Is the pion the hologram of a vibrating string?

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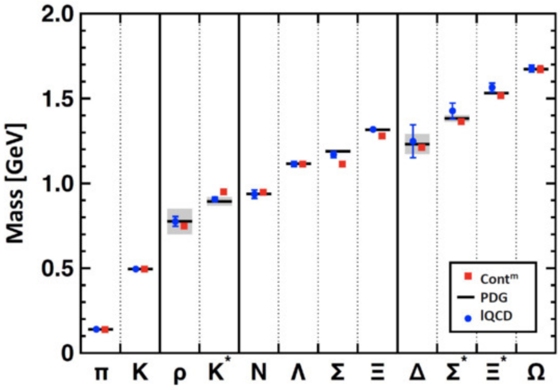
Seminar at IP2I Lyon - Pôle "Théorie des 2 Infinis"

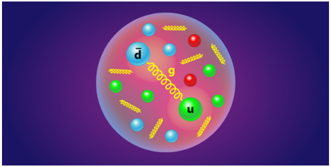
based on: PRD 111, 034024 (2025), arXiv:2501.00526 in collaboration with Jeff Forshaw (Manchester)

The lightest hadron in Nature



Craig Roberts, Symmetry 2020, 12(9), 1468

QCD bound state and pseudo-Goldstone boson



https://alanstonebraker.com

- Gell-Mann-Oakes-Renner (GMOR)

$$f_\pi^2 M_\pi^2 = 2m_q |\langle \bar{q} q \rangle| + \mathcal{O}(m_q^2)$$

  - ⋆ f_π: pion decay constant, M_π: pion mass
  - ⋆ $\langle q\bar{q} \rangle$: quark condensate, m_q: light quark mass

M. Gell-Mann, R. J. Oakes, and B. Renner, Phys. Rev. 175, 2195 (1968)

Pion data

Precise data on non-perturbative observables

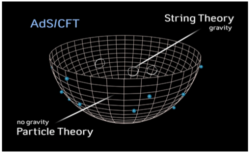
- mass, decay constant, charge radius, radiative decay width

	M_π [MeV]	f_π [MeV]	r_π [fm]	$\Gamma_{\pi\to\gamma\gamma}$ [eV]
PDG	139	130.2 ± 1.7	0.659 ± 0.004	7.82 ± 0.23

- EM form factor data at low Q^2 (CERN, 1986)

Holographic duality

't Hooft, Susskind, Maldacena (1990s)



https://rationalisingtheuniverse.org/2018/11/27/ads-cft-a-deep-duality/

- String theory in higher dimensional anti de Sitter (curved) spacetime has same information content as conformal QFT at the (flat) spacetime boundary
- Weak-Strong coupling duality
- Gravity dual for QCD (not conformal) unknown

QCD has an underlying conformal symmetry

- In the chiral limit and at tree level, the QCD Lagrangian

$$\mathcal{L}_{QCD} = \bar{\Psi}(i\gamma^{\mu} D_{\mu} - \not{m_q})\Psi - \frac{1}{4} G^{a}_{\mu\nu} G^{a\mu\nu}$$

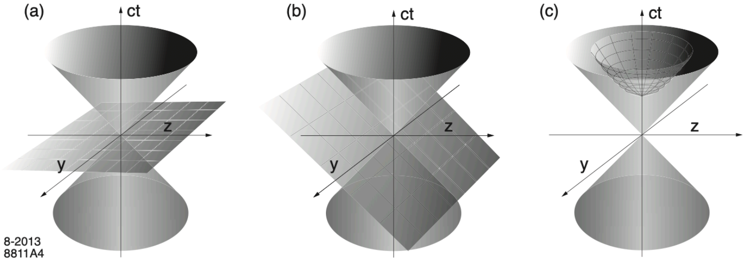
then has no mass scale and QCD action

$$S_{\text{QCD}} = \int \mathrm{d}^4 x \mathcal{L}_{\text{QCD}}$$

is conformally invariant

- Quantum loops $\rightarrow$ Anomalous conformal symmetry breaking: Λ_{QCD}

Light-front quantization



- Light-front time: $x^+ = x^0 + x^3$
- Light-front energy: $x^- = x^0 - x^3$
- Transverse coordinates: (x^1, x^2)

P.M.A Dirac, Rev. Mod. Phys. 21 (1949) 392-399

Bound states in LFQCD

S. J. Brodsky, H-C Pauli and S. Pinsky, Phys. Rept. 301 (1998) 299-486

LF Heisenberg Equation

$$H_{\mathrm{LF}}^{\mathrm{QCD}}|\Psi\rangle = M^2|\Psi\rangle$$

↓ Fock expansion

$$|\Psi\rangle = \sum_n |n\rangle\langle n|\Psi\rangle$$

↓ Hide higher Fock states

$$H_{\mathrm{eff}}|q\bar{q}\rangle = M^2|q\bar{q}\rangle$$

↓ Project in momentum space

LF Schrödinger Equation

$$\left(\frac{\mathbf{k}^2+m_q^2}{x(1-x)} + U_{\mathrm{eff}}\right)\Psi(x,\mathbf{k}_\perp) = M^2\Psi(x,\mathbf{k}_\perp)$$

Separation of variables: transverse and longitudinal

$$\Psi(x, \mathbf{b}) = \int \frac{\mathrm{d}^2\mathbf{k}}{(2\pi)^2} \Psi(x, \mathbf{k})$$

- $x = k^+/P^+$ is the light-front momentum fraction of quark
- $\mathbf{b}$ is transverse distance between quark and antiquark
- Normalization:

$$\int \mathrm{d}^2\mathbf{b}\mathrm{d}x |\Psi(x, \mathbf{b})|^2 = 1$$

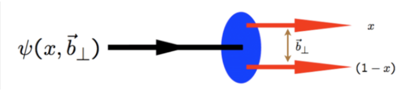


Figure: Stan Brodsky

Factorization

- Key variable

$$\zeta = \sqrt{x(1-x)}\mathbf{b}$$

- Factorization of wavefunction

$$\Psi(x,\zeta) = \frac{\phi(\zeta)}{\sqrt{2\pi\zeta}} e^{iL\varphi} X(x) \qquad M^2 = M_\perp^2 + M_\parallel^2$$

- Transverse: underlying conformal symmetry

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2} + \frac{4L^2-1}{4\zeta^2} + U_\perp\right]\phi(\zeta) = M_\perp^2 \phi(\zeta) \qquad \int_0^\infty \mathrm{d}\zeta\, |\phi(\zeta)|^2 = 1$$

- Longitudinal: captures conformal symmetry breaking

$$\left[\frac{m_q^2}{x(1-x)} + U_\parallel\right] X(x) = M_\parallel^2 X(x) \qquad \int_0^1 \frac{\mathrm{d}x}{x(1-x)} |X(x)|^2 = 1$$

Holographic dictionary

$$\left[-\frac{\mathrm{d}^2}{\mathrm{d}\zeta^2}+\frac{4L^2-1}{4\zeta^2}+U_\perp\right]\phi(\zeta)=M_\perp^2\phi(\zeta) \qquad \int_0^\infty \mathrm{d}\zeta\,|\phi(\zeta)|^2=1$$

maps onto EOM of freely propagating spin-J string modes in warped AdS_5 if

- $\zeta = z_5$
- $L^2 = (\mu_5 R)^2 + (2-J)^2$

where

$$U_\perp(\zeta)=\frac{1}{2}\varphi''(z_5)+\frac{1}{4}\varphi'(z_5)^2+\frac{2J-3}{2z_5}\varphi'(z_5)$$

with $\varphi(z_5)$ is a dilaton field in AdS_5.

Review: S. J. Brodsky, G. F. de Téramond, H. G. Dosch, J. Erlich, Phys. Rept. 584 (2015) 1-105

Uniqueness of a quadratic dilaton

- A quadratic dilaton, $\varphi = \kappa^2 z_5^2$, gives

$$U_\perp(\zeta) = \kappa^4\zeta^2 + 2\kappa^2(J-1)$$

- (Only) harmonic potential preserves the underlying conformal invariance

S. J. Brodsky, G. F. de Téramond, H. G. Dosch, Phys. Lett. B729 (2014) 3-8

Predicting a massless pion

Solving the holographic LF Schrödinger Equation

$$\left(-\frac{d^2}{d\zeta^2} - \frac{1-4L^2}{4\zeta^2} + U_\perp(\zeta)\right)\phi(\zeta) = M_\perp^2\phi(\zeta)$$

with

$$U_\perp(\zeta) = \kappa^4\zeta^2 + 2\kappa^2(J-1)$$

gives

$$M_\perp^2 = (4n+2L+2)\kappa^2 + 2\kappa^2(J-1)$$

- For pion: $n=0, L=0, J=0,\ M_\pi = M_\perp = 0$
- To generate pion mass in accordance with GMOR: longitudinal dynamics

Connecting to 't Hooft $(1+1)$-dim, large N_c QCD

$$\left[\frac{m_q^2}{x(1-x)} + U_\parallel\right] X(x) = M_\pi^2 X(x) \qquad \int_0^1 \frac{\mathrm{d}x}{x(1-x)} |X(x)|^2 = 1$$

$$\chi(x) = \frac{X(x)}{\sqrt{x(1-x)}}$$

$$\left[\frac{m_q^2}{x(1-x)} + V_\parallel\right] \chi(x) = M_\pi^2 \chi(x)$$

$$V_\parallel = \frac{1}{\sqrt{x(1-x)}} U_\parallel \sqrt{x(1-x)} \qquad \int_0^1 \mathrm{d}x\, |\chi(x)|^2 = 1$$

This is the 't Hooft Equation if

$$V_\parallel = -g^2 \mathcal{P} \int_0^1 \mathrm{d}y \frac{\chi(y) - \chi(x)}{(x-y)^2} \qquad [g] = \mathrm{mass}$$

Chabysheva & Hiller, Annals Phys. 337, 143 (2013)

Li & Vary model for $V_{\parallel}$

1. 't Hooft, Nucl. Phys. B75, 461 (1974)

$$\frac{m_q^2 - g^2}{x(1-x)}\chi(x) - g^2 \mathcal{P}\int_0^1 \mathrm{d}y \frac{\chi(y)}{(x-y)^2} = M_\pi^2 \chi(x)$$

2. Li & Vary, Phys. Lett. B 758, 118 (2016)

$$\frac{m_q^2}{x(1-x)}\chi(x) - \sigma^2 \partial_x (x(1-x)\partial_x)\chi(x) = M_\pi^2 \chi(x)$$

- Very different analytical forms
- Yet completely degenerate in describing pion data with $\chi(x) \sim (x(1-x))^\beta$, $\beta = m_q/\sigma$ or $\beta \sim m_q/g$

Ahmady et al., Li & Vary, Brodsky & de Téramond, Weller & Miller (2021-23)

An underlying link in AdS_3

- Ansatz

$$\left(\frac{m_q^2 - g^2}{x(1-x)}\right)\chi(x) - g^2\mathcal{P}\int_0^1 \mathrm{d}y\,\frac{\chi(y)}{(x-y)^2} - \sigma^2\partial_x(x(1-x)\partial_x)\chi(x) = M_\pi^2\chi(x)$$

- Holographic dictionary

$$\boxed{g^2 = m_q^2 + \sigma^2/4 \qquad g_s = g^2/(4\sigma^2) \qquad \mu^2 = M_\pi^2/(4\sigma^2)}$$

- Vegh's string equation in pure AdS_3 (2023)

$$-\mathcal{P}\int_0^1 \mathrm{d}y\,\frac{\chi(y)}{(x-y)^2} - 4g_s\sqrt{x(1-x)}\,\partial_x^2\sqrt{x(1-x)}\,\chi(x) = \mu^2\chi(x)$$

g_s is dimensionless string coupling: conformal invariance restored !

Observables

After integrating transverse d.o.f

$$M_\pi^2 = \int_0^1 \mathrm{d}x\ \chi^*(x) \left[\frac{m_q^2}{x\bar{x}} + V_\parallel \right] \chi(x)$$

$$f_\pi = \frac{\sqrt{6}}{\pi}\kappa \int_0^1 \mathrm{d}x \sqrt{x\bar{x}}\chi(x) \qquad r_\pi^2 = \frac{3}{2\kappa^2} \int_0^1 \mathrm{d}x \frac{(1-x)}{x} |\chi(x)|^2$$

$$F_\pi(Q^2) = \int_0^1 \mathrm{d}x\ |\chi(x)|^2\ \exp\left(-\frac{\bar{x}}{x} \frac{Q^2}{4\kappa^2} \right)$$

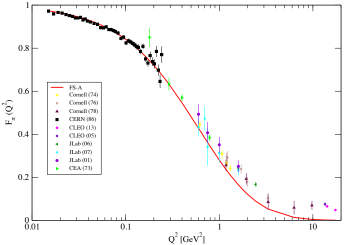
$$Q^2 F_{\pi\gamma}(Q^2) = \frac{2\kappa}{\sqrt{3}\pi} \int_0^1 \mathrm{d}x \sqrt{x\bar{x}}\chi(x)$$

$$\times \int_0^\infty \mathrm{d}b_\perp (m_q b_\perp) K_1(m_q b_\perp) \exp\left(-\frac{\kappa^2 x\bar{x} b_\perp^2}{2} \right) Q J_1(b_\perp \bar{x} Q)$$

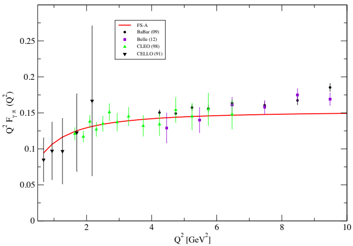
$$\Gamma_{\gamma\gamma} = \frac{\pi}{4}\alpha_{\mathrm{em}}^2 M_\pi^3 |F_{\pi\gamma}(0)|^2$$

Extracting the parameters from data

- Reproduce exactly M_π, f_π and r_π
- Predicts excellent agreement for form factors at low Q^2
- Predicts $\Gamma_{\gamma\gamma} = 7.0$ eV. PDG: 7.82 ± 0.22 eV



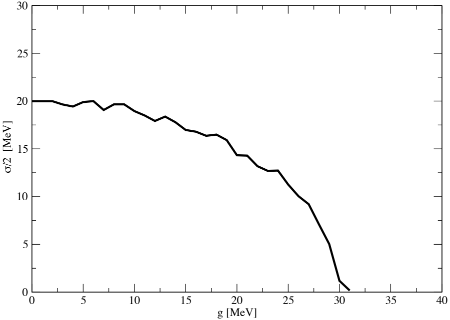
Elastic EM form factor



Transition Form Factor

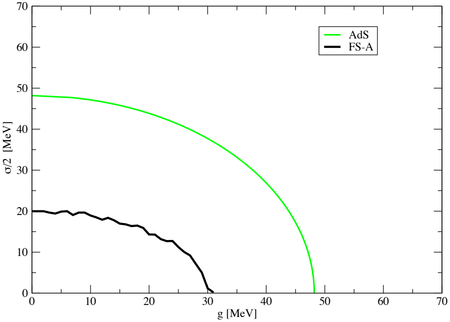
Degeneracy in parameter space

$$\kappa = 423 \text{ MeV}$$



- Degenaracy extends beyond 't Hooft and Li-Vary limits
- m_q varies slightly along curve: $m_q = [48, 60]$ MeV

Comparing to holographic prediction



- $\kappa = 423$ MeV, $m_q = 60$ MeV
- Correlation extends beyond 't Hooft and Li-Vary limits
- Green: Holographic prediction $g^2 = m_q^2 + \sigma^2/4$

Similarity transformations

- Eigenvalue invariant under transformation

$$V_{\parallel} \to (x(1-x))^{n/2} \, V_{\parallel} \, \frac{1}{(x(1-x))^{n/2}}$$

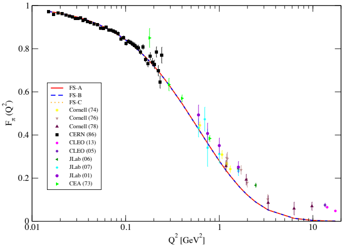
- Eigenfunction transforms as

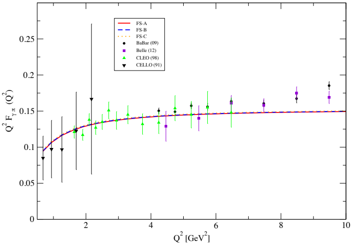
$$\chi(x) \to (x(1-x))^{n/2} \chi(x)$$

- Before: $n = 0$
- Now:
  - ⋆ $n < 0$: bad chiral limit behaviour
  - ⋆ $n > 2$: discarded by data
  - ⋆ Model A: $n = 0$, Model B: $n = 1$, Model C: $n = 2$

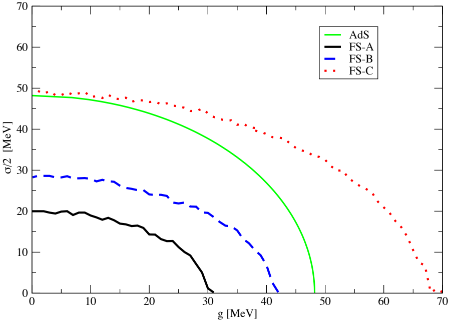
Completely degenerate in fitting data

- Reproduce exactly M_π, f_π and r_π
- Predict $\Gamma_{\gamma\gamma} = 7.0, 7.2, 7.4$ eV for Models A, B, C. PDG: 7.82 ± 0.22 eV
- Predict excellent agreement for form factors at low Q^2





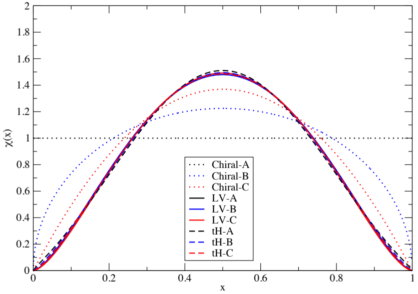
Degeneracy lifted in parameter space



- $\kappa = 423$ MeV, $m_q = [47, 60]$ MeV
- Green: Holographic prediction $g^2 = m_q^2 + \sigma^2/4$
- Model A, Model B, Model C
- Model C: quantitative agreement with holography in weak string coupling limit

Extracted longitudinal mode

- Regardless of the AdS_3 connection, the extracted longitudinal mode, $\chi(x)$, is much more peaked than previously found in literature
- Important consequences for pion Distribution Amplitude (DA) and Parton Distribution Function (PDF)



Concluding remarks

- Intriguing agreement with AdS_3 prediction for weak string coupling with Model C
- Holographic pion DA and PDF: work in progress

Acknowledgements

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