Investigation of Non-Symmetric Black Hole Mimickers

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Ongoing project in collaboration with Sergey Solodukhin

Black holes and their symmetries
IDP Tours

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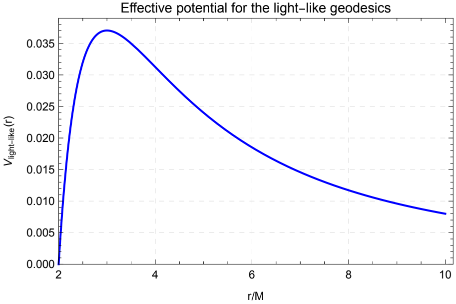
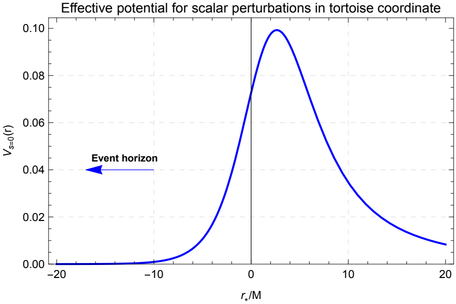
Outline

Motivation

We can not directly examine an event horizon of a black hole by optical and gravitational wave experiments [Cardoso et al. Phys. Rev. Lett. **117** (2016)]





Metric may change around $r \sim 2M$, for instance due to the quantum corrections.

Damour – Solodukhin symmetric wormhole [Damour and Solodukhin, Phys. Rev. D **76** (2007), 024016] :

$$ds^2 = -(g_{\text{sch}}(\rho) + b^2)\ dt^2 + d\rho^2 + r^2(\rho)(\ d\theta^2 + \sin^2\theta\ d\varphi^2), \quad b \ll 1 \tag{1}$$

where $d\rho = \pm dr/\sqrt{1 - 2M/r}$ is proper length.

$\Delta t \sim \ln 1/b^2$

Metric modifications

General metric

$$ds^2 = -g(\rho)\ dt^2 + d\rho^2 + r^2(\rho)(\ d\theta^2 + \sin^2\theta\ d\varphi^2) \tag{2}$$

Our assumptions:

- In the physical region $\rho > 0$ the functions $g(\rho)$ and $r(\rho)$ are given by their classical form plus some small corrections
- Function $g(\rho)$ is non-vanishing for any finite value of ρ, at $\rho = 0$ its value is small
- In the inner region both functions $g(\rho)$ and $r(\rho)$ may have either small or large deviations from the classical Z_2 configuration
- The global spacetime is geodesically complete both for null geodesics and the time-like geodesics
- The global spacetime is free from the curvature singularities

Geodesic completeness

Equation for the radial geodesics:

$$\frac{d\rho}{d\lambda} = \frac{\sqrt{E^2 - \varkappa g(\rho)}}{\sqrt{g(\rho)}} \tag{3}$$

Equation for the rotating geodesics:

$$\frac{d\rho}{d\lambda} = \frac{1}{\sqrt{g(\rho)}} \sqrt{E^2 - g(\rho)\left(\varkappa + \frac{L^2}{r^2(\rho)}\right)} \tag{4}$$

$\varkappa = 0$ (light – like), 1 (time – like).
In the inner region:

$$g(\rho)|_{\rho \to -\infty} \sim g_0(-\rho)^{\kappa_1}\,, \quad r(\rho)|_{\rho \to -\infty} \sim r_0(-\rho)^{\kappa_2} \tag{5}$$

The completeness of the radial null geodesics imposes constraint only on function $g(\rho)$:

$$\Delta\lambda = \frac{1}{E}\int^{-\infty} \sqrt{g(\rho)} = \infty \;\Rightarrow\; \kappa_1 \geq -2 \tag{6}$$

Geodesic completeness

Light – like geodesics $V_{\text{light-like}}(\rho) = \left.\dfrac{g(\rho)}{r^2(\rho)}\right|_{\rho\to-\infty} \sim (-\rho)^{\kappa_1 - 2\kappa_2}$

$A1$: $\kappa_1 - 2\kappa_2 < 0$ the potential tends to vanish asymptotically (a non-radial null geodesic reaches asymptotic infinity)

$A2$: $\kappa_1 = 2\kappa_2$ the potential approaches a constant (a non-radial null geodesic reaches asymptotic infinity for sufficiently small impact)

$A3$: $\kappa_1 - 2\kappa_2 > 0$ the potential diverges in the asymptotic region (a non-radial null geodesic never reaches infinity for any values of E/L)

Time-like geodesics $V_{\text{time-like}}(\rho) = g(\rho)\left(\varkappa + \dfrac{L^2}{r^2(\rho)}\right)$

$B1$: $\kappa_1 > 0$ no time-like geodesic with any value of L can reach infinity in the inside region

$B2$: $\kappa_1 \leq 0$ and $\kappa_1 - 2\kappa_2 > 0$ only radial geodesic $(L = 0)$ can reach the asymptotic region

$B3$: $\kappa_1 < 0$ and $\kappa_1 = 2\kappa_2$ geodesics with sufficiently large ratio E/L may reach infinity

$B4$: $\kappa_1 < 0$ and $\kappa_1 - 2\kappa_2 < 0$ any geodesic reaches infinity for any values of E and L

Curvature regularity

Ricci scalar:

$$R = -\frac{g''}{g} + \frac{1}{2}\left(\frac{g'}{g}\right)^2 - \frac{2r'g'}{rg} - \frac{4r''}{r} + 2\left(\frac{1-r'^2}{r^2}\right) \tag{7}$$

Square of the Riemann tensor:

$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = 2\frac{g''}{g} - \left(\frac{g'}{g}\right)^2 + 8\left(\frac{r'}{r}\right)^2 + 2\left(\frac{r'g'}{rg}\right)^2 + 4\left(\frac{1}{r^2} - \frac{r'^2}{r^2}\right)^2 \tag{8}$$

Asymptotic behavior:

$$R = \frac{2}{r_0\rho^{2\kappa_2}} + O\left(\frac{1}{\rho^2}\right) < \infty \Rightarrow \kappa_2 \geq 0$$

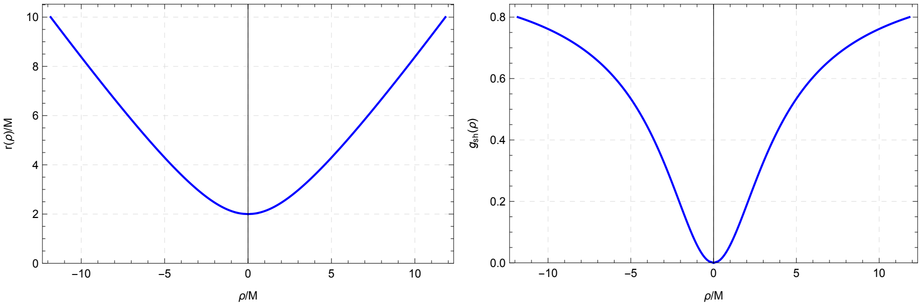
$$R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu} = \frac{4}{r_0^4\rho^{4\kappa_2}} - \frac{8\kappa_2^2}{\rho^{2\kappa_2+2}} + O\left(\frac{1}{\rho^2}\right) < \infty \Rightarrow \kappa_2 \geq 0 \tag{9}$$

Thus

$$\kappa_1 \geq -2\,, \quad \kappa_2 \geq 0 \tag{10}$$

In all cases $r(\rho)|_{\rho\to\pm\infty} \sim \pm\rho$ is as in the Schwarzschild space-time.

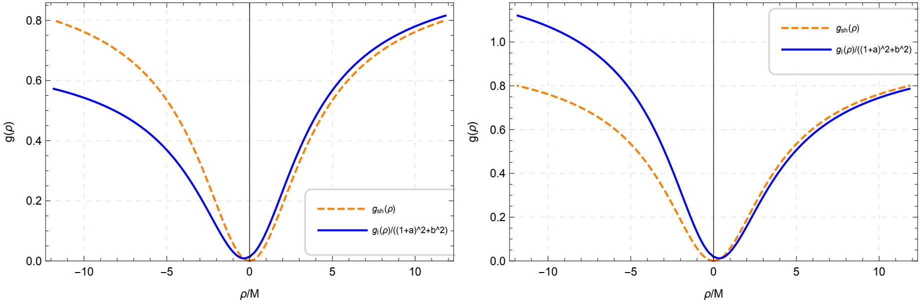
Classical $g(\rho)$ and $r(\rho)$ functions



Metric functions $r(\rho)$ and $g(\rho)$ as functions of the proper coordinate ρ.

Model I

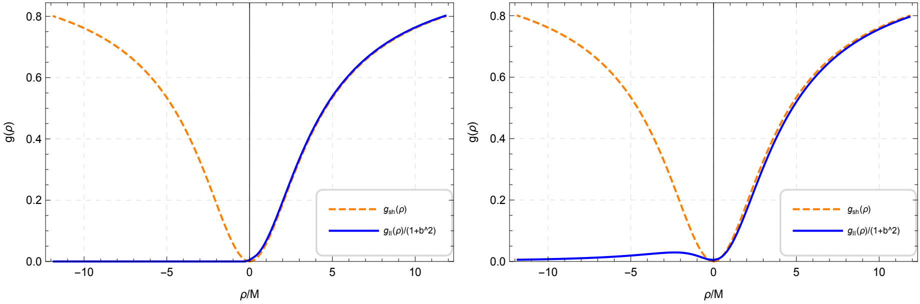
$$g(\rho) = \left(\sqrt{g_{\text{sch}}(\rho)} + a\right)^2 + b^2 \Big|_{\rho \to -\infty} \sim \text{const}, \quad g_{\text{sch}}(\rho) = 1 - \frac{2M}{r(\rho)}\Big|_{\rho \sim 0} \sim \frac{\rho^2}{16M^2} \Rightarrow$$
$$\Rightarrow \sqrt{g_{\text{sch}}(\rho)}\Big|_{\rho \sim 0} \sim \frac{\rho}{4M} \tag{11}$$



Left panel: $a = 0.08$ and $b = 0.1$. Right panel: $a = -0.08$ and $b = 0.1$

Model II

$$g(\rho) = \left(g_{\text{sch}}(\rho) + b^2\right)\Delta(\rho)\Big|_{\rho\to-\infty} \sim (-\rho)^{-2}, \quad \Delta(\rho) = \frac{1}{2}\left(1 + \frac{\rho}{\sqrt{\rho^2+\rho_0^2}}\right) \qquad (12)$$



Left panel: $\rho_0 = 0.01$ and $b = 0.1$. Right panel: $\rho_0 = 10$ and $b = 0.1$

We are mostly interested in monotonically case. For small values of parameter b:

$$\rho_0 < 3\sqrt{3}b + 6\sqrt{3}b^3 - 15\sqrt{3}b^5 + \dots$$

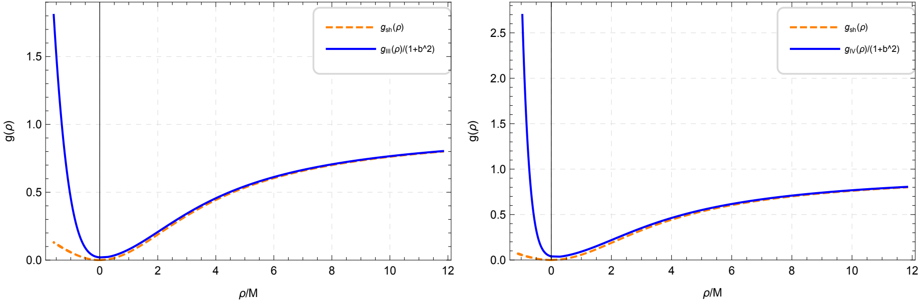
Models III and IV

Model III:

$$g(\rho) = \left(g_{\text{sch}}(\rho) + b^2\right) \Delta^{-1}(\rho)\Big|_{\rho\to-\infty} \sim (-\rho)^2 \tag{13}$$

Model IV:

$$g(\rho) = \left(g_{\text{sch}}(\rho) + b^2\right) \Delta^{-n}(\rho)\Big|_{\rho\to-\infty} \sim (-\rho)^{2n}, \quad n > 1 \tag{14}$$



Left panel is for model III with $\rho_0 = 1$ and $b = 0.1$. Right panel is for model IV with $\rho_0 = 1$, $b = 0.1$ and $n = 2$

They look similar but they have different properties!

Travel time

If $g(\rho)$ has a minimum then there is notion of the travel time.

$$\Delta t = \int_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{g(\rho)}} \sim$$

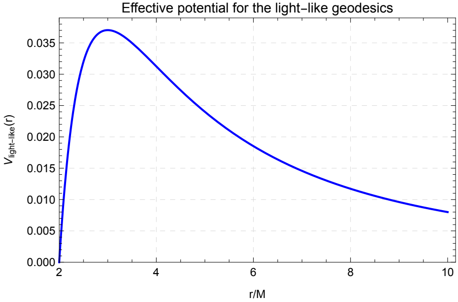
$$\sim \int_{\rho_1}^{\rho_2} \frac{d\rho}{\sqrt{g(\rho_{\text{min}}) + \frac{1}{2}g''(\rho_{\text{min}})(\rho - \rho_{\text{min}})^2}} \sim \frac{1}{2}\sqrt{\frac{2}{g''(\rho_{\text{min}})}} \ln \frac{1}{g(\rho_{\text{min}})} + \dots \qquad (15)$$

- Model I: $g(\rho) = (\sqrt{g_{\text{sch}}(\rho)} + a)^2 + b^2 \quad \Rightarrow \quad \Delta t \sim \frac{4M}{(1-a^2)^2} \ln \frac{1}{b^2}$
- Model III: $g(\rho) = (g_{\text{sch}}(\rho) + b^2)\Delta^{-1}(\rho) \quad \Rightarrow$

  $$\Delta t \sim \frac{1}{2}\sqrt{\frac{2}{g''(\rho_{\text{min}})}} \ln \frac{1}{(b^2 + 2b^2\rho_0/(\rho_0^2 + 8b^2))\Delta^{-1}(\rho_{\text{min}})} \sim \ln \frac{1}{b^2}$$

- Model IV: $g(\rho) = (g_{\text{sch}}(\rho) + b^2)\Delta^{-n}(\rho) \quad \Rightarrow$

  $$\Delta t \sim \frac{1}{2}\sqrt{\frac{2}{g''(\rho_{\text{min}})}} \ln \frac{1}{(b^2 + 2b^2 n\rho_0/(\rho_0^2 + 8b^2 n))\Delta^{-n}(\rho_{\text{min}})} \sim \ln \frac{1}{b^2}$$



$$\left(\frac{d\rho}{d\lambda}\right)^2 = \frac{L^2}{g(\rho)}\left(\frac{1}{p^2} - \underbrace{\frac{g(\rho)}{r^2(\rho)}}_{V_{\text{eff}}(\rho)}\right) \tag{16}$$

Photon sphere = maximum of $V_{\text{eff}}(\rho)$

$$\left.\frac{d\rho}{d\lambda}\right|_{\rho_{\text{ph}}} = 0, \quad \left.\frac{d^2\rho}{d\lambda^2}\right|_{\rho_{\text{ph}}} = 0 \quad \Rightarrow$$

$$\frac{1}{p_{\text{ph}}^2} = \frac{g(\rho_{\text{ph}})}{r^2(\rho_{\text{ph}})}, \quad \left.\frac{d}{d\rho}V_{\text{eff}}(\rho)\right|_{\rho_{\text{ph}}} = 0 \tag{17}$$

$$\sin^2\alpha_{\text{sh}} = p_{\text{ph}}^2\frac{g(\rho_{\text{o}})}{r^2(\rho_{\text{o}})} = \frac{r^2(\rho_{\text{ph}})}{g(\rho_{\text{ph}})}\frac{g(\rho_{\text{o}})}{r^2(\rho_{\text{o}})} \quad \Rightarrow$$

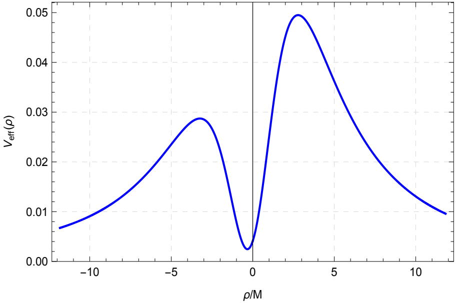
Shadow radius:

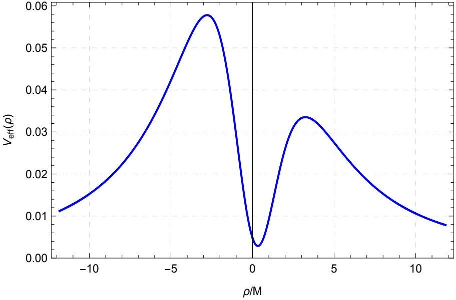
$$R_{\text{sh}}/M = r(\rho_{\text{o}})/M\sin\alpha_{\text{sh}} = r(\rho_{\text{ph}})/M\sqrt{\frac{g(\rho_{\text{o}})}{g(\rho_{\text{ph}})}} \tag{18}$$

ρ_{o} is the position of an observer

Shadow of the Model I

$g(\rho) = (\sqrt{g_{\text{sch}}(\rho)} + a)^2 + b^2$





The effective potentials for massless geodesics. The left panel is presented for $a = 0.08$ and the right for $a = -0.08$. In both cases $b = 0.1$

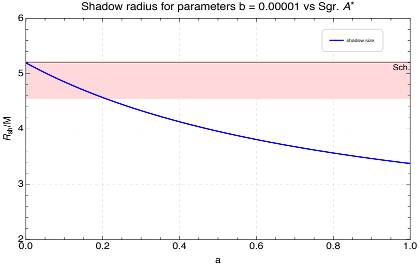
$$\text{M87}^* : \quad 4.31 \leq R_{\text{sh}}/M \leq 6.08$$

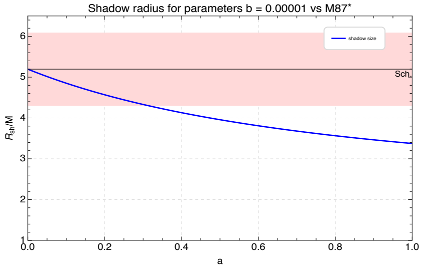
$$\text{Sgr.A}^* : \quad 4.55 \leq R_{\text{sh}}/M \leq 5.22$$

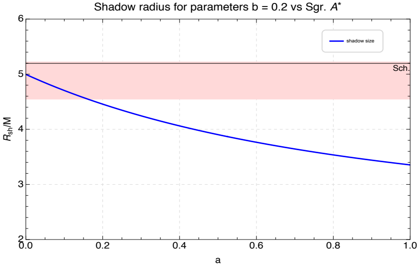
[Vagnozzi *et al.*Class. Quant. Grav. **40** (2023) no.16, 165007],[Akiyama *et al.* Astrophys. J. Lett. **875**, no.1, L6 (2019)], [Akiyama *et al.* Astrophys. J. Lett. **930**, no.2, L17 (2022)]

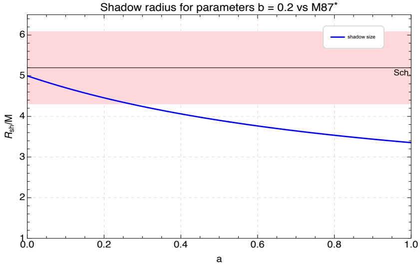
Shadow of the Model I

$a > 0$ case:

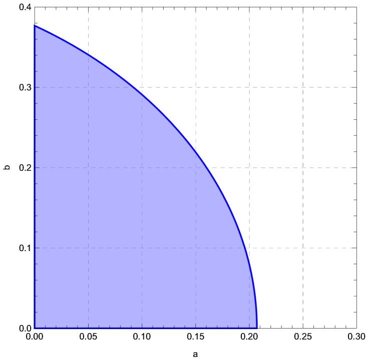


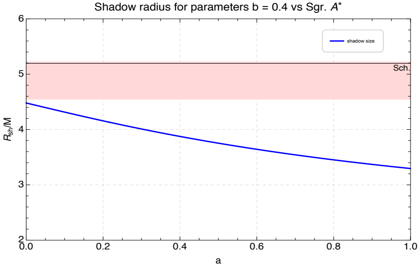


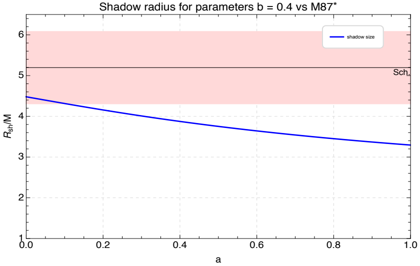




Shadow of the Model I

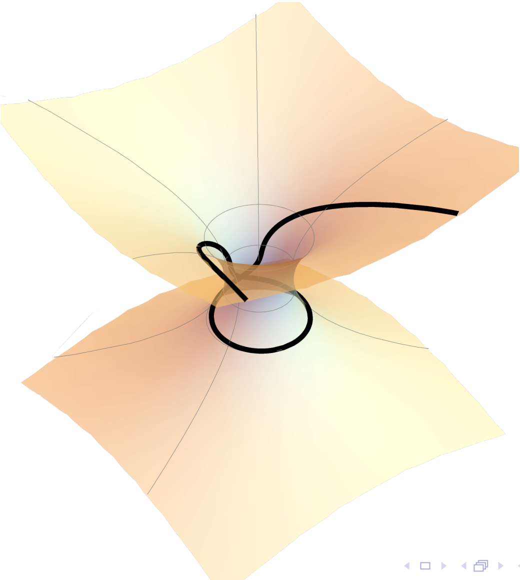




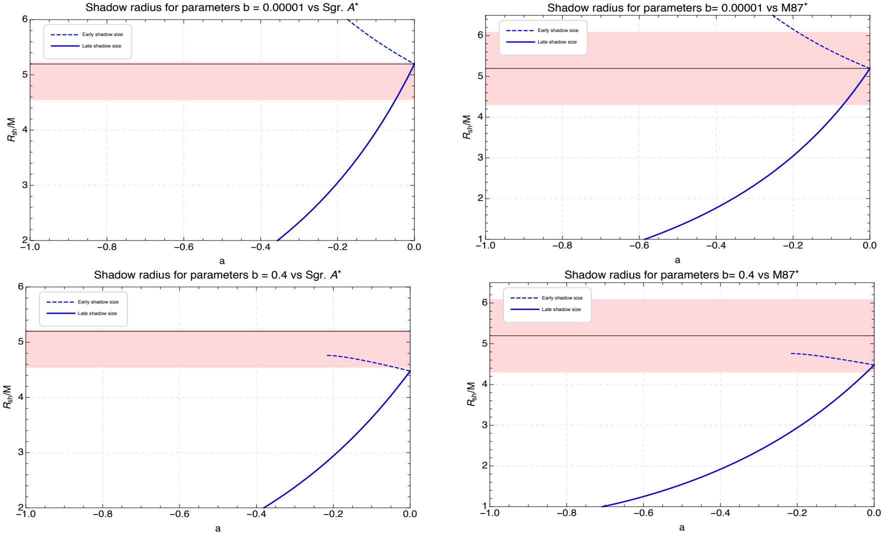


Shadow of the Model I

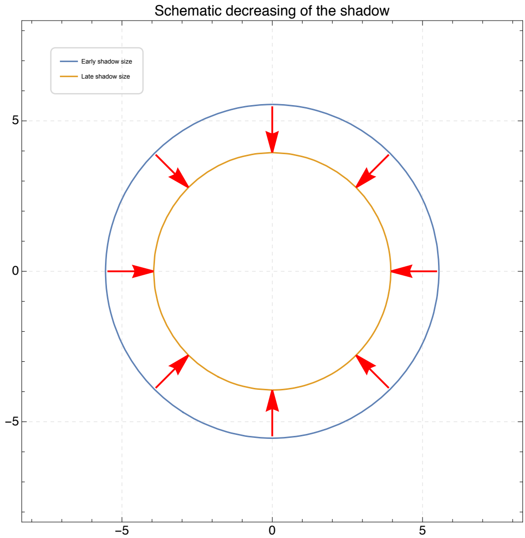
$a < 0$ case



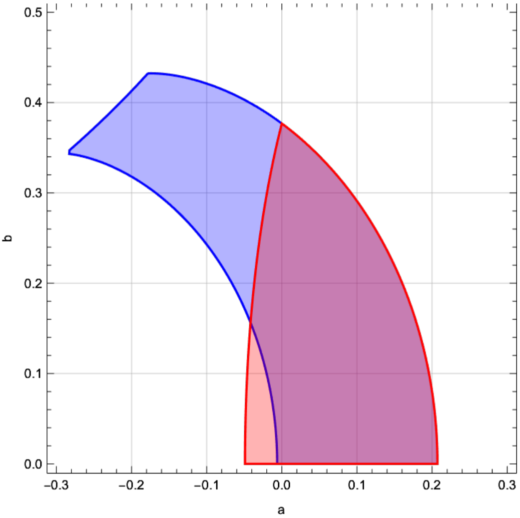
Shadow of the Model I



Decreasing of the shadow of the Model I



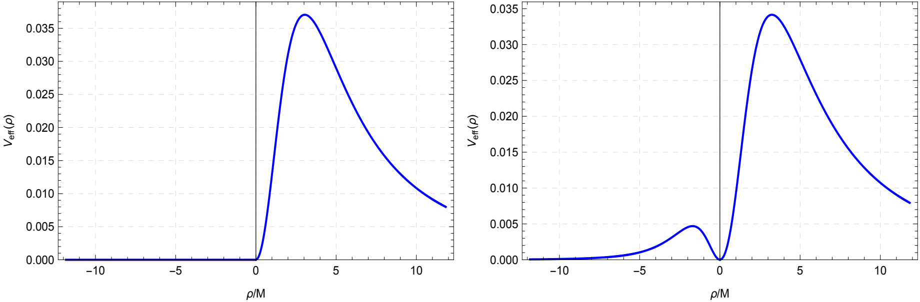
Constraint of the Model I



Constraint on the parameters a and b from the shadows

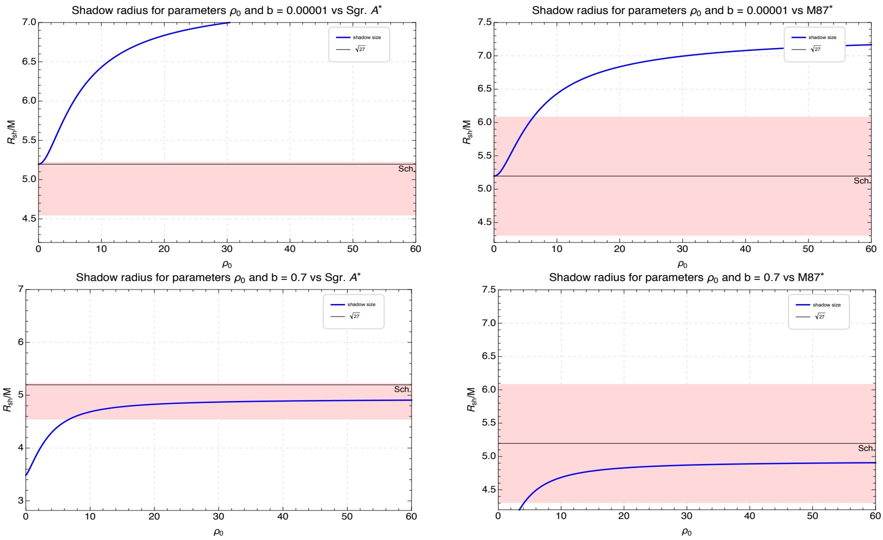
Shadow of the Model II

$$g(\rho) = (g_{\mathsf{sch}}(\rho) + b^2)\Delta(\rho)$$

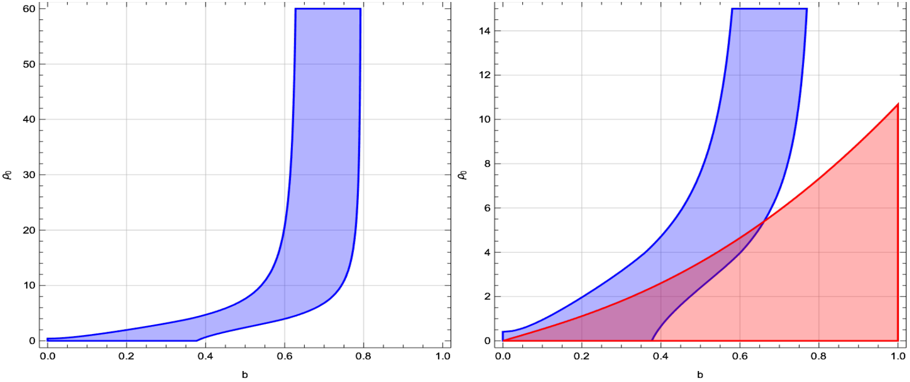


The effective potentials for massless geodesics. The left panel is presented for $\rho_0 = 0.01$ and the right for $\rho_0 = 10$. In both cases $b = 0.1$

Shadow of the Model II



Shadow of the Model II

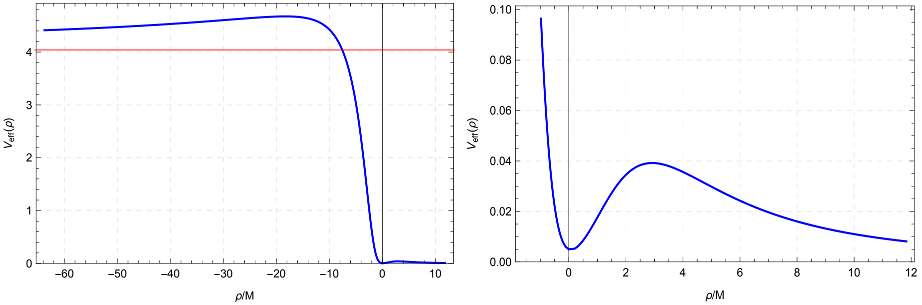


Constraint on the parameters ρ_0 and b from the shadows

Shadow of the Model III

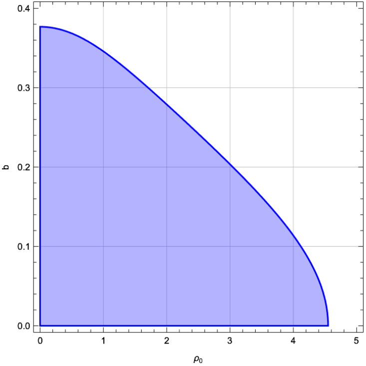
$g(\rho) = (g_{\text{sch}}(\rho) + b^2)\Delta^{-1}(\rho)$

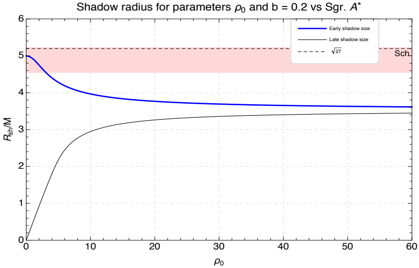
$$V_{\text{eff}}(\rho) = \left.\frac{g(\rho)}{r^2(\rho)}\right|_{\rho\to-\infty} \to \frac{(1+b^2)}{(-\rho)^2}\frac{4(-\rho)^2}{\rho_0^2} = \frac{4(1+b^2)}{\rho_0^2} \tag{19}$$

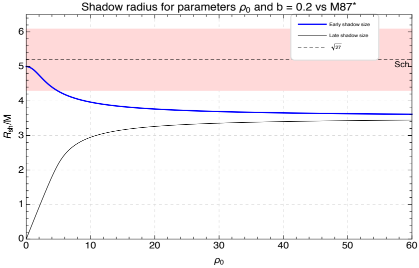


The effective potentials for massless geodesics. The left panel is presented for $\rho_0 = 1$ and the right is zoomed version of the left one. In both cases $b = 0.1$. The red line is constant that the effective potential approaches.

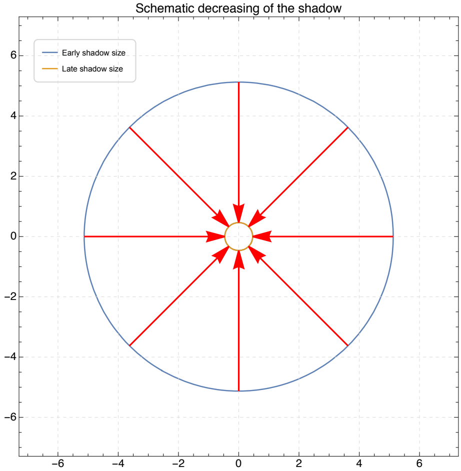
Shadow of the Model III







Decreasing of the shadow of the Model III

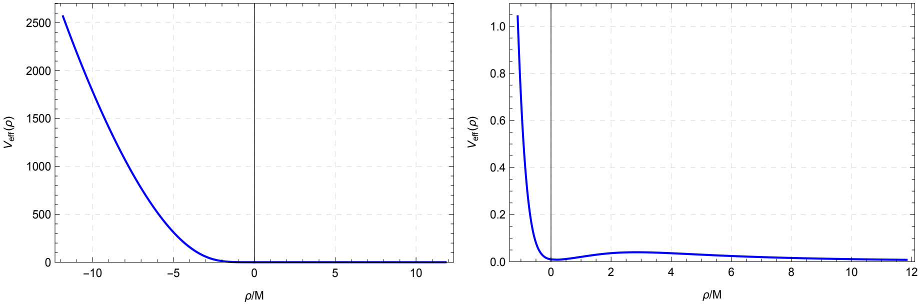


Size of the small circle: $\sim \dfrac{\rho_0}{2}$

Shadow of the Model IV

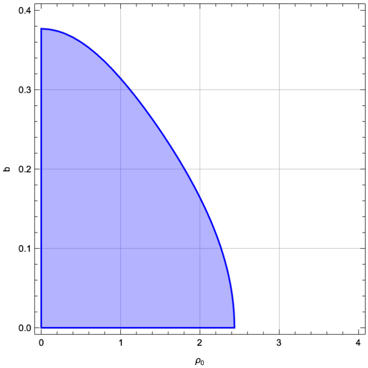
$g(\rho) = (g_{\text{sch}}(\rho) + b^2)\Delta^{-2}(\rho)$

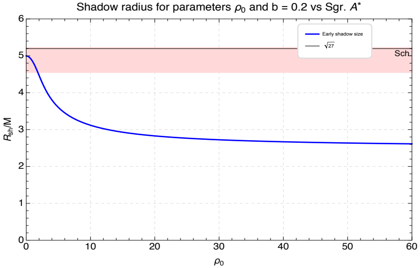
$$V_{\text{eff}}(\rho) = \left.\frac{g(\rho)}{r^2(\rho)}\right|_{\rho\to-\infty} \to \frac{(1+b^2)}{(-\rho)^2}\frac{16(-\rho)^4}{\rho_0^4} \sim \rho^2 \tag{20}$$

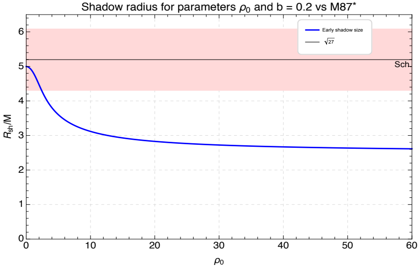


The effective potentials for massless geodesics. The left panel is presented for $\rho_0 = 1$ and the right is zoomed version of the left one. In both cases $b = 0.1$.

Shadow of the Model III







A brief overview of quasinormal modes (QNM)

QNM $=$ a dissipative characteristic.
Let x be some coordinate

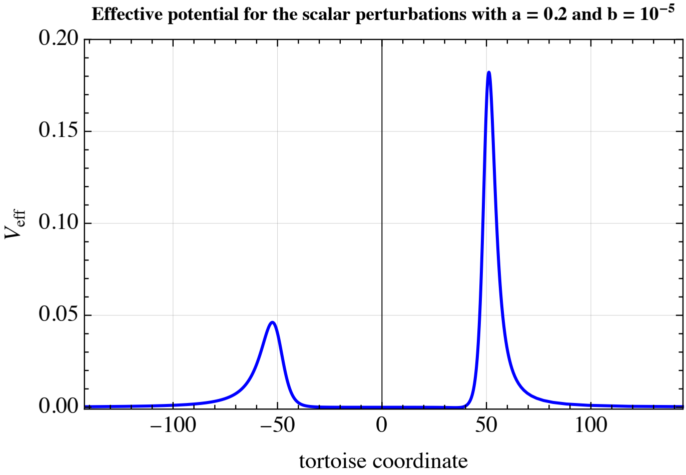
$$ds^2 = -g(x)dt^2 + \frac{dx^2}{f(x)} + r^2(x)d\Omega^2, \quad \Rightarrow$$

$$dz_{\text{tort}} = \frac{dx}{\sqrt{g(x)f(x)}} - \text{tortoise coordinate}, \quad z_{\text{tort}} \in (-\infty, \infty) \tag{21}$$

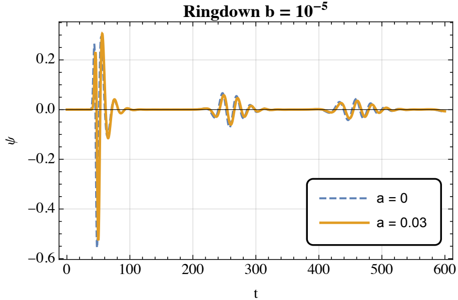
s – spin perturbations:

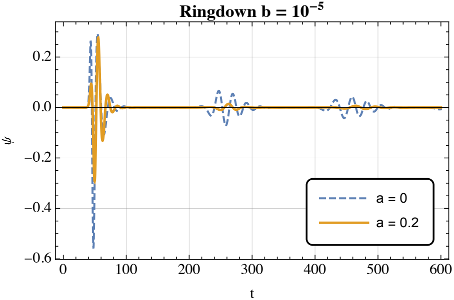
$$\begin{cases} \frac{\partial^2}{\partial t^2}\Psi(t, z_{\text{tort}}) - \frac{\partial^2}{\partial z_{\text{tort}}^2}\Psi(t, z_{\text{tort}}) + V_{\text{s}}(z_{\text{tort}})\Psi(t, z_{\text{tort}}) = 0 \\ \Psi(z_{\text{tort}}) \sim e^{i\omega(t - z_{\text{tort}})}, \quad z_{\text{tort}} \to \infty \\ \Psi(z_{\text{tort}}) \sim e^{-i\omega(t - z_{\text{tort}})}, \quad z_{\text{tort}} \to -\infty \end{cases}$$

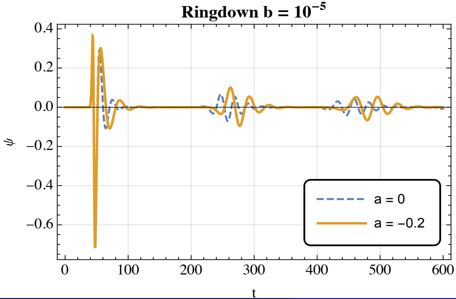
Echoes and QNM of the Model I

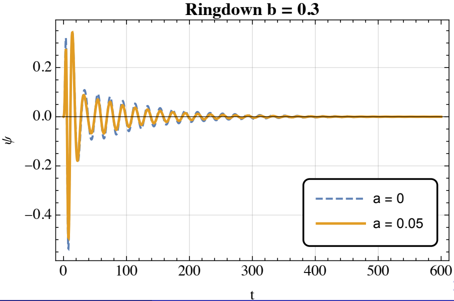


Echoes of the Model I

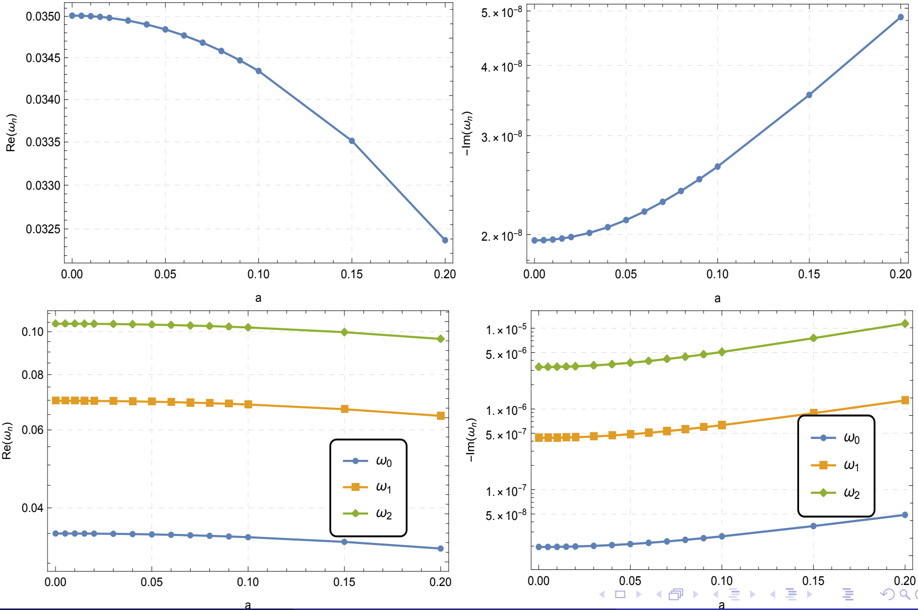








QNM of the Model I

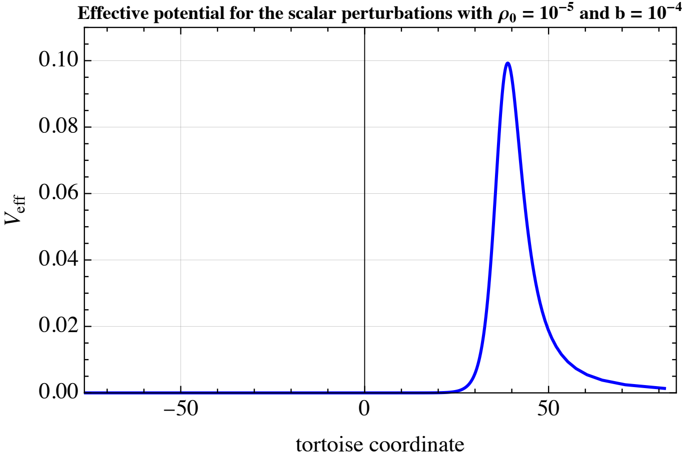


QNM of the Model I

n	a =0	a =0.01	a =0.02
0	0.0350092 - 1.9529*10^{-8}i	0.0350025 - 1.9597*10^{-8}i	0.0349824 - 1.9799*10^{-8}i
1	0.0698196 - 4.3914*10^{-7}i	0.0698062 - 4.4099*10^{-7}i	0.0697659 - 4.4654*10^{-7}i
2	0.104259 - 3.3048*10^{-6}i	0.104238 - 3.3218*10^{-6}i	0.104178 - 3.3728*10^{-6}i
3	0.138188 - 1.6173*10^{-5}i	0.138161 - 1.6272*10^{-5}i	0.138079 - 4.463*10^{-5}i

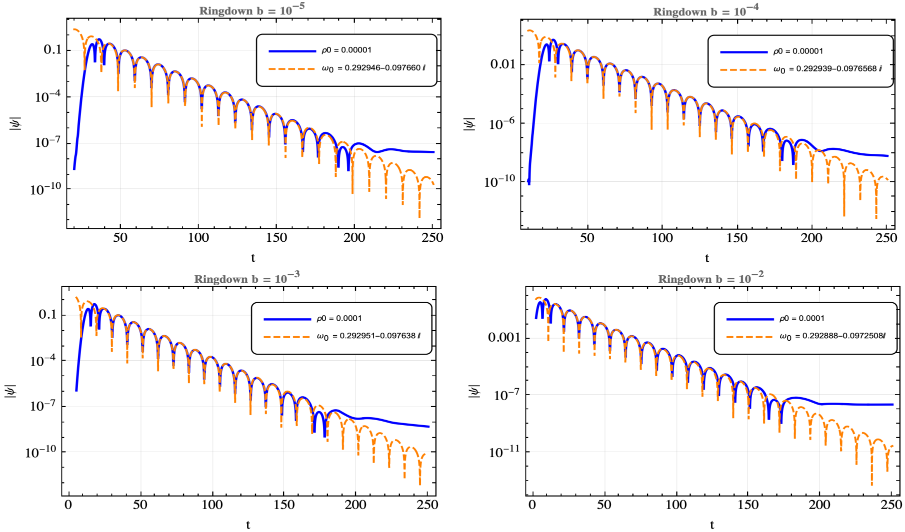
n	a =0.05	a =0.1	a =0.2
0	0.0348415 - 2.1226*10^{-8}i	0.0343408 - 2.6427*10^{-8}i	0.0323728 - 4.8763*10^{-8}i
1	0.0694845 - 4.857*10^{-7}i	0.0684837 - 6.3007*10^{-7}i	0.0645508 - 1.2763*10^{-7}i
2	0.103754 - 3.734*10^{-6}i	0.102248 - 5.083*10^{-6}i	0.0963285 - 1.1407*10^{-6}i
3	0.137509 - 1.87*10^{-5}i	0.135481 - 2.6749*10^{-5}	0.127516 - 6.5947*10^{-5}i

Ringdown and QNM of the Model II



Ringdown and QNM of the Model II

$$\omega_{\mathsf{sch},0} = 0.292936 - 0.097660i$$



Conclusions and next steps

Conclusions

- We analyzed the constraints on the asymptotic behavior of Schwarzschild metric deformations and discussed several test metrics.
- An effect of shadow decreasing over long observation times was discovered, which results from the presence of an asymmetric potential — that is, it serves as an optical analogue of the echo effect.
- It was found that the metric II is the most suitable for mimicking, as it allows one to reproduce not only the shadow but also the ringdown signal of a Schwarzschild black hole.

Next steps

- To analyze the ringdown stage and quasinormal modes of Models III and IV.
- Proceed to the study of rotating mimickers.

Thank you for your attention!

Hyperboloid method

Numerical computations $\Rightarrow$

$$\begin{cases} \text{Artificial boundary } |z_{\text{tort}}| = L \quad + \quad \text{Radiation boundary conditions} \\ \\ \text{Compact coordinate } x \quad + \quad \text{New time coordinate} \end{cases}$$

Let $x \in [a,\ b]$ be some compact coordinate

$$\begin{cases} z_{\text{tort}} = z_{\text{tort}}(x) \\ \\ t = \tau - H(x) \end{cases} \Rightarrow \frac{\partial}{\partial t} = \frac{\partial}{\partial \tau}, \quad \frac{\partial}{\partial z_{\text{tort}}} = \frac{1}{z'_{\text{tort}}(x)}\frac{\partial}{\partial x} + \frac{H'(x)}{z'_{\text{tort}}(x)}\frac{\partial}{\partial \tau}$$

Equation for the perturbation $\Rightarrow$

$$-\left(1 - \left(\frac{H'(x)}{z'_{\text{tort}}(x)}\right)^2\right)\partial_{\tau\tau}\Psi(t,\ x) - \frac{z''_{\text{tort}}(x)}{z'_{\text{tort}}(x)}\partial_x\Psi(t,x) + \frac{1}{(z'_{\text{tort}}(x))^2}\partial_{xx}\Psi(t,\ x) +$$

$$+ \frac{1}{z'_{\text{tort}}(x)}\partial_x\left(\frac{1}{z'_{\text{tort}}(x)}\right)\partial_\tau\Psi(t,\ x) + 2\frac{H'(x)}{z'_{\text{tort}}(x)}\partial_{\tau x}\Psi(t,\ x) - V(x)\Psi(t,\ x) = 0 \quad (22)$$

Hyperboloid method

Let introduce:

$$p_{\tau\tau}(x) = z'_{\text{tort}}(x) - \frac{H'(x)^2}{z'_{\text{tort}}(x)}, \quad p_{\tau x}(x) = \frac{H'(x)}{z'_{\text{tort}}(x)}, \quad p_{\tau}(x) = \partial_x \left(\frac{H'(x)}{z'_{\text{tort}}(x)} \right)$$

$$p_{xx}(x) = \frac{1}{z'_{\text{tort}}(x)}, \quad p_x(x) = -\frac{z''_{\text{tort}}(x)}{z'_{\text{tort}}(x)^2}, \quad \hat{V}(x) = z'_{\text{tort}}(x)\ V(x)$$

Main equation

$$-p_{\tau\tau}(x)\ \partial_{\tau\tau}\Psi(t,\ x)\ +\ 2p_{\tau x}(x)\ \partial_{\tau x}\Psi(t,\ x)\ +\ p_{xx}(x)\ \partial_{xx}\Psi(t,\ x)\ +$$

$$+\ p_x(x)\ \partial_x\Psi(t,\ x)\ +\ p_\tau\ \partial\tau\Psi(t,\ x)\ -\ \hat{V}(x)\Psi(t,\ x) = 0 \qquad (23)$$

Hyperboloid method

Let $\Phi(t,\ x) = \partial_\tau \Psi(t,\ x)$

$$\partial_\tau \underbrace{\begin{pmatrix} \Psi(t,\ x) \\ \Phi(t,\ x) \end{pmatrix}}_{U} = i \underbrace{\frac{1}{i} \begin{pmatrix} 0 & 1 \\ \mathbf{L}_1 & \mathbf{L}_2 \end{pmatrix}}_{\mathbf{L}} \begin{pmatrix} \Psi(t,\ x) \\ \Phi(t,\ x) \end{pmatrix}$$

$$\mathbf{L}_1 = \frac{1}{p_{\tau\tau}(x)} \left[\partial_x \left(p_{xx}(x) \partial_x \right) - \hat{V}(x) \right], \quad \mathbf{L}_2 = \frac{1}{p_{\tau\tau}(x)} \left[2 p_{\tau x}(x) \partial_x + \partial_x p_{\tau x}(x) \right] \tag{24}$$

QNM as eigenvalues

$$U(t,\ x) = e^{i\omega t} \hat{U}(x) \quad \Rightarrow \quad \mathbf{L} \hat{U} = \omega \hat{U} \tag{25}$$

The operator $\mathbf{L}$ is not self-adjoint in the energy norm. Therefore, its eigenvalues are complex.

Energy norm: $E(\Psi) = \int_{\Sigma_\tau} T_{ab}(\Psi) t^a n^b d\Sigma_\tau = \frac{1}{2} \int_a^b \left(p_{\tau\tau}(x) |\Phi| + p_{xx}(x) |\partial_x \Psi| + \hat{V}(x) |\Psi| \right) dx$

$|| \hat{U} || = || \begin{pmatrix} \Psi \\ \Phi \end{pmatrix} || = E(\Psi)$

Hyperboloid method

Let introduce:

$$u(t,\ x) = \partial_x \Psi(t,\ x), \quad \pi(t,\ x) = p_{\tau\tau}(x)\ \partial_\tau \Psi(t,\ x)\ -\ p_{\tau x}(x)\partial_x \Psi(t,\ x) \tag{26}$$

Evolution equation

$$\begin{cases} \partial_\tau \pi = \partial_x \left[\dfrac{1}{p_{\tau\tau}} (u\ +\ p_{\tau x}\pi) \right] \\ \partial_\tau u = \partial_x \left[\dfrac{1}{p_{\tau\tau}} (p_{\tau x} u\ +\ \pi) \right] \\ \partial_\tau \Psi = \dfrac{1}{p_{\tau\tau}} (p_{\tau x} u\ +\ \pi) \\ \\ + \text{ initial conditions} \end{cases} \tag{27}$$

Minimal gauge for the height function

$$z_{\text{tort}}(x) = \underbrace{z_{\text{tort}}^{(\text{sing},+)}(x)}_{\text{singular at } x\to b} + \underbrace{z_{\text{tort}}^{(\text{sing},-)}(x)}_{\text{singular at } x\to a} + z_{\text{tort}}^{\text{reg}}(x)$$

i) The in – out strategy. Consider ingoing null geodesics $v = t + z_{\text{tort}}(x)$

$$\left.\frac{dv}{dx}\right|_{x\to a} \sim 2z_{\text{tort},\,x}^{(\text{sing},-)}(x) \Rightarrow v \sim \tau + 2z_{\text{tort}}^{(\text{sing},-)}(x) \Rightarrow t = \tau - \underbrace{\left(-2z_{\text{tort},\,x}^{(\text{sing},-)}(x) + z_{\text{tort}}(x)\right)}_{H^{\text{in - out}}(x)}$$

$$H^{\text{in - out}}(x) = z_{\text{tort}}^{(\text{sing},+)}(x) - z_{\text{tort}}^{(\text{sing},-)}(x) + z_{\text{tort}}^{\text{reg}}(x)$$

ii) The out – in strategy. Consider outgoing null geodesics $u = t - z_{\text{tort}}(x)$

$$\left.\frac{du}{dx}\right|_{x\to b} \sim -2z_{\text{tort},\,x}^{(\text{sing},+)}(x) \Rightarrow v \sim \tau - 2z_{\text{tort}}^{(\text{sing},+)}(x) \Rightarrow t = \tau - \underbrace{\left(2z_{\text{tort},\,x}^{(\text{sing},+)}(x) - z_{\text{tort}}(x)\right)}_{H^{\text{out - in}}(x)}$$

$$H^{\text{out - in}}(x) = z_{\text{tort}}^{(\text{sing},+)}(x) - z_{\text{tort}}^{(\text{sing},-)}(x) - z_{\text{tort}}^{\text{reg}}(x)$$

Echoes and QNM of the Model I

Compact coordinate:

$$-a \;+\; b\sinh x = \sqrt{g_{\mathsf{sch}}(\rho)} \quad\Rightarrow\quad x \in \left[\operatorname{arcsinh}\frac{a-1}{b},\ \operatorname{arcsinh}\frac{a+1}{b}\right] \tag{28}$$

Tortoise coordinate:

$$z_{\mathsf{tort},\ x}(x) = \frac{4}{\left(1-(b\sinh x - a)^2\right)^2} = \underbrace{\frac{2-(b\sinh x - a)}{(1-(b\sinh x - a))^2}}_{z^{(\mathsf{sing},+)}_{\mathsf{tort},\ x}(x)} + \underbrace{\frac{2+(b\sinh x - a)}{(1+(b\sinh x - a))^2}}_{z^{(\mathsf{sing},-)}_{\mathsf{tort},\ x}(x)} \tag{29}$$

Height function:

$$H_x(x) = \frac{2-(b\sinh x - a)}{(1-(b\sinh x - a))^2} - \frac{2+(b\sinh x - a)}{(1+(b\sinh x - a))^2} \tag{30}$$

$$\begin{aligned} &\text{Old characteristics:} \quad z_{\mathsf{tort}}(x) \;\pm\; t = \text{const} \\ &\text{New characteristics:} \quad z_{\mathsf{tort}}(x) \;\pm\; (\tau \;-\; H(x)) = \text{const} \end{aligned} \tag{31}$$

Ringdown and QNM of the Model II

Compact coordinate:

$$b \sinh x = \sqrt{g_{\mathsf{sch}}(\rho)} \quad \Rightarrow \quad x \in \left[-\operatorname{arcsinh}\frac{1}{b},\ \operatorname{arcsinh}\frac{1}{b}\right] \tag{32}$$

Tortoise coordinate:

$$z_{\mathsf{tort},\ x}(x) = \frac{4}{\left(1 - b^2 \sinh^2 x\right)^2} \frac{1}{\sqrt{\Delta(\rho(x))}} = \underbrace{\frac{2 - b \sinh x}{(1 - b \sinh x)^2}}_{z^{(\mathsf{sing},+)}_{\mathsf{tort},\ x}(x)} +$$

$$+ \underbrace{\frac{4}{\left(1 - b^2 \sinh^2 x\right)^2} \left[-\frac{2\rho(x)}{\rho_0} - \frac{3}{4}\frac{\rho_0}{\rho(x)}\right]}_{z^{(\mathsf{sing},-)}_{\mathsf{tort},\ x}(x)} + z^{reg}_{\mathsf{tort},x}(x) \tag{33}$$

Height function:

$$H_x(x) = \frac{4}{\left(1 - b^2 \sinh^2 x\right)^2} \frac{1}{\sqrt{\Delta(\rho(x))}} - 2\frac{2 - b \sinh x}{(1 - b \sinh x)^2} \tag{34}$$

$\rho(x) = \dfrac{2b \sinh x}{1 - b^2 \sinh^2 x} + \ln \dfrac{1 + b \sinh x}{1 - b \sinh x}$