T^2/Q^2)$ corrections stay below $\lesssim 5\%$ up to $q_T \lesssim Q/3 \ldots Q/2$

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2) Apply strategy 1 to $\mathrm{d}\sigma^{(0)}/\mathrm{d}q_T$

$$\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}q_T} = \left[\sum_{a,b} H_{ab} \times B_a \otimes B_b \otimes S\right](L = \ln q_T/Q)$$

$$F(\alpha_s, L) = F(\alpha_s) \exp \int_0^L \mathrm{d}L' \left\{\Gamma_{\mathrm{cusp}}[\alpha_s(L')]\, L' + \gamma_F[\alpha_s(L')]\right\}$$

- q_T dependence is predicted by resummation in terms of several independent (scalar) series
- Boundary conditions and anomalous dimensions of RGE for each function

TNPs for p_T Spectrum: After the Dust has Settled.

Brake things down to independent perturbative series, e.g. at N^{2+1}LL

- 5 scalar series (plus a few more we can neglect here for simplicity)

$$\Gamma(\alpha_s) = \alpha_s\,\hat\Gamma_0 + \alpha_s^2\,\hat\Gamma_1 + \alpha_s^3\,\hat\Gamma_2 + \alpha_s^4\,\Gamma_3(\theta_3^\Gamma)$$

$$\gamma_\mu(\alpha_s) = \alpha_s\,\hat\gamma_{\mu 0} + \alpha_s^2\,\hat\gamma_{\mu 1} + \alpha_s^3\,\gamma_{\mu 2}(\theta_2^{\gamma_\mu})$$

$$\gamma_\nu(\alpha_s) = \alpha_s\,\hat\gamma_{\nu 0} + \alpha_s^2\,\hat\gamma_{\nu 1} + \alpha_s^3\,\gamma_{\nu 2}(\theta_2^{\gamma_\nu})$$

$$H(\alpha_s) = \Big|\hat c_0 + \alpha_s\,\hat c_1 + \alpha_s^2\,c_2(\theta_2^H)\Big|^2$$

$$\tilde S(\alpha_s) = \Big[\hat{\tilde S}_0 + \alpha_s\,\hat{\tilde S}_1 + \alpha_s^2\,\tilde S_2(\theta_2^S)\Big]^2$$

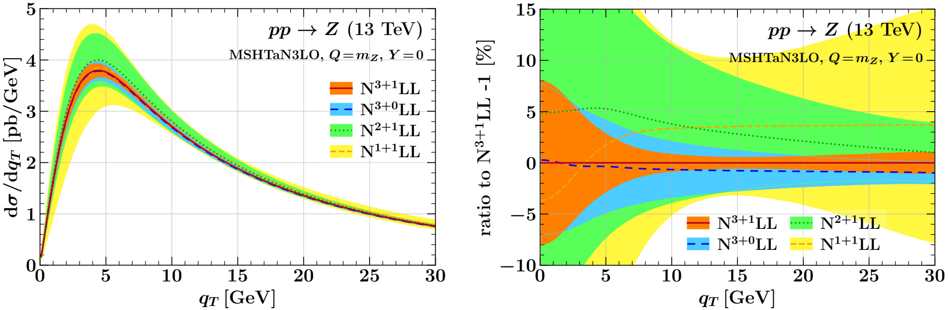
- Up to 5 one-dimensional functional series for beam functions (plus several more for DGLAP splitting functions)

$$\tilde b_i(x, \alpha_s) = \sum_j \int \frac{\mathrm{d}z}{z}\Big[\hat I_{ij,0}(z) + \hat I_{ij,1}(z) + I_{ij,2}(z, \theta_2^{B_{ij}})\Big]\, f_j\Big(\frac{x}{z}\Big)\,,$$

  - Currently use known functional form: $I_{ij,n}(z, \theta_n^{B_{ij}}) = \frac{3}{2}\,\theta_n^{B_{ij}}\,\hat I_{ij,n}(z)$
  - In the future use strategy 2 to parameterize z dependence

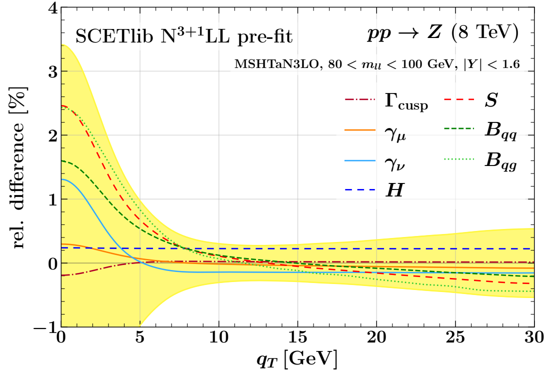
Results for Drell-Yan p_T Spectrum.

Comparing different orders at 95% “theory CL” ($\boldsymbol{\Delta\theta_n = 2}$)



- Uncertainties reduce as we go to higher order (by construction)

Uncertainty Breakdown.

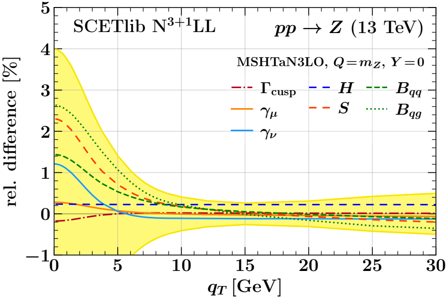


Separately varying each TNP by $\pm\Delta\theta_n = 1$

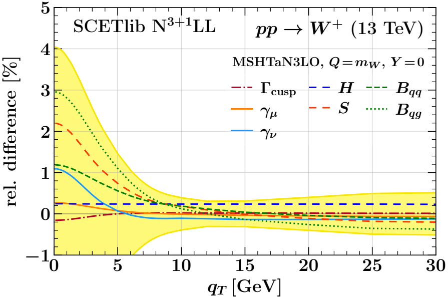
- TNPs provide breakdown into independent uncertainty sources with correct shape
  - Encodes correct point-by-point correlations
  - Importantly, carries over to p_T^ℓ and other decay kinematics

Correlations between W and Z.

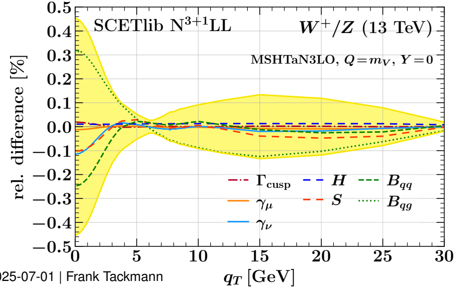
relative impacts for Z



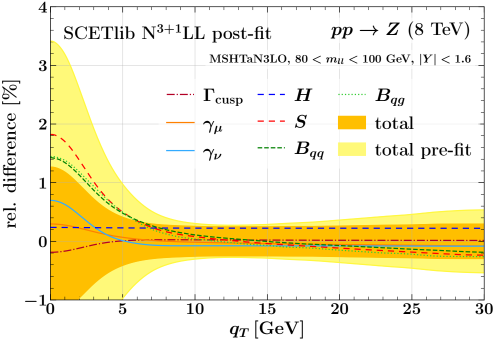
relative impacts for W

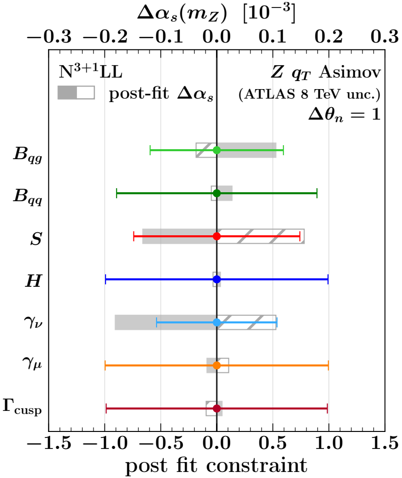


relative impacts on W/Z



- Caveats apply: These are only the (formerly) leading perturbative unc.
- Subleading effects can become important (or even dominant) now
  - Quark mass effects
  - EW corrections
  - Power corrections

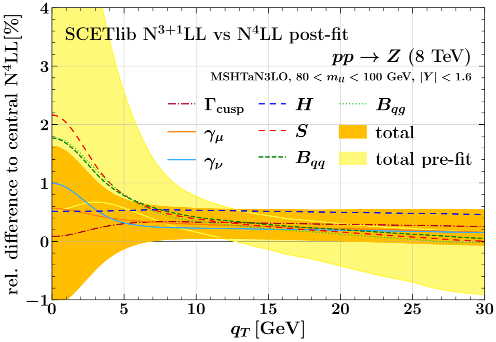


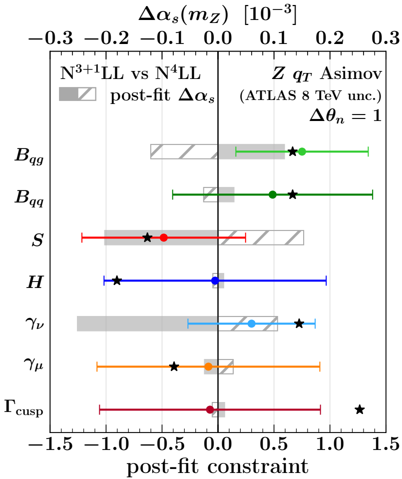


Fit to pseudodata generated from N^{3+1}LL

- Include prior Gaussian theory constraint $\theta_n = 0 \pm 1$
- Data provides nontrivial constraints on TNPs
  - Post-fit prediction has reduced theory uncertainties
  - Induces nontrivial post-fit correlations

Profile against True Higher Order.

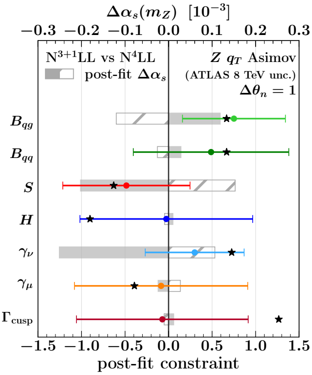




Fit to pseudodata generated from true N^{4}LL

- Simulates fit to real data (which contains all-order result)
  - TNPs are pulled toward their true values
  - Post-fit prediction gets corrected toward true result

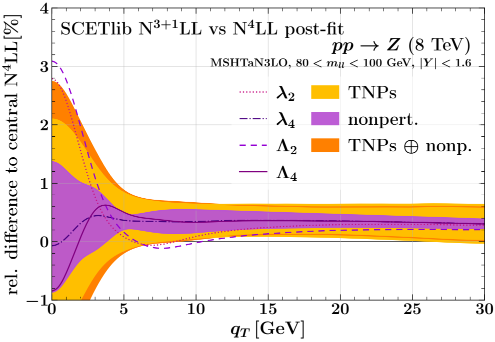
Relaxing the Prior Theory Constraint.

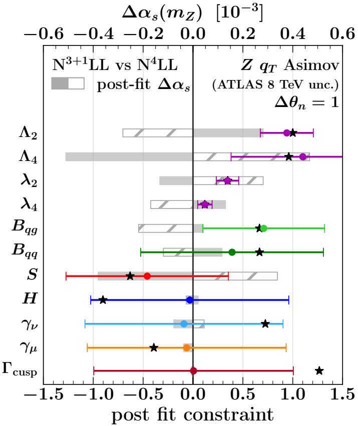


Data is able to sufficiently constrain TNPs by itself

- Reduces dependence on prior theory constraint (and associated potential bias)
  - Post-fit constraints on TNPs become even more consistent with true values
- Uncertainty on final result almost unchanged

Bonus: Including Nonperturbative Effects.





For $1/b_T \sim q_T \gg \Lambda_{\rm QCD}$ nonpert. effects can be systematically expanded in an OPE

$$\tilde{f}_i(x, b_T, \mu, Q) = \tilde{f}_i^{(0)}(x, b_T, \mu, Q)\left\{1 + b_T^2\left[\Lambda_{2,i}(x) + \lambda_2 \ln\frac{b_T Q}{b_0}\right] + \mathcal{O}(\Lambda_{\rm QCD}^4 b_T^4)\right\}$$

- Also include quadratic and quartic OPE coefficients in the fit
  - TNPs still pulled toward their true values but less constrained

Summary.

Interpretation of precision measurements requires *meaningful* theory uncertainties which includes in particular proper theory correlations

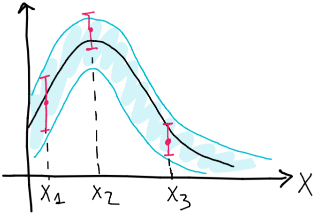
Scale variations become insufficient once theory unc. $\sim$ experimental unc

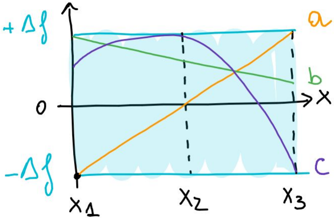
- Neither particularly reliable nor can they do correlations

Theory nuisance parameters

- Provide truly parametric theory uncertainties that
  - ✓ Encode correct correlations
  - ✓ Can be consistently propagated everywhere (fits, MCs, neural networks, ...)
  - ✓ Can be consistently profiled and constrained by data
- Bonus: Can fully benefit from all known partial higher-order information
- First successful applications to resummed Drell-Yan p_T spectrum
  - Implemented in SCETlib (available upon request, hopefully fully public soon)
  - Precision W-mass measurement by CMS [→ see Kenneth's talk]
- First promising applications to PDF fits and fixed-order predictions [MSHT20aN3LO (Gowan et al.) '22; Poncelet, Lim '24]

Additional Slides





	ρ_{12}	ρ_{13}	ρ_{23}
a	0	−1	0
b	+1	+1	+1
c	+1	−1	−1

Repeat fit for each individual scale variation and take envelope of results

- Amounts to trying out various correlation models for the same total uncertainty band
  - None of the trial variations provides a realistic correlation model
  - Individual variations are not meaningful (which is why we take their envelope)
- Best we can do with scale variations
  - Perform as many variations as we can to “fill out” the band, hoping to include at least one that happens to give sufficiently conservative estimate
  - And/or identify conceptually “independent” subsets of variations and add their envelopes

TNPs for Drell-Yan p_T Spectrum: More Details.

Apply strategy 2 with $\varepsilon = q_T^2/Q^2$ ($Q \equiv \sqrt{m_{\ell\ell}}$)

$$\frac{\mathrm{d}\sigma}{\mathrm{d}^4 q} = \frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}^4 q} \times \left[1 + \mathcal{O}\left(\frac{q_T^2}{Q^2}\right)\right]$$

- Power corrections stay below $\lesssim 5\%$ up to $q_T \lesssim Q/3 \ldots Q/2$
- Leading-power term is subject of q_T factorization and resummation

$$\frac{\mathrm{d}\sigma^{(0)}}{\mathrm{d}^4 q} = \frac{1}{2E_{\mathrm{cm}}^2} L_{VV'}(q^2) \sum_{a,b} H_{VV'\,ab}(q^2, \mu)$$

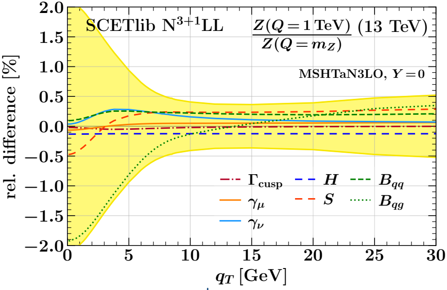
$$\times \int \frac{\mathrm{d}^2 \vec{b}_T}{(2\pi)^2} e^{\mathrm{i}\vec{b}_T \cdot \vec{q}_T}\, \widetilde{B}_a(x_a, b_T, \mu, \nu/Q)\, \widetilde{B}_b(x_b, b_T, \mu, \nu/Q) \tilde{S}(b_T, \mu, \nu)$$

$$x_{a,b} = \frac{Q}{E_{\mathrm{cm}}} e^{\pm Y}$$

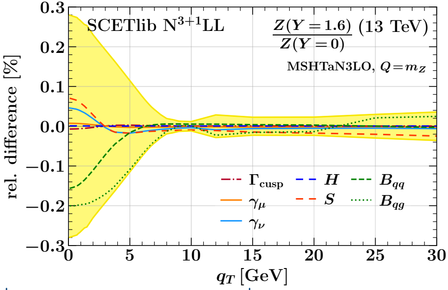
- Here $VV' = \{\gamma\gamma, \gamma Z, Z\gamma, ZZ, W^+W^+, W^-W^-\}$
- Factorization allows us to apply strategy 1 to $q_T(b_T)$, q^2, V
- Also allows us to factorize $x_{a,b}$ dependence and apply strategy 2 to it

Results for Drell-Yan p_T Spectrum: Other Ratios.

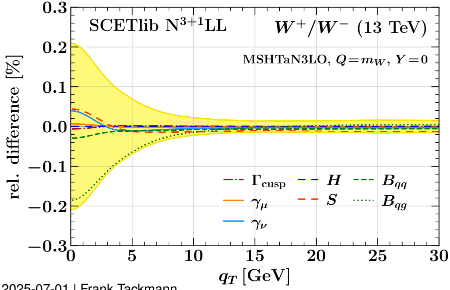
$Z(Q = 1\,\text{TeV})/Z(Q = m_Z)$



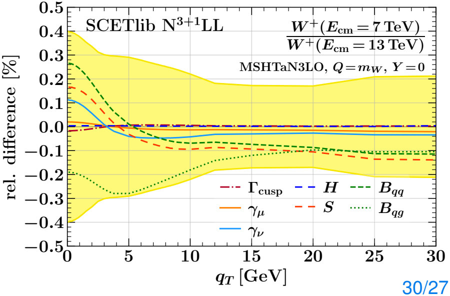
$Z(Y = 1.6)/Z(Y = 0)$



W^+/W^-



$W^+(E_{\text{cm}} = 7\,\text{TeV})/W^+(E_{\text{cm}} = 13\,\text{TeV})$



Asimov Fit Setup.

[For details see Cridge, Marinelli, FT; arXiv:2506.13874]

We perform Asimov fits to (unfluctuated) pseudodata

- Standard method to study expected uncertainties in a controlled setting
  - Unobscured by statistical fluctuations and subleading effects
- Goals: Demonstrate TNPs and estimate expected uncertainties in $\alpha_s(m_Z)$
  - Can consistently drop subleading effects in both pseudodata and theory model (power corrections, quark mass effects, EW corrections)
  - They are needed to fit the real data, but are irrelevant for estimating the dominant uncertainties

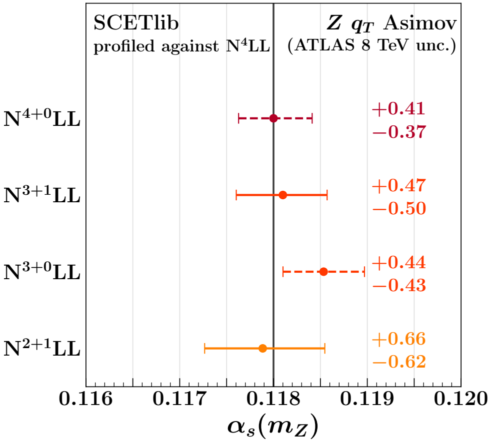
Pseudodata

- Central value given by central SCETlib prediction with $\alpha_s(m_Z) = 0.118$
- Exp. uncertainties and correlations from ATLAS 8 TeV inclusive $Z\ p_T$ measurement [Eur. Phys. J. C 84 (2024) 315 [arXiv: 2309.09318]]
- Same bins and cuts as used by ATLAS $\alpha_s(m_Z)$ determination [arXiv:2309.12986]

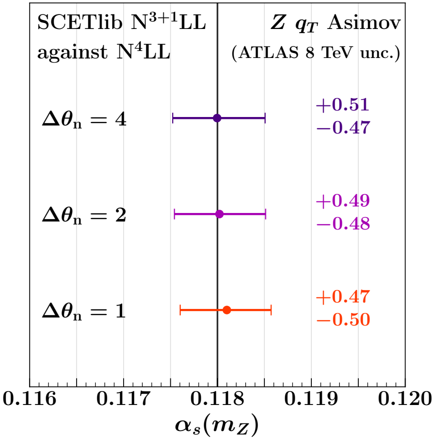
Results for α_s with Profiling TNPs.

[For details see Cridge, Marinelli, FT; arXiv:2506.13874]

different pert. orders



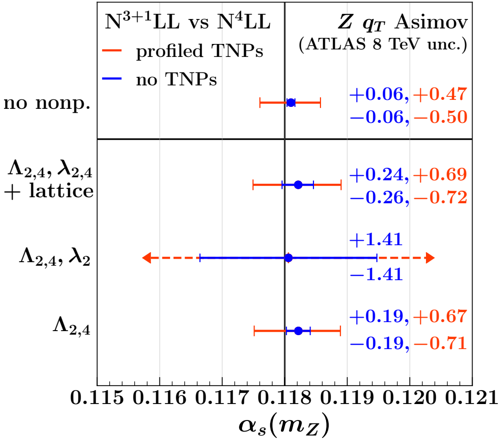
relaxing theory constraint



- Important: Not all sources of theory uncertainties included yet

α_s Results with TNPs and Nonpert. Parameters.

[For details see Cridge, Marinelli, FT; arXiv:2506.13874]



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European Research Council

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