BMS particles

Black holes and their symmetries (04/07/25),

Yannick Herfray, Institut Denis Poisson Tours

Based on

X. Bekaert -- L. Donnay -- YH, 2024

X. Bekaert -- YH, 2025

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1- Introduction and Motivations

- In a recent work *(Bekaert -- Donnay -- YH, 2024)* we investigated
  BMS particles
  i.e. unitary irreducible representations (UIR) of the BMS group.
  See also (Bekaert -- YH, 2025).

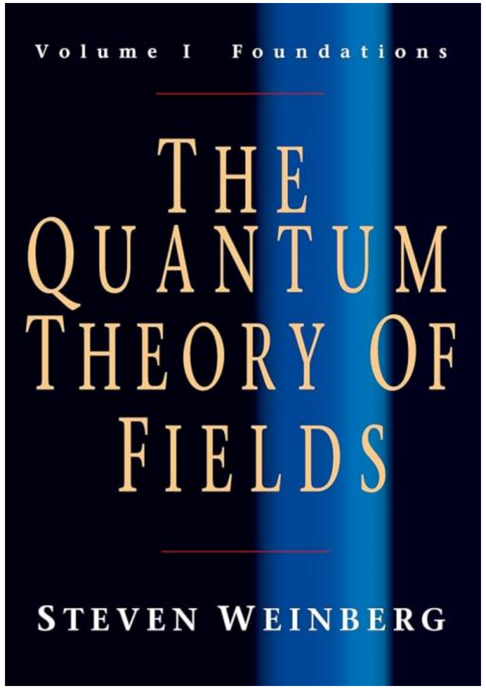
- Representations of BMS_4 group have been classified by McCarthy *(McCarthy, 1972 -- 1978)*; *with important contributions of Girardello–Parravicini (1974).*

- Concept and name are borrowed from *Barnich – Oblak (2014, 2015) and B. Oblak's thesis* 'BMS Particles in Three Dimensions'.

Why representation theory ?

Why representation theory ?

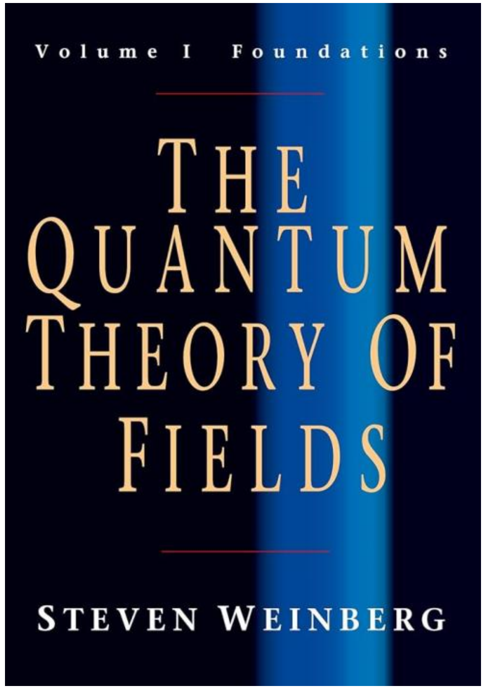
- Fundamental perspective:



UIR of Poincaré give
fundamental derivation / definition
of particles in QFT (Weinberg's QFT Vol 1).

Why representation theory ?

- Fundamental perspective: “ Hilbert space + Symmetry = UIR ”



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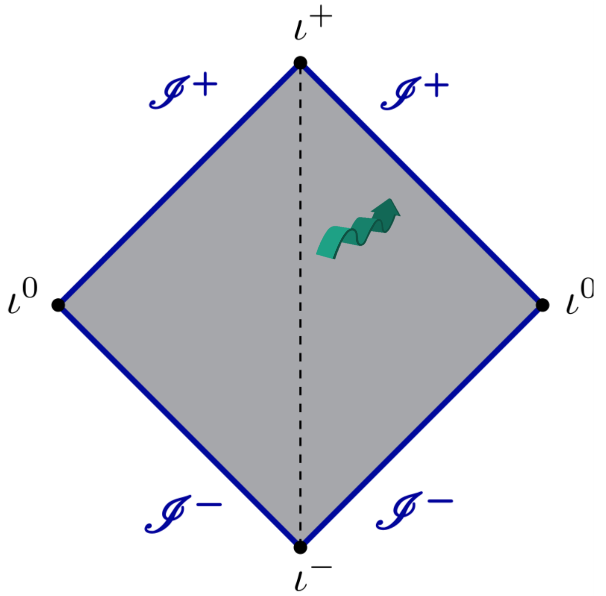
- Fundamental perspective: “ Hilbert space + Symmetry = UIR ”

- Exhaustivity: “Are higher or continuous spins particles out there?”

- Investigating dualities
  e.g. AdS/CFT
  “Fields which appear different can in fact be the same as UIR”

BMS ?

(Bondi -- Van der Burg -- Metzner -- Sachs 62)



- = the asymptotic symmetry group of asymptotically flat spacetimes

- = infinite dimensional extension of Poincaré

$$ISO(3,1) \simeq \mathbb{R}^4 \rtimes SO(3,1) \quad \Longrightarrow \quad BMS_4 \simeq \mathcal{C}^\infty(S^2) \rtimes SO(3,1)$$

Why BMS representation theory ?

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Strominger (2014) has taught us that the BMS group is the group of symmetry of gravity S-matrix (and not Poincaré group).

"Weinberg's soft theorems *Weinberg (65)*
can be understood as *Ward identities* for BMS asymptotic symmetries *Bondi--Van der Burg--Metzner-Sachs (62)*."

➡ **The gravitational S-matrix is BMS invariant** *Strominger (2014)*

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Why BMS representation theory ?

The gravitational S-matrix is BMS invariant *Strominger (2014)*

" Hilbert space + Symmetry = UIR " → States of S-matrix = BMS particles

Exhaustivity → Are all of them meaningful? What's the physics?

Investigating dualities → Dual states for candidate flat holography = BMS particles

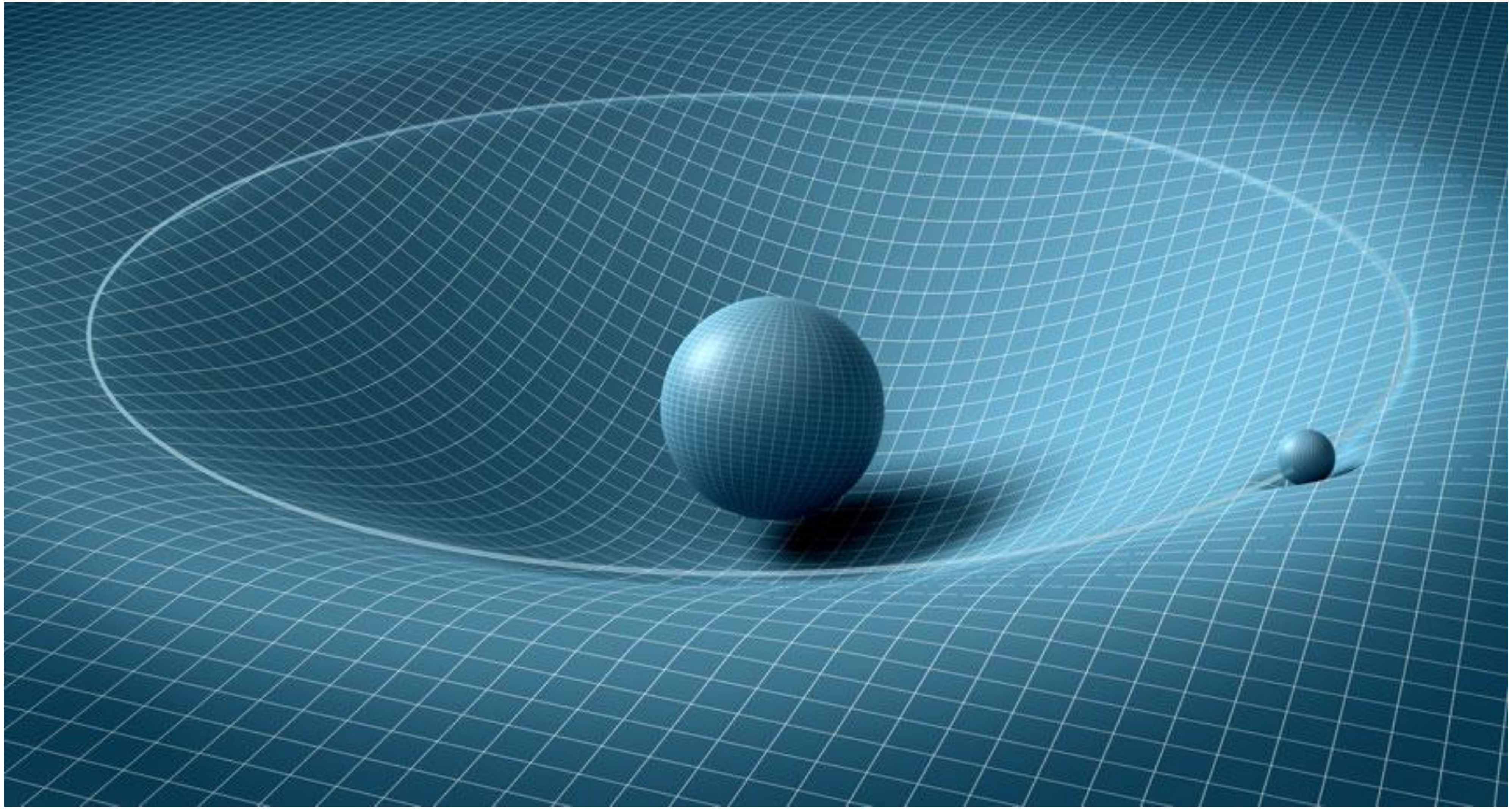
Bekaert -- Donnay -- YH 2024,   Bekaert – YH 2025

Bekaert -- Donnay -- YH 2024, *Bekaert – YH 2025*

The main objective of this project was to reconsider the classical results of *(McCarthy, 1972 – 1978)* in light of the new physical understanding which arose in the last years:

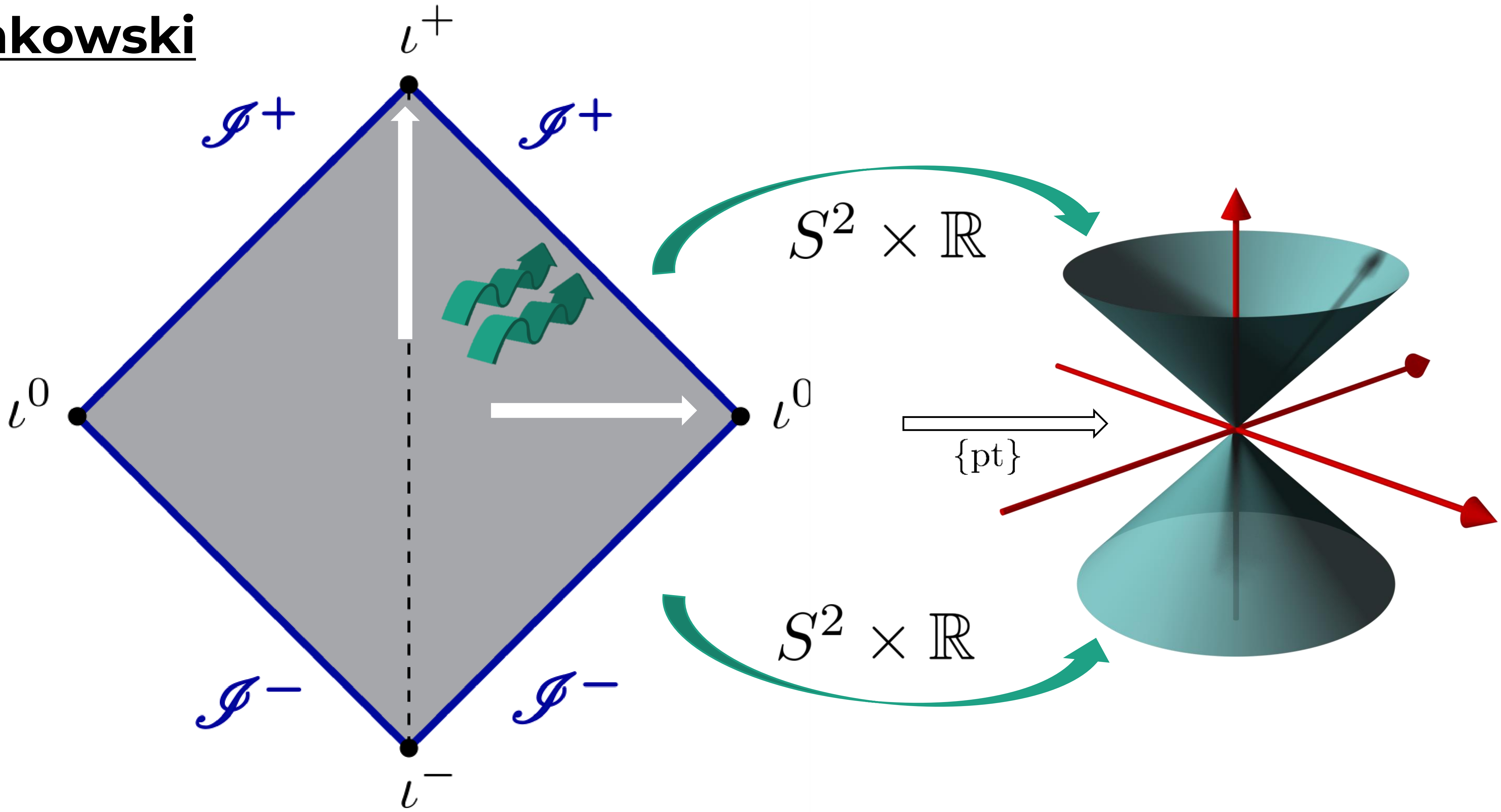
- A new decomposition of supermomenta in hard/soft parts
- Explicit realization of
  - generic BMS wavefunctions
  - branching rules under Poincaré
- Physical interpretation of BMS particles as a superposition of Poincaré particles in all possible gravity vacua.

2- Asymptotics in General Relativity



Conformal diagram

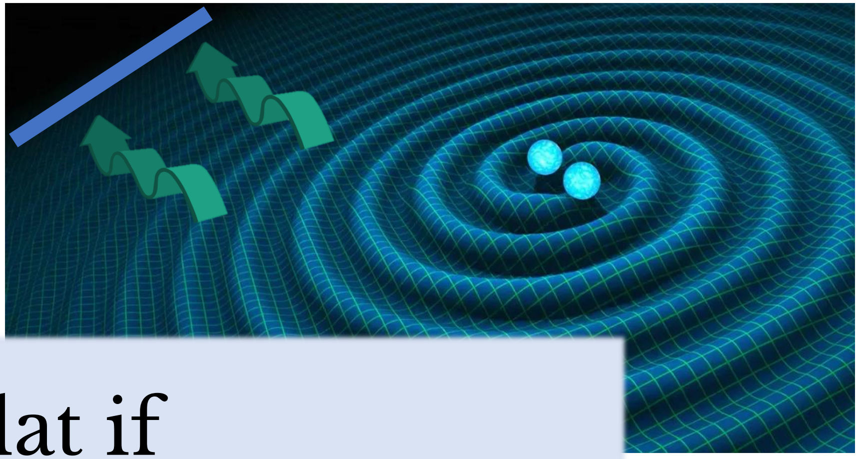
Minkowski



Definition :

Asymptotic flatness at null infinity

Penrose (1963)



A spacetime $(\tilde{M}, \tilde{g})$, is asymptotically flat if

- There exists a spacetime with boundary (M, g)
- A "boundary defining" function Ω : $\Omega|_\partial = 0, \quad d\Omega|_\partial \neq 0$
- The interior of M is isometric to $\tilde{M}$ with $\tilde{g} = \frac{1}{\Omega^2} g$
- $\tilde{g}$ is Einstein
- The normal $n^\mu = \Omega^{-2} \tilde{g}^{\mu\nu} \nabla_\nu \Omega \big|_\partial$ is null $n^2 = 0$

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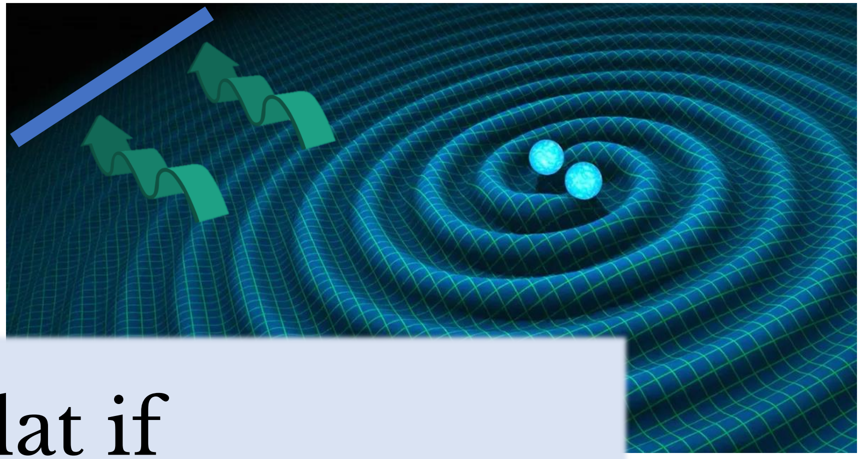

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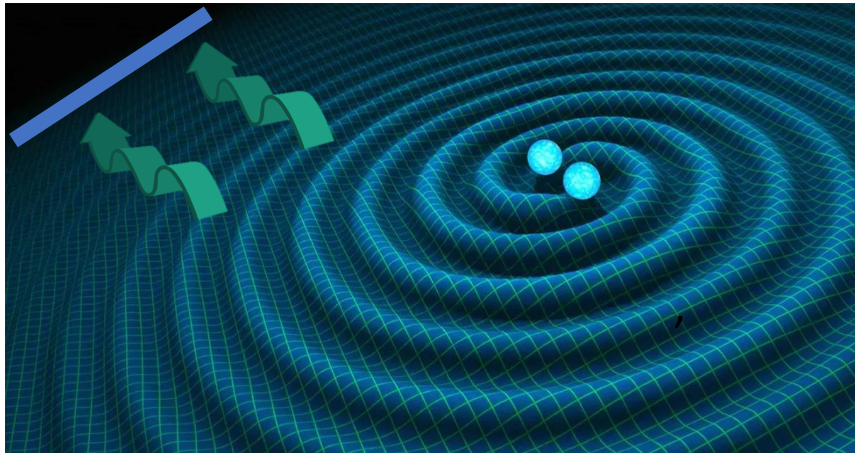
Vast class of asymptotically flat spacetimes **exists**

Friedrich (1986), Christodoulou—Klainerman (1993), Chrusciel—Delay (2002), Hintz-Vasy (2020)

Adapted coordinates :

BMS coordinates

(Bondi -- Van der Burg -- Metzner -- Sachs 62)



One can always choose a coordinate system $(u, \Omega, z, \bar{z})$ such that

$$\tilde{g} = \frac{1}{\Omega^2} \Big[\quad 2dud\Omega + \tilde{h}_{z\bar{z}}(z,\bar{z})dzd\bar{z} + \Omega\Big(C_{zz}(u,z,\bar{z})dzdz + c.c. \Big) + \mathcal{O}(\Omega^2) \Big]$$

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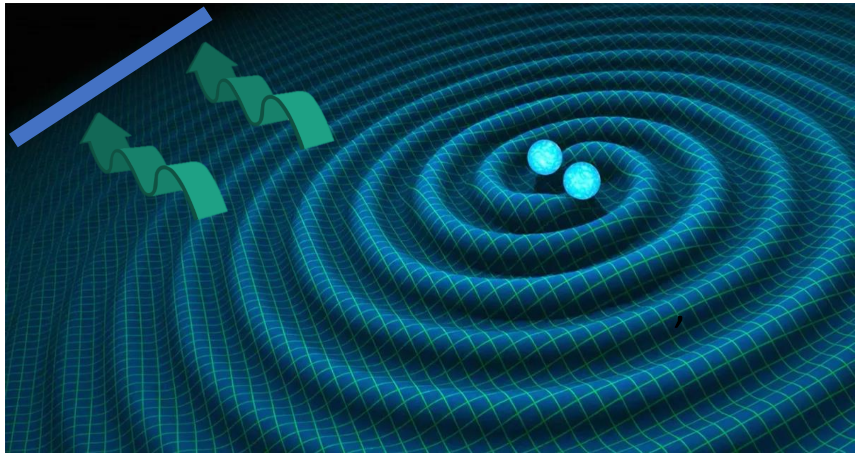

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"Universal" boundary geometry

$$(\tilde{h} = \tilde{h}_{z\bar{z}}(z,\bar{z})dzd\bar{z}, \tilde{n} = \partial_u)$$

"asymptotic shear", encodes the dynamical part of the geometry

3- The BMS group

Geometric characterization of the BMS group:

- The group of *asymptotic symmetries* of asymptotically flat spacetime

- The group of conformal symmetries of null infinity (conformal Carroll symmetries)

The BMS group $BMS_4 \simeq \mathcal{E}[1](S^2) \rtimes SO(3,1)$

is the group of asymptotic symmetry of asymptotically flat space-time

$$\tilde{g} = \frac{1}{\Omega^2}\left[2du d\Omega + \tilde{h}_{AB}(x) dx^A dx^B + \Omega\left(C_{AB}(u,x) dx^A dx^B \right) + \mathcal{O}(\Omega^2)\right]$$

$$\xi^\mu \partial_\mu = \left(\mathcal{T}(z,\bar{z}) + \frac{u}{2}\left(\partial_z \mathcal{Y}^z + \partial_{\bar{z}} \bar{\mathcal{Y}}^{\bar{z}}\right)\right)\partial_u + \mathcal{Y}^z(z)\partial_z + \bar{\mathcal{Y}}^{\bar{z}}(z)\partial_{\bar{z}} + O(\Omega)$$

(Infinitesimal) diffeomorphism along null infinity

(Infinitesimal) asymptotic symmetry

generated by $\left(\mathcal{T}(z,\bar{z}) \, , \, \mathcal{Y}^z(z) \right) \in \mathcal{E}[1] \rtimes SL(2,\mathbb{C})$

The BMS group $BMS_4 \simeq \mathcal{E}[1](S^2) \ltimes SO(3,1)$

Important remarks

- The Poincaré group $ISO(3,1) \simeq \mathbb{R}^4 \ltimes SO(3,1)$

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$\mathbb{R}^{3,1} \subset C^\infty(S^2)$

Super-translations: $$\mathcal{T}(z,\bar{z}) = \sum_{l,m}^{l=\infty} \mathcal{T}_{l,m} Y_{l,m}(z,\bar{z})$$

Translations: $$T^\mu \simeq T^0 Y_{0,0}(z,\bar{z}) + \sum_{m=-1}^{m=1} T^m Y_{1,m}(z,\bar{z})$$

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- The Poincaré group $ISO(3,1) \simeq \mathbb{R}^4 \rtimes SO(3,1)$

sits inside BMS: $ISO(3,1) \subset BMS_4$

- This inclusion is **not unique**.

→ **Infinitely many non-equivalent Poincaré groups inside BMS**

Following *Ashtekar (1984)* we refer to such a choice of $ISO(3,1) \subset BMS_4$ as a choice of gravity vacuum. → $\dfrac{BMS}{ISO(3,1)}$

The BMS group $BMS_4 \simeq \mathcal{E}[1](S^2) \rtimes SO(3,1)$

Important remarks

Nevertheless, there is a canonical, $SO(3,1)$ invariant, inclusion

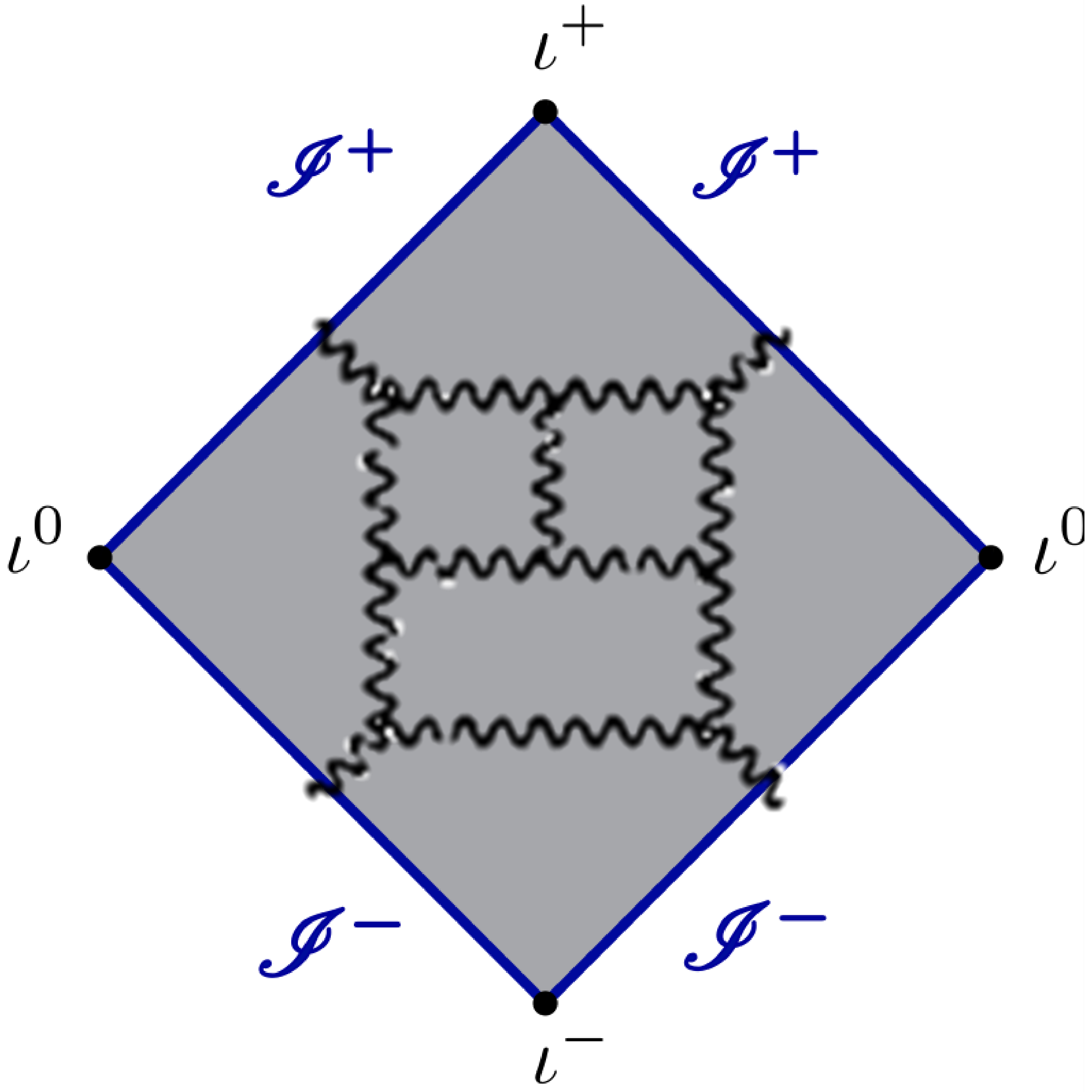
$$\mathbb{R}^4 \hookrightarrow \mathcal{E}[1](S^2)$$

$$q_\mu \left| \begin{array}{ccc} \mathbb{R}^{3,1} & \hookrightarrow & \mathcal{E}[1] \\ T^\mu & \mapsto & \mathcal{T}(z,\bar{z}) = T^\mu q_\mu(z,\bar{z}) \end{array} \right.$$

$$q^\mu(z,\bar{z}) = \left(1+|z|^2, \quad z+\bar{z}, \quad -i(z-\bar{z}), \quad 1-|z|^2\right)$$

4- BMS and the S-matrix: Asymptotic states

$$\langle \mathrm{out}|S|\mathrm{in}\rangle$$



Asymptotically flat spacetimes give a natural geometrical setup to the “interaction picture” of QFT :

Asymptotically free states are “ at infinity ”.

Scalar field :

$$\Phi(X) \;=\; \frac{\kappa}{(2\pi)^3}\int \frac{d^3\boldsymbol{p}}{2p^0}\,(\boldsymbol{p})\left(e^{-ix^\mu p_\mu(\boldsymbol{p})}\,a(\boldsymbol{p}) + e^{ix^\mu p_\mu(\boldsymbol{p})}a^\dagger(\boldsymbol{p})\right)$$

$$p^\mu = \begin{pmatrix} p^0 \\ \boldsymbol{p} \end{pmatrix} = \omega q^\mu(\zeta,\bar{\zeta}) \qquad\qquad q^\mu(\zeta,\bar{\zeta}) = \left(\,1+|\zeta|^2,\quad \zeta+\bar{\zeta},\quad -i(\zeta-\bar{\zeta}),\quad 1-|\zeta|^2\,\right)$$

$$= \frac{i\kappa}{16\pi^3}\int_0^\infty \omega d\omega \int_{\mathbb{CP}^1} d\zeta d\bar{\zeta}\;\left(e^{-i\omega x^\mu q_\mu(\zeta,\bar{\zeta})}\,a(\omega,\zeta,\bar{\zeta}) + e^{i\omega x^\mu q_\mu(\zeta,\bar{\zeta})}a^\dagger(\omega,\zeta,\bar{\zeta})\right)$$

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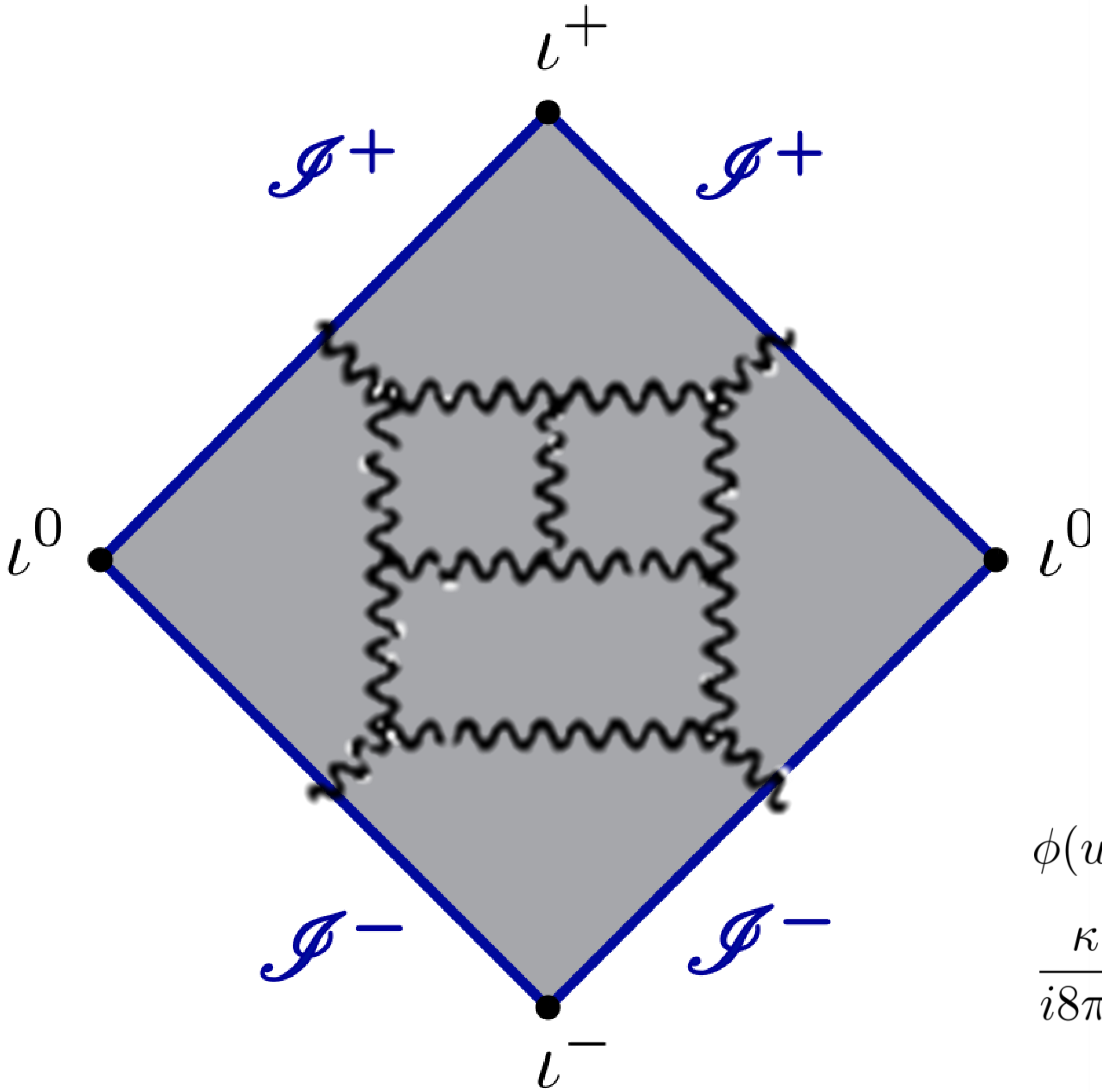
$$\Phi(X) \underset{r\to\infty}{\sim} r^{-1} \phi(u, z, \bar{z}) + O\left(r^{-2}\right)$$

$$\phi(u, z, \bar{z}) = \frac{\kappa}{i8\pi^2} \int_0^\infty \omega d\omega \left(e^{-i\omega u} a(\omega, z, \bar{z}) - e^{i\omega u} a^\dagger(\omega, z, \bar{z})\right)$$



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What did we gain ?

$$\phi(u,z,\bar{z}) = \frac{\kappa}{i8\pi^2}\int_0^\infty \omega d\omega \left(e^{-i\omega u}\, a(\omega,z,\bar{z}) - e^{i\omega u} a^\dagger(\omega,z,\bar{z})\right)$$

... is a **BMS (unitary irreducible) representation** (as a field on $\mathscr{I} = \mathbb{R}\times S^2$).

$$\left(\mathcal{T}(z,\bar{z})\,,\begin{pmatrix} a & b \\ c & d\end{pmatrix}\right) \in \mathcal{E}[1] \rtimes SL(2,\mathbb{C}) = BMS_4$$

$$C_{zz}(u,z,\bar{z}) \longmapsto C_{z'z'}(u',z',\bar{z}')$$

$$a(\omega,z,\bar{z}) \underset{BMS_4}{\longmapsto} e^{-i\omega\mathcal{T}(z,z)} a(\omega',z',\bar{z}')$$

$$u' = u + \mathcal{T}(z,\bar{z})$$

$$z' = \frac{a+bz}{c+dz}$$

Sachs (62)

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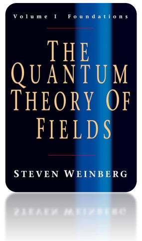
6- BMS UIR : induced reps method



Poincaré

BMS

$$\left(T^\mu\,,\,M^\mu{}_\nu\right) \in \mathbb{R}^4 \rtimes SO(3,1)$$

1. Define momenta as dual to translations

$$p_\mu \in \left(\mathbb{R}^4\right)^* \qquad \langle p, T\rangle := p_\mu T^\mu$$

2. Fix an **orbit** of the Lorentz group $\mathcal{O}_k \subset \left(\mathbb{R}^4\right)^*$

In practice fix $k_\mu \in \left(\mathbb{R}^4\right)^*$ and consider

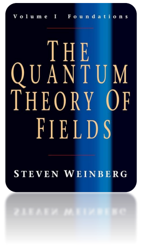
$$p_\mu := k_\nu M^\nu{}_\mu \in \mathcal{O}_k \simeq \frac{SO(3,1)}{\ell_k}$$

ℓ_k	$SU(2)$	$ISO(2)$	$SL(2,\mathbb{R})$	$SO(3,1)$
$\mathcal{O}_k$	H^3	$\mathbb{R}\times S^2$	dS_3	$\{0_\mu\}$

3. Poincaré particle = wavefunction with support on a given orbit of the Lorentz group

$$\mathcal{O}_k = \frac{SO(3,1)}{\ell_k}$$

+ spin (UIR of little group)



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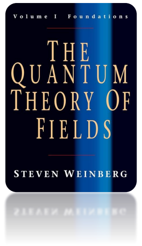

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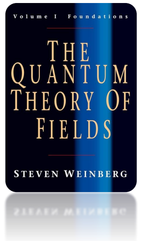
[McCarthy] : there are **11** possible **little groups**

$\ell_\mathcal{K} = stab(\mathcal{K}(z,\bar z))$ with mass-shell of dimensions $Dim\left(\mathcal{O}_\mathcal{K}\right) \in \{0,3,4,5,6\}$



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BMS particle = wavefunction with support on a given orbit of the Lorentz group $\mathcal{O}_\mathcal{K} = \dfrac{SO(3,1)}{\ell_\mathcal{K}}$

+ BMS spin (UIR of BMS little group)

McCarthy (1975):

there are 11 possible connected little groups:

- 4 from Wigner

$$SL(2,\mathbb{C})$$

$$ISO(2) \quad SU(2) \quad SL(2,\mathbb{R})$$

- 7 extra little groups

$$\mathbb{R}^2 \quad \mathbb{R}^+ \times U(1)$$

$$\mathbb{R}^+ \quad U(1) \quad \mathbb{R} \quad \mathbb{R}^+$$

$$\{e\}$$

Representatives $\varphi(z,\bar{z}) := \mathcal{K}(z,\bar{z})$ for the corresponding orbits are given by :

TABLE 2. INVARIANT VECTORS OF CONNECTED SUBGROUPS OF G

$$\Phi(G) = 0$$

$$\Phi(D) = 0$$

$\Phi(\Delta)$

$$\varphi = K$$

$$\hat{\varphi} = K|z|^{-6} + A\delta^{2,2} + C\delta$$

$$\Phi(J) = 0 \qquad \Phi(K) = 0$$

$\Phi(SU(2))$

$$\varphi = K(1+|z|^2)^{-3}$$

$$\hat{\varphi} = K(1+|z|^2)^{-3}$$

$\Phi(SL(2,R))$

$$\varphi = K\left(\frac{z-\bar{z}}{\mathrm{i}}\right)^{-3} + A\delta^2\left(\frac{z-\bar{z}}{\mathrm{i}}\right)$$

$$\hat{\varphi} = K\left(\frac{\bar{z}-z}{\mathrm{i}}\right)^{-3} + A\delta^2\left(\frac{\bar{z}-z}{\mathrm{i}}\right)$$

$\Phi(\Sigma)$

$$\varphi = K$$

$$\hat{\varphi} = K|z|^{-6} + A\delta^{2,2} + B\delta^{2,0} + \bar{B}\delta^{0,2} + C\delta$$

$\Phi(P)$

$$\varphi = K|z|^{-3}$$

$$\hat{\varphi} = K|z|^{-3}$$

$\Phi(\Omega)$

$$\varphi = K\left(\frac{z-\bar{z}}{\mathrm{i}}\right)^{-3} + A\delta^2\left(\frac{z-\bar{z}}{\mathrm{i}}\right)$$

$$\hat{\varphi} = K\left(\frac{\bar{z}-z}{\mathrm{i}}\right)^{-3} + A\delta^2\left(\frac{\bar{z}-z}{\mathrm{i}}\right)$$

$\Phi(\Pi)$

$$\varphi = r^{-3}\beta(\theta) + A\delta^{1,0} + \bar{A}\delta^{0,1}$$

$$\hat{\varphi} = r^{-3}\beta(-\theta) + B\delta^{1,0} + \bar{B}\delta^{0,1}$$

$$(z = r\mathrm{e}^{\mathrm{i}\theta})$$

$\Phi(\Gamma)$

$$\varphi = \beta(r)$$

$$\hat{\varphi} = r^{-6}\beta(r^{-1})$$

$$(z = r\mathrm{e}^{\mathrm{i}\theta})$$

$\Phi(\Lambda)$

$$\varphi = \beta(z+\bar{z})$$

$$\hat{\varphi} = |z|^{-6}\beta(z^{-1}+\bar{z}^{-1}) + A\delta^{2,2} + B\delta^{2,0} + \bar{B}\delta^{0,2} + C\delta$$

$\Phi(Q)$

$$\varphi = r^{-3}\beta(\theta)$$

$$\hat{\varphi} = r^{-3}\beta(-\theta)$$

$$(z = r^{(1+\mathrm{i}\rho)}\mathrm{e}^{\mathrm{i}\theta})$$

Take home message for physics:

McCarthy's classification of UIR yields a list of little groups

→ 11 different connected little groups

→ a quadratic Casimir invariant playing the role of mass

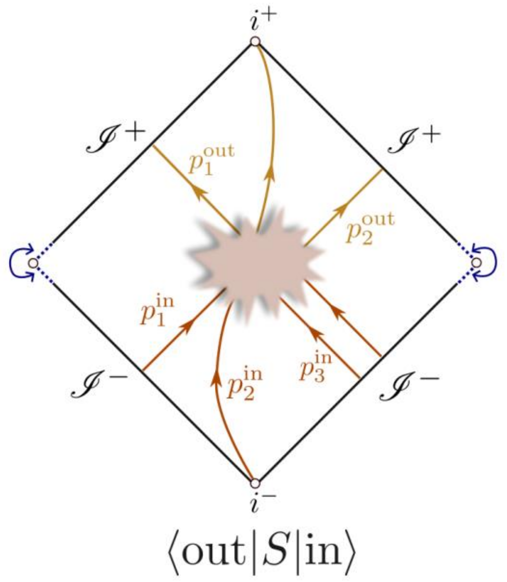
A classification results (similar to learning that mass and spin are important things) but....

What do the states look like ? What's the physics ? Does it relate to IR?

7- BMS UIR : hard reps

Hard BMS UIRs

It is known since Sachs (1962) and Longhi – Materassi (1999) that usual Poincaré UIR lift to BMS reps: they give the hard BMS UIRs.

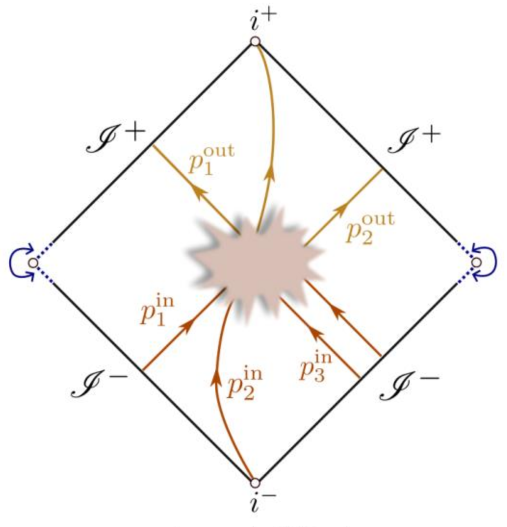


$$\Phi(X) = \int \frac{d^3p}{(2\pi)^3 2p^0}\left[a(p)e^{ip\cdot X} + a(p)^\dagger e^{-ip\cdot X}\right]$$

- massless scalar near $\mathscr{I}$

- massive scalar near $i^{\pm}$

Hard BMS UIRs



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$$\Phi(X) = \int \frac{d^3p}{(2\pi)^3 2p^0}\left[a(p)e^{ip\cdot X} + a(p)^\dagger e^{-ip\cdot X}\right]$$

- massless scalar near $\mathscr{I}$

$$\phi(u,\zeta) \propto \int_0^{+\infty} d\omega \left[a(\omega,\zeta,\bar\zeta)e^{-i\omega u} - a(\omega,\zeta,\bar\zeta)^\dagger e^{i\omega u}\right]$$

$$a(p) \underset{BMS_4}{\longmapsto} e^{i\omega\mathcal{T}(\zeta,\bar\zeta)}a(p) \qquad p_\mu = \omega q_\mu(\zeta,\bar\zeta)$$

- massive scalar near $i^\pm$

$$\phi(s,\mathbf{k}) \propto a(\mathbf{k})e^{-ims}$$

$$a(\mathbf{k}) \underset{BMS_4}{\longmapsto} e^{-\int d^2z \frac{\mathcal{T}(z,\bar z)m^4}{(q(z,\bar z)\cdot p(\mathbf{k}))^3}} a(\mathbf{k})$$

Hard BMS UIRs

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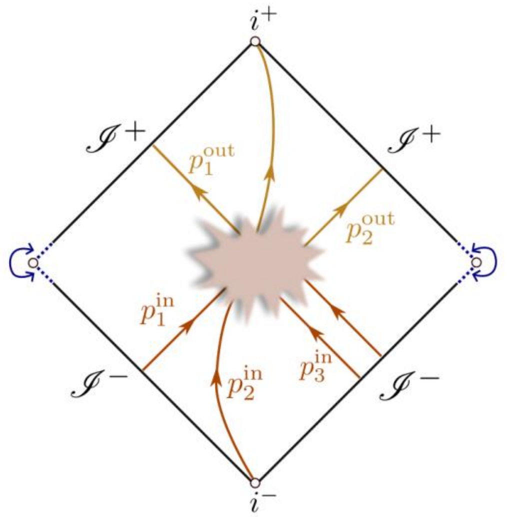

$\langle \text{out}|S|\text{in}\rangle$

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Hard rep

$$a(p) \underset{BMS_4}{\longmapsto} e^{i\langle P\mathcal{T}\rangle}a(p)$$

Hard massless

$$p^\mu = \omega q^\mu(\zeta,\bar\zeta)$$

$$P(z,\bar z) = \omega\delta^{(2)}(z-\zeta)$$

Hard massive

$$P(z,\bar z) = -\frac{m^4}{\pi}\left(q(z,\bar z)\cdot p\right)^{-3}$$

Non linearity of hard supermomenta

The map, $p^\mu \longrightarrow \begin{cases} P(z,\bar{z}) = \omega\delta^{(2)}(z-\zeta) \\ P(z,\bar{z}) = -\frac{m^4}{\pi}\left(q(z,\bar{z})\cdot p\right)^{-3} \end{cases}$ $p_\mu = \omega q_\mu(\zeta,\bar{\zeta})$

from momenta to hard supermomenta, is nonlinear.

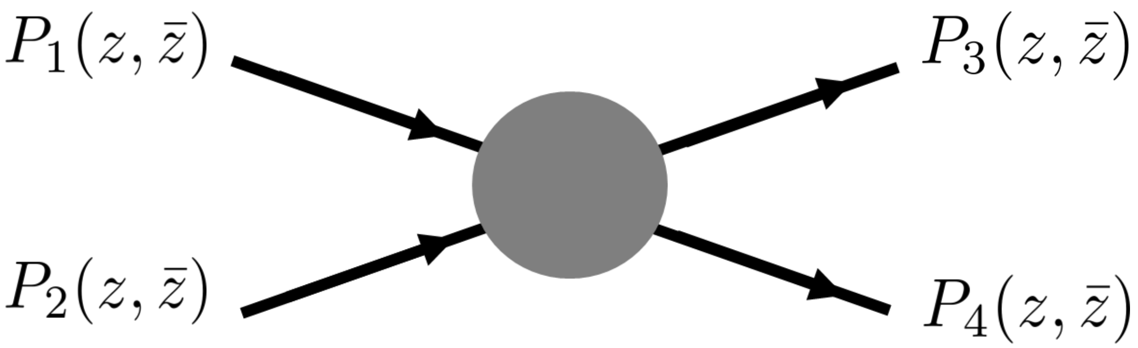
Consequence:

Non linearity of hard supermomenta

The map, $p^\mu \longrightarrow \begin{cases} P(z,\bar{z}) = \omega \delta^{(2)}(z-\zeta) \\ P(z,\bar{z}) = -\frac{m^4}{\pi}\left(q(z,\bar{z})\cdot p\right)^{-3} \end{cases}$ $\quad p_\mu = \omega q_\mu(\zeta,\bar{\zeta})$

from momenta to hard supermomenta, is nonlinear.

Consequence:



This is the reason for IR divergences in QFT

Momentum conservation <u>does not imply</u> conservation of hard supermomentum:

$$p_1^\mu + p_2^\mu = p_3^\mu + p_4^\mu \qquad\qquad P_1(z,\bar{z}) + P_2(z,\bar{z}) \neq P_3(z,\bar{z}) + P_4(z,\bar{z})$$

Chatterjee – Lowe (2017) for massive reps

8- BMS UIR : Beyond hard reps

Supermomenta decomposition

Key result

Soft part **Hard part**

$$\mathcal{P}(z,\bar{z}) = \boxed{\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}} + \boxed{P(z,\bar{z})}$$

This decomposition is unique.
It is Lorentz invariant but nonlinear.

Bekaert – Donnay – YH (2024)
Bekaert – YH (2025)

Supermomenta decomposition

Key result

Soft part **Hard part**

$$\mathcal{P}(z,\bar{z}) = \boxed{\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}} + \boxed{P(z,\bar{z})}$$

This decomposition is unique. It is Lorentz invariant but nonlinear.

$\simeq$ 'memory'

$\simeq$ extra IR degrees of freedom

$$\mathcal{N}(z,\bar{z}) \in \mathcal{E}[1]/\mathbb{R}^{3,1}$$

massive case ($p^2 < 0$)

$$P(z,\bar{z}) = -\frac{m^4}{\pi}\left(q(z,\bar{z})\cdot p\right)^{-3}$$

massless case ($p^2 = 0$) $p^\mu = \omega q^\mu(\zeta,\bar{\zeta})$

$$P(z,\bar{z}) = \omega \delta^{(2)}(z-\zeta)$$

Bekaert – Donnay – YH (2024)
Bekaert – YH (2025)

Supermomenta decomposition

Key result

Soft part **Hard part**

$$\mathcal{P}(z,\bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N} + P(z,\bar{z})$$

This decomposition is unique. It is Lorentz invariant but nonlinear.

Important property:

the projection of supermomenta to momenta

$$\mathcal{P}(z,\bar{z}) \mapsto p_\mu = \int d^2 z\, \mathcal{P}(z,\bar{z}) q_\mu(z,\bar{z})$$

gives

$$\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N} \mapsto p_\mu = 0$$

$$P(z,\bar{z}) \mapsto p_\mu$$

massive case ($p^2 < 0$)

$$P(z,\bar{z}) = -\frac{m^4}{\pi} (q(z,\bar{z}) \cdot p)^{-3}$$

massless case ($p^2 = 0$) $p^\mu = \omega q^\mu(\zeta,\bar{\zeta})$

$$P(z,\bar{z}) = \omega \delta^{(2)}(z-\zeta)$$

Bekaert – Donnay – YH (2024)
Bekaert – YH (2025)

9- BMS UIR : BMS wavefunctions

BMS wavefunctions

Bekaert – Donnay – YH (2024)

$$|\Psi\rangle = \int \mathcal{D}\mathcal{P}\, \Psi(\mathcal{P})|\mathcal{P}\rangle$$

eigenstate of the supertranslations generator

Soft part / Hard part

$$\mathcal{P}(z,\bar{z}) = \underbrace{\partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}}_{\text{Soft part}} + \underbrace{P(z,\bar{z})}_{\text{Hard part}}$$

BMS wavefunctions

Bekaert – Donnay – YH (2024)

$$|\Psi\rangle = \int \mathcal{D}\mathcal{P}\, \Psi(\mathcal{P})|\mathcal{P}\rangle$$

eigenstate of the supertranslations generator

Soft part — Hard part

$$\mathcal{P}(z,\bar{z}) = \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N} + P(z,\bar{z})$$

$$|\Psi\rangle = \int d^3p \int \mathcal{D}\mathcal{N}\; \Psi(p; \partial^2\mathcal{N})|p; \partial^2\mathcal{N}\rangle$$

seems formal but the integrals* are **finite-dimensional** on $\ell_p/\ell_{\mathcal{P}}$

wavefunctions with support only on a given orbit of the Lorentz group

* see *Bekaert – YH (2025)* for explicit examples

Conclusion

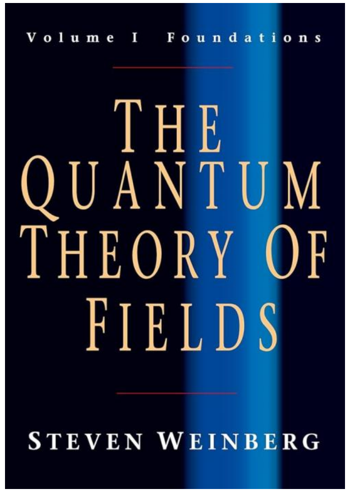
- Usual asymptotic states of QFT are BMS representations (Hard UIR)

$$\mathcal{P}(z,\bar{z}) = P(z,\bar{z})$$

→ however they cannot by themselves fulfill supermomentum conservations → IR divergences

- A fully BMS invariant extension of QFT has a chance to define infrared finite S-matrix elements

→ **will require a generic notion of BMS particles !** →



$$\mathcal{P}(z,\bar{z}) = P(z,\bar{z}) + \partial_z^2 \partial_{\bar{z}}^2 \mathcal{N}$$

Asymptotic one-particle states should belong to UIRs of the BMS group
Multi-particle states should be tensor products thereof:

$$|p_1, \partial^2 \mathcal{N}_1\rangle \otimes \cdots \otimes |p_n, \partial^2 \mathcal{N}_n\rangle$$

Bekaert – Donnay – YH (2025)

Thank you for your attention !

Decomposition of supermomenta *Bekaert—Donnay—YH (2024)*

This decomposition is Lorentz invariant but non-linear.

$$\underbrace{\mathcal{P}_1(z,\bar{z}) = P_1(z,\bar{z})}_{\simeq p_1^\mu} + \partial_z^2\partial_{\bar{z}}^2\mathcal{N}_1 \qquad \underbrace{\mathcal{P}_2(z,\bar{z}) = P_2(z,\bar{z})}_{\simeq p_2^\mu} + \partial_z^2\partial_{\bar{z}}^2\mathcal{N}_2$$

$$\Rightarrow \quad \mathcal{P}_1(z,\bar{z}) + \mathcal{P}_2(z,\bar{z}) = \underbrace{P_3(z,\bar{z})}_{\simeq p_3^\mu} + \partial_z^2\partial_{\bar{z}}^2\left(\mathcal{N}_1 + \mathcal{N}_2 - \frac{1}{\pi}\mathcal{S}\right)$$

Where $P_3(z,\bar{z})$ is the hard supermomentum for $p_3^\mu := p_1^\mu + p_2^\mu$

and $\mathcal{S} = q\cdot p^1 \ln|q\cdot p^1| + q\cdot p^2 \ln|q\cdot p^2| - q\cdot p^3 \ln|q\cdot p^3|$

BMS little group / Poincaré little group

Let $\mathcal{P}(z,\bar{z})$ be a supermomentum, its BMS little group is the stabilizer

$$\ell_{\mathcal{P}} = Stab(\mathcal{P}) \subset SO(3,1)$$

its Poincaré little group is the stabilizer of the corresponding momentum

$$\ell_p = Stab(p_\mu) \subset SO(3,1) \qquad p_\mu = \pi_\mu(\mathcal{P})$$

it follows from $p_\mu = \pi_\mu(\mathcal{P})$ that

$$\ell_{\mathcal{P}} \subset \ell_p \subset SO(3,1)$$

$\frac{\ell_p}{\ell_{\mathcal{P}}}$ encodes the extra d.o.f. of BMS particles

BMS wavefunctions

Bekaert—Donnay—YH (2024)

$$\mathcal{O}_{\mathcal{K}} = \frac{SO(3,1)}{\ell_{\mathcal{K}}} \rightarrow \mathcal{O}_k = \frac{SO(3,1)}{\ell_k}$$

BMS wavefunctions live here

"=" $\mathcal{O}_k \times \frac{\ell_k}{\ell_{\mathcal{K}}}$

Poincaré wavefunctions live here

$$\Psi\big(\mathcal{P}\big)$$

$$\Psi\big(p\big)$$

$$= \Psi\Big(p; \partial_{\bar{z}}^2 \partial_z^2 (\mathcal{N}_p \cdot L)\Big) \qquad [L] \in F \simeq \frac{\ell_k}{\ell_{\mathcal{K}}}$$

Example

$$\ell_{\mathcal{K}} = \mathbb{R}^2 \quad \ell_k = ISO(2) \quad F \simeq \frac{\ell_k}{\ell_{\mathcal{K}}} = U(1)$$

$$\mathcal{K}(z,\bar{z}) = \delta^{(2)}(z,\bar{z}) + \left(\partial_z^2 \delta^{(2)}(z,\bar{z}) + c.c.\right)$$

$$\Psi\left(\mathcal{P}\right) = \Psi\left(\omega, \zeta, \bar{\zeta}; e^{i\alpha}\right)$$

BMS wavefunctions

$$\Psi(p) = \Psi\left(\omega, \zeta, \bar{\zeta}\right)$$

Poincaré wavefunctions

BMS wavefunctions

Bekaert—Donnay—YH (2024)

Half Fourier transform

$$\mathcal{P}(z,\bar{z}) \in \mathcal{O}_{\mathcal{K}}$$

$$\Psi\big(\mathcal{P}\big) = \Psi\Big(p, \partial_z^2(\mathcal{N}_p \cdot L)\Big)$$

$$(p, L) \in \mathcal{O}_k \times \frac{\ell_k}{\ell_{\mathcal{K}}}$$

$$\Psi\big(p, \partial_z^2 \mathcal{C}\big) = \int_{\frac{\ell_p}{\ell_{\mathcal{P}}}} dL \;\; e^{i\langle \partial_z^2 \partial_z^2 (\mathcal{N}\cdot L), \mathcal{C}\rangle} \; \Psi\Big(p, \partial_z^2(\mathcal{N} \cdot L)\Big)$$

'on the mass-shell' of the soft charge

Function on $\mathcal{O}_k \times \dfrac{BMS}{ISO(3,1)}$ ← Gravity vacua

BMS wavefunctions

Bekaert—Donnay—YH (2024)

$$|\Psi\rangle = \int \mathcal{D}(\partial_z \mathcal{C}) \int d^3p \;\; \Psi\left(p, \partial_z^2 \mathcal{C}\right) \, |p; \partial_z^2 \mathcal{C}\rangle$$

sum over $\mathcal{O}_k \times \frac{BMS}{ISO(3,1)}$ ⟵ Gravity vacua

Contrasting with the literature

- **Dressed states** (``coherent states'', ``attaching soft clouds'', etc.) [Chung '65][Kibble '68] [Dollard '71][Kulish, Fadeev '70]
  - ✓ IR divergences cancel in S-matrix elements but
    - the construction has an intrinsic ambiguity
    - the states themselves are now IR divergent [Hannesdottir, Schwartz '19]

- **Dressed states + BMS** [Kapec, Perry, Raclariu, Strominger '17][Akhoury, Choi '17]

  Each Fadeev-Kulish state has 'large gauge/BMS' charge = 0

  Too restrictive! It is enough to require the total conservation of BMS charges.

  → Constructed eigenstates of the soft charge and showed the equivalence to FK construction.

From the BMS particle point of view

the dressed state construction ≈ **shoehorn BMS states** into the conventional QFT language.

→ Dressed states are not momentum eigenstates, and thus not supermomentum eigenstates. They do not belong to a separable, Lorentz invariant, Hilbert space. [Prabhu, Satishchandran, Wald '22]

Weinberg's theorem as a Ward identity

- *Weinberg (65)* 's soft theorem

$$0 = \partial_{\bar{z}}^2 \left(\left(\sum_i \frac{(\epsilon(z,\bar{z}) \cdot p_i)^2}{q(z,\bar{z}) \cdot p_i} \right) \langle \text{out} | \hat{S} | \text{in} \rangle + \lim_{\omega \to 0} \omega \, \langle \text{out} | [a(\omega, z, \bar{z}), \hat{S}] | \text{in} \rangle \right)$$

- **Is equivalent to the Ward identity for supertranslation** ***Strominger et al (2014)***

$$0 = \langle \text{out} | [\hat{\mathcal{P}}(z,\bar{z}), \hat{S}] | \text{in} \rangle$$

$$\hat{\mathcal{P}}(z,\bar{z}) = \int \omega d\omega \, a^\dagger(\omega, z, \bar{z}) a(\omega, z, \bar{z}) + \partial_{\bar{z}}^2 \left(\lim_{\omega \to 0} \omega a(\omega, z, \bar{z}) \right)$$

$$= \int du \Big(|\partial_u C_{zz}|^2 + \partial_{\bar{z}}^2 \left(\partial_u C_{zz} \right) \Big) \quad = \int du \, (\partial_u m)$$

$$C_{zz}(u, z, \bar{z}) \propto \int_0^\infty \omega d\omega \left(e^{-i\omega u} a_+(\omega, z, \bar{z}) - e^{i\omega u} a_-^\dagger(\omega, z, \bar{z}) \right)$$