The Memory Effect: from numerical to analytical

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Q.-L. Zhao, M. Elbistan, J. Balog

Abstract: The motion of particles hit by a burst of gravitation waves is studied. With Gaussian seed solution and its derivatives only numerical solutions are obtained, however ingeneous approximations of the wave profile by (derived) Pöschl-Teller and Scarf potentials allow to find analytic solutions. The displacement effect Memory Effect of Zel'dovich, Polnarev is recovered when the wave parameters take certain "magic" values.

P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, "Soft gravitons and the memory effect for plane gravitational waves," Phys. Rev. D **96** (2017) no.6, 064013 [arXiv:1705.01378 [gr-qc]].

P. M. Zhang, P. A. Horvathy, *"Displacement within velocity effect in gravitational wave memory,"* Annals Phys. **470** (2024), 169784 [arXiv:2405.12928 [gr-qc]].

P. M. Zhang, Q. L. Zhao, J. Balog and P. A. Horvathy, *"Displacement memory for flyby,"* Annals Phys. **473** (2025), 169890 [arXiv:2407.10787 [gr-qc]].

P. M. Zhang, Q. L. Zhao, M. Elbistan and P. A. Horvathy, "Gravitational wave memory: further examples," [arXiv:2412.02705 [gr-qc]].

P. M. Zhang, Z. K. Silagadze and P. A. Horvathy, *"Flyby-induced displacement: analytic solution,"* Phys. Lett. **B** 868 (2025) 139687 [arXiv:2502.01326 [gr-qc]].

E. Catak, M. Elbistan and M. Mullahasanoglu, "Displacement memory effect from supersymmetry," Eur. Phys. J. Plus **140** (2025) no.6, 540 [arXiv:2504.05043 [gr-qc]].

Road map:

- Memory Effect

  velocity VM Ehlers-Kundt, Braginsky-Thorne . . . p.3

  displacement DM (Zel'dovich-Polnarev) p.4

- Sandwich waves p.5

- Geodesics in Brinkmann coordinates p.6.

- Gaussian profile p.8

- Pöschl-Teller potential p.10

- GW in D = 3+1 dim p.13

- DM/VM for flyby p.14

- Derived Pöschl-Teller for flyby p.16

- Analytical solution for dPT p.18

- Scarf approximation p.19

- Hodograph p.23

- SUSY QM & Memory p.26

- Conclusion p. 33

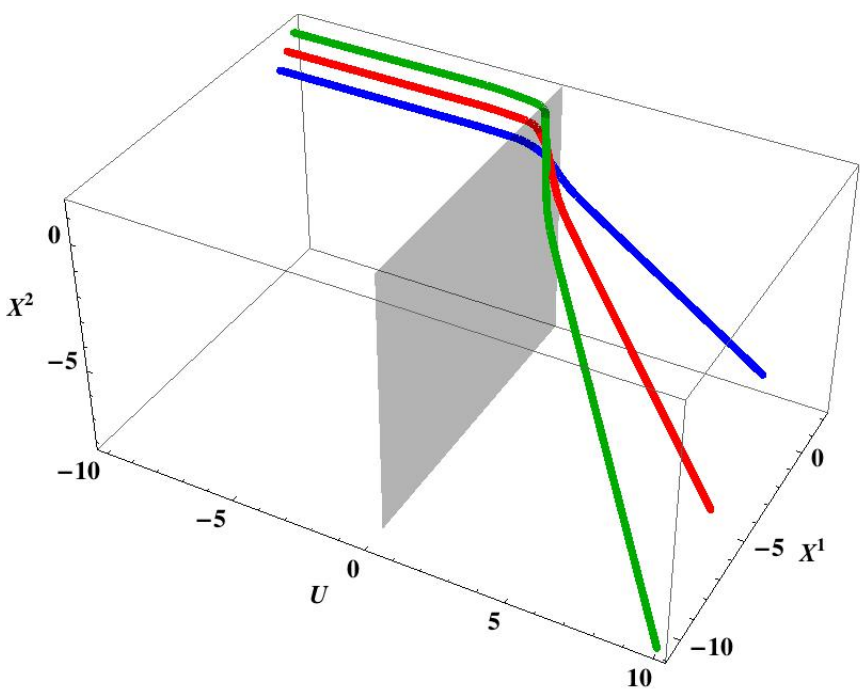
Memory effect

Velocity Effect VM

J. Ehlers and W. Kundt, *"Exact solutions of the gravitational field equations," 1962*,

Braginsky and Thorne, *"Gravitational-wave burst with memory and experimental prospects," Nature (London)* **327** *123 (1987).*

Grishchuk and Polnarev, *"Gravitational wave pulses with 'velocity coded memory'," Sov. Phys. JETP* **69** *(1989) 653 [Zh. Eksp. Teor. Fiz.* **96** *(1989) 1153].*



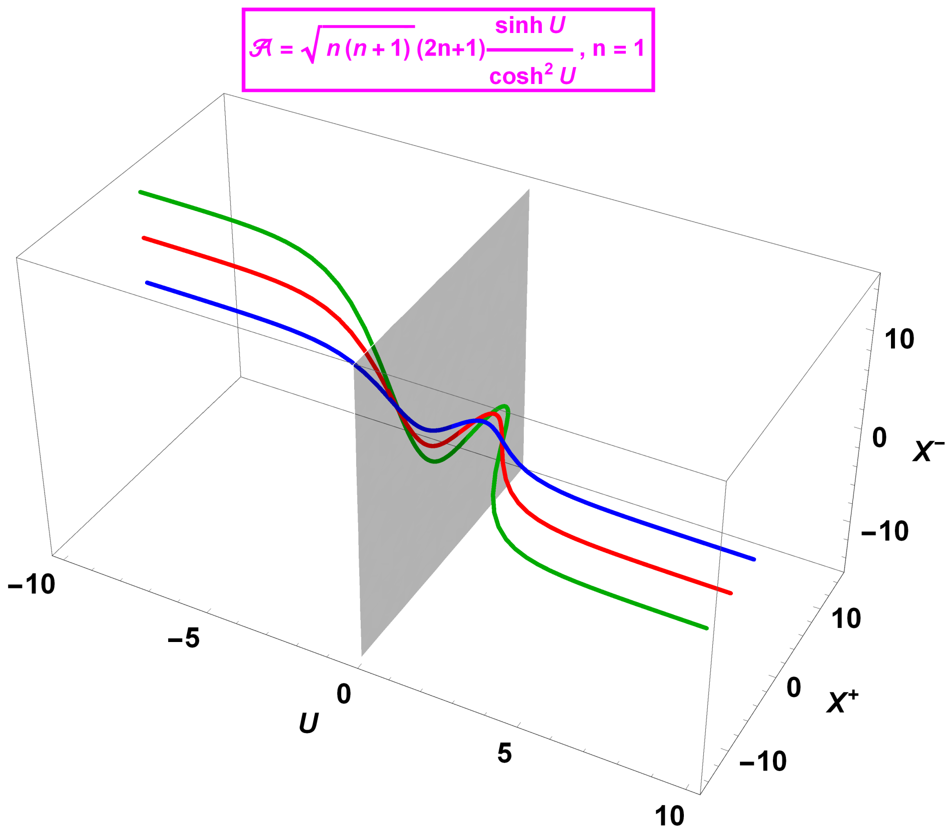
Particles hit by GW fly apart with non-zero constant velocity. "billiard"

Displacement Effect DM

Zel'dovich, Polnarev

"Radiation of gravitational waves by a cluster of super-dense stars," Astron. Zh. **51**, *30 (1974)*

> *. . . [for] two noninteracting bodies [. . .] the distance should change, and this effect might possibly serve as a nonresonance detector. [. . .] their relative velocity will become vanishingly small as* **flyby** *concludes.*

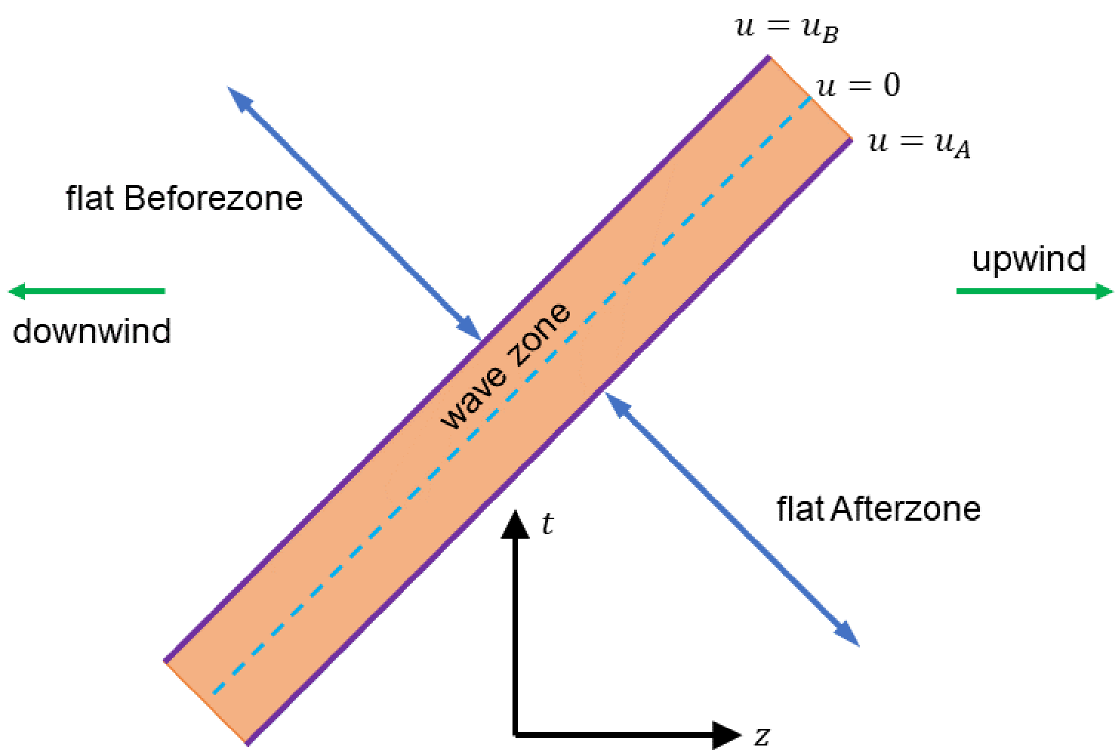


DM for "magic" wave amplitudes.

Braginsky & Grishchuk

"Kinematic resonance and the memory effect in free mass gravitational antennas," Zh. Eksp. Teor. Fiz. **89** *744-750 (1985)*

Sandwich wave: Spacetime non-flat only in short interval $u_B \leq u \leq u_A$ of retarded time [Wave-zone]. Flat both in Beforezone $u < u_B$ that the wave has not reached yet, and in Afterzone $u_A < u$ where has already passed,



(u flows from left to right, whereas wave advances from right to left.)

Geodesics in Brinkmann* coordinates

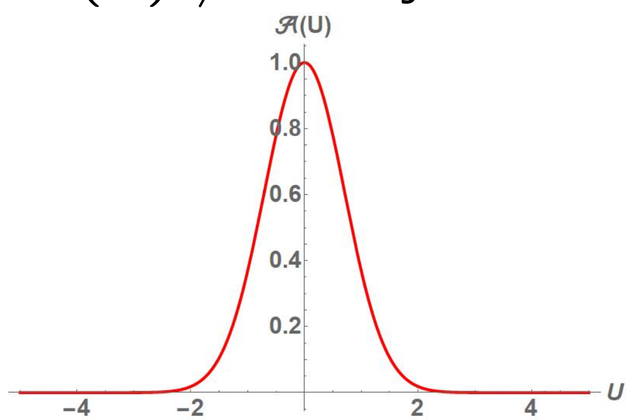
(toy model in $D = 1$ space + 2 lighlike dimensions.) plane GW

$$g_{\mu\nu}X^{\mu}X^{\nu} = dX^2 + 2dUdV - \mathcal{A}(U)X^2dU^2 \quad (1)$$

X = transverse, U, V light-cone coords.

Sandwich wave: $\mathcal{A}(U) \neq 0$ only in "wave zone"

$U_B < U < U_A$.



For non-tachyonic geodesic: Jacobi invariant

$$\mathfrak{m}^2 = g_{\mu\nu}\dot{X}^{\mu}\dot{X}^{\nu} = \text{const} \leq 0\,. \quad (2)$$

Massive: $\mathfrak{m}^2 < 0$, Lightlike $\mathfrak{m}^2 = 0$.

* M. W. Brinkmann, "Einstein spaces which are mapped conformally on each other," Math. Ann. **94** (1925) 119–145.

Lightlike geodesics $\mathfrak{m}^2 = 0$:

$$\frac{d^2X}{dU^2} + \frac{1}{2}\mathcal{A}(U)X = 0\,, \tag{3a}$$

$$\frac{d^2V}{dU^2} - \frac{1}{4}\frac{d\mathcal{A}}{dU}(X)^2 - \tfrac{1}{2}\mathcal{A}\,\frac{d(X^2)}{dU} = 0\,. \tag{3b}$$

$V(U)$ horizontal lift of $X(U)$ Coordinate X decoupled from V. Projection into transverse space is V-independent. Conversely, lightlike geo determined by eqn. (3a) with U viewed as Newtonian time *.

Eqn (3a) Sturm-Liouville $\Rightarrow$ no analytic solution in general.

*L. P. Eisenhart, *"Dynamical trajectories and geodesics"*, Annals. Math. **30** 591-606 (1928).

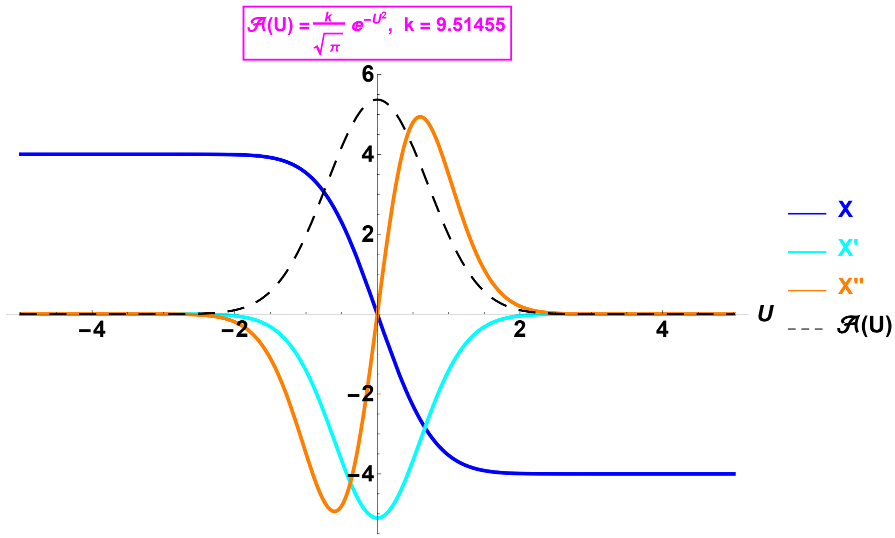
C. Duval, G. Burdet, H. P. Kunzle and M. Perrin, *"Bargmann Structures and Newton-cartan Theory,"* Phys. Rev. D **31** (1985), 1841-1853

C. Duval, G. W. Gibbons and P. Horvathy, *"Celestial mechanics, conformal structures and gravitational waves,"* Phys. Rev. D **43** (1991), 3907-3922 [arXiv:hep-th/0512188 [hep-th]].

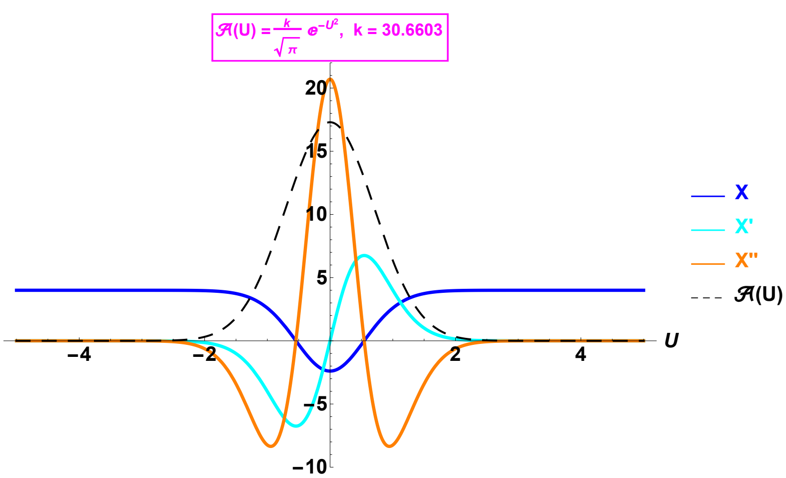
Gaussian profile

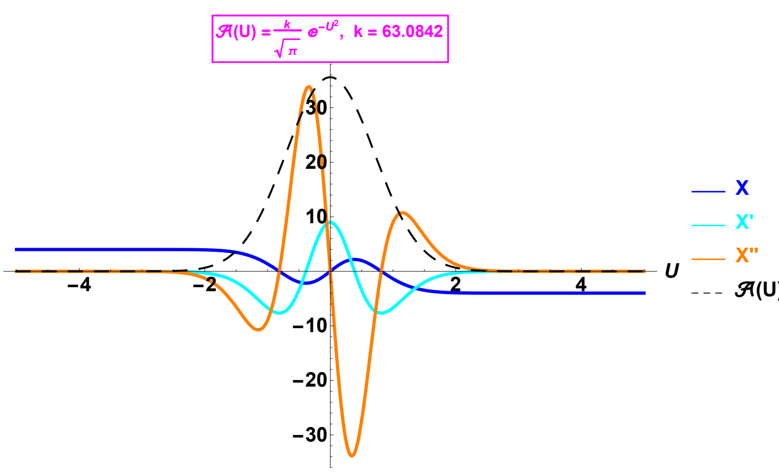
$$\mathcal{A}^G(U) = \frac{k}{\sqrt{\pi}} e^{-U^2}, \tag{4}$$

Duval et al $\rightsquigarrow$ for general amplitude k particles fly apart with non-zero velocity : **VM** . **No DM ?** Surprise:



Fine-tuning amplitude, $k = k_{crit}$; $\Rightarrow$ **DM** *with* $\mathrm{m} = 1$ *half-wave.* X : **trajectory** , dX/dU : **velocity** , $\mathbf{d^2X}/dU^2$: **force**.





DM *with* $\mathrm{m} = 2$ *and* $\mathrm{m} = 3$ **half-waves** *as trajectories.*

“Miracle” explained by : at (approximate) boundaries of Wavezone $U_B < U < U_A$ both

velocity & force vanish

Outside Wavezone motion governed by Newton’s 1st law.

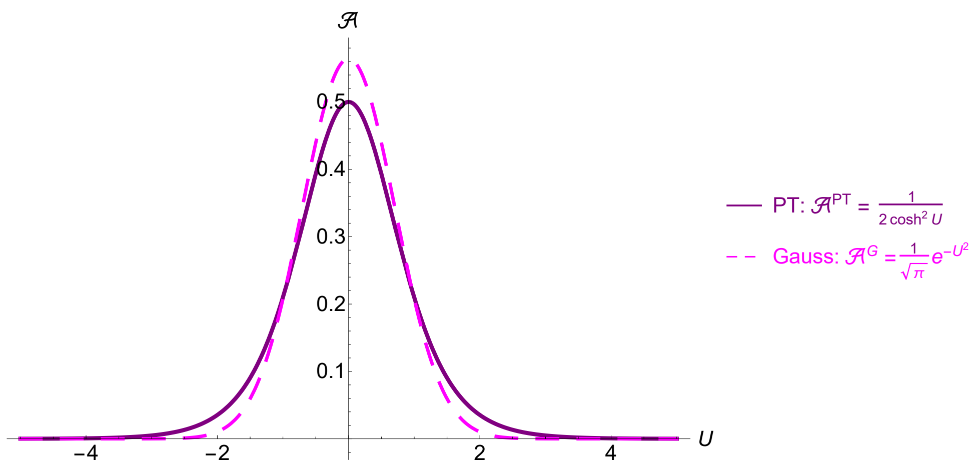
DM trajectories consist of $\mathbf{m}$ half-waves. Outgoing position depends on parity:

$$X_{out} = (-1)^m X_{in} \,. \tag{5}$$

Pöschl-Teller profile

No analytic solutions for Gauss. Shape of $\mathcal{A}^G$ reminiscent of Pöschl-Teller (PT) potential,

$$\mathcal{A}^{PT}(U) = \frac{k}{2\cosh^2 U}\,, \qquad (6)$$



Gaussian bell (dashed) well approximated by Pöschl-Teller potential (6) *(solid line), for which* (3a) *has analytic solutions.*

$$k_m = 4m(m+1) \qquad (7)$$

$$\boxed{\frac{d^2X}{dU^2} + \frac{m\,(m+1)}{\cosh^2 U}X = 0\,.} \qquad (8)$$

$\psi \rightsquigarrow X, \quad x \rightsquigarrow U$ time-indept Schrödinger eqn with $E = 0$ energy. (cf. pp.28-32).

Particle at rest before burst arrives:

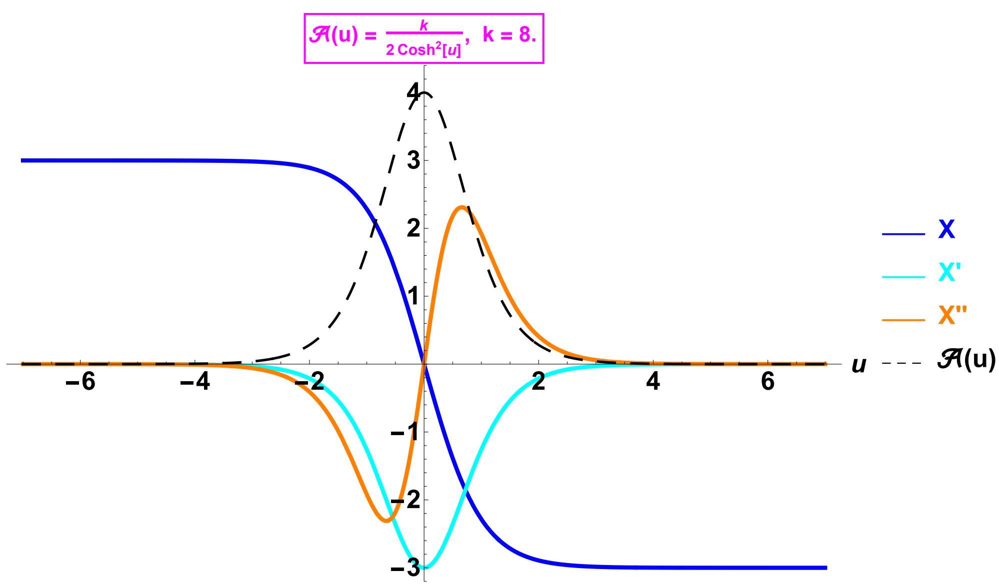
$$X(U=-\infty)=X_0, \quad \dot{X}(U=-\infty)=0\,. \tag{9}$$

Put $t=\tanh(U)$ into (8) $\rightsquigarrow$ Legendre eqn,

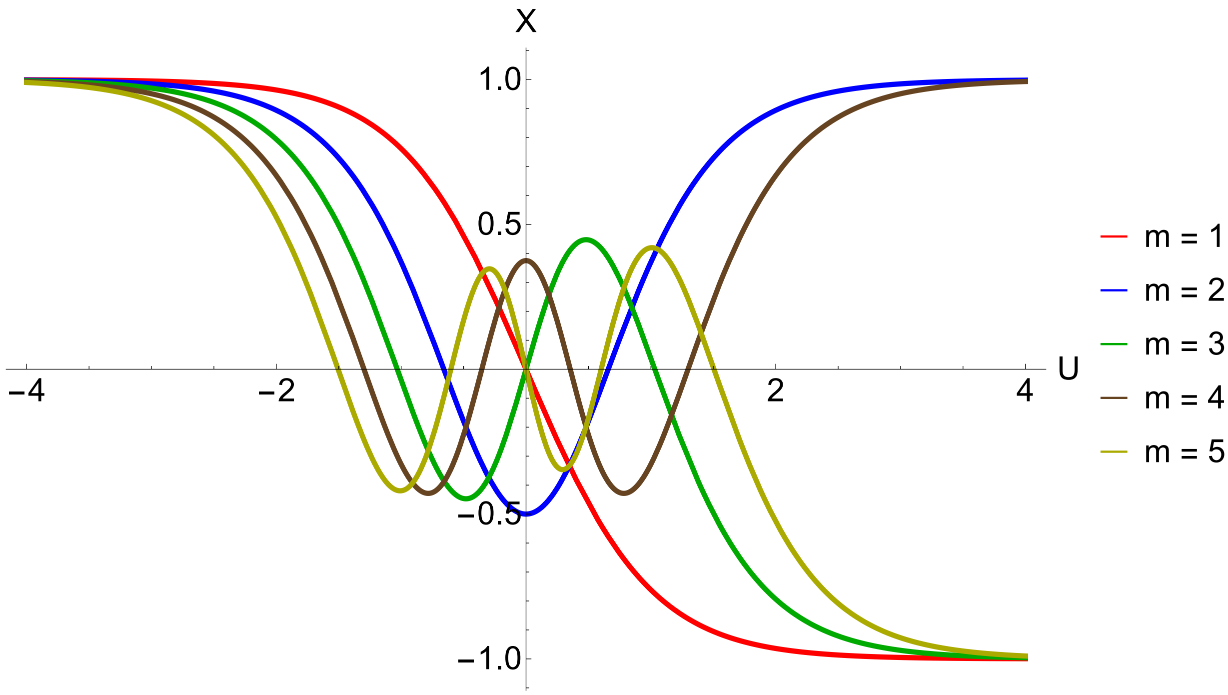
$$\left(1-t^2\right)\frac{d^2X}{dt^2}-2t\frac{dX}{dt}+m\,(m+1)\,X=0\,. \tag{10}$$

DM requires $X(U)\to$ const for $U\to\infty$ $\Rightarrow$ solution extends to $t=\pm1$ $\rightsquigarrow$ m positive integer $\Rightarrow$ solution propto Legendre polynomial,

$$X_m(U)=P_m(\tanh U), \quad m=1,\,2,\,\dots\,, \tag{11}$$

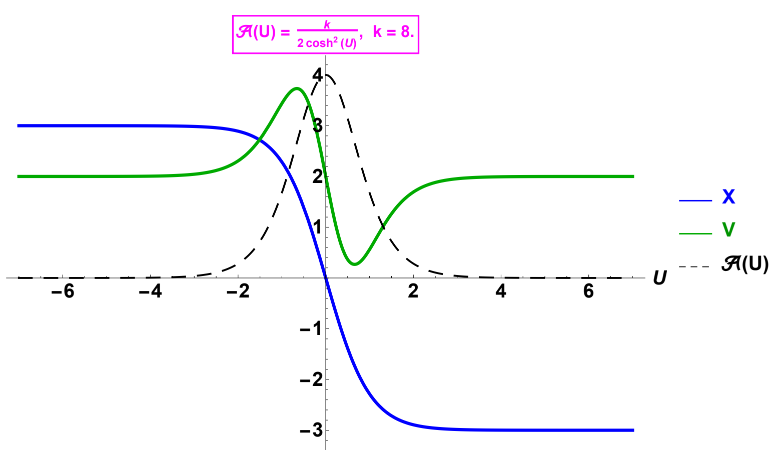


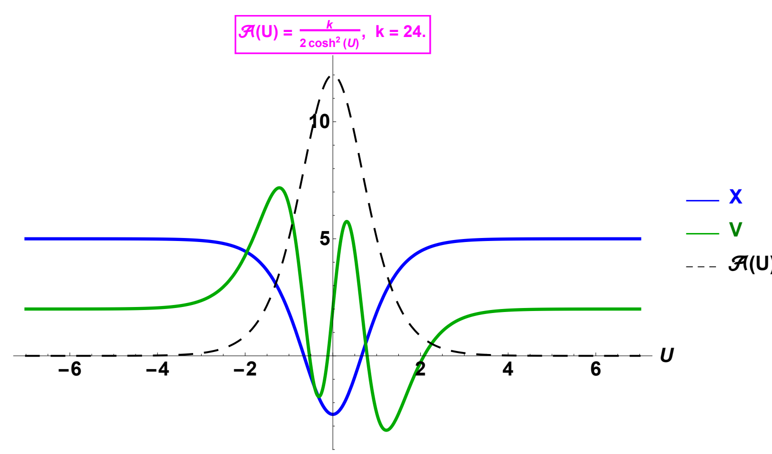
For Pöschl-Teller with $k_1=8$, transverse trajectory consistent with **DM** . $\mathbf{m=1}$.



DM *trajectories for PT profile with* $\mathbf{m = 1, \ldots, 5}$ *half-waves.* $X_{out} = (-1)^m X_{in} \sim$ *time-inversion symmetry*

N.B. "vertical" component ~ given by minus classical Hamiltonian action calculated along transverse trajectory





GW in $3+1$ dimensions

Coords X^1, X^2, U, V, with U, V lightlike. Metric :

$$\delta_{ij}\, dX^i dX^j + 2dU dV + K_{ij}(U) X^i X^j\, dU^2, \quad (12a)$$

$$K_{ij}(U)X^iX^j = \tfrac{1}{2}\mathcal{A}_+(U)\Big((X^1)^2 - (X^2)^2\Big) + \mathcal{A}_\times(U)\, X^1 X^2, \quad (12b)$$

Lightlike Geodesics for linearly polarized ($\mathcal{A}_\times \equiv 0$) GW :

$$\frac{d^2X^+}{dU^2} - \frac{1}{2}\mathcal{A}(U)X^+ = 0, \quad (13a)$$

$$\frac{d^2X^-}{dU^2} + \frac{1}{2}\mathcal{A}(U)X^- = 0, \quad (13b)$$

$$\frac{d^2V}{dU^2} + \frac{1}{4}\frac{d\mathcal{A}}{dU}\Big((X^+)^2 - (X^-)^2\Big) + \mathcal{A}\Big(X^+\frac{dX^+}{dU} - X^-\frac{dX^-}{dU}\Big) = 0\,. \quad (13c)$$

$X^{1,2}$-components decoupled from V. Projection of $4D$ worldline to transverse plane (X^1, X^2) independent of $V(U)$. Eqn (13c) $\sim$ horizontal lift.

In Eisenhart-Duval framework: (13a)-(13b) $\sim$ repulsive/attractive harmonic force with (time-dept) frequency

$$\omega^2(U) = \mathcal{A}(U)$$

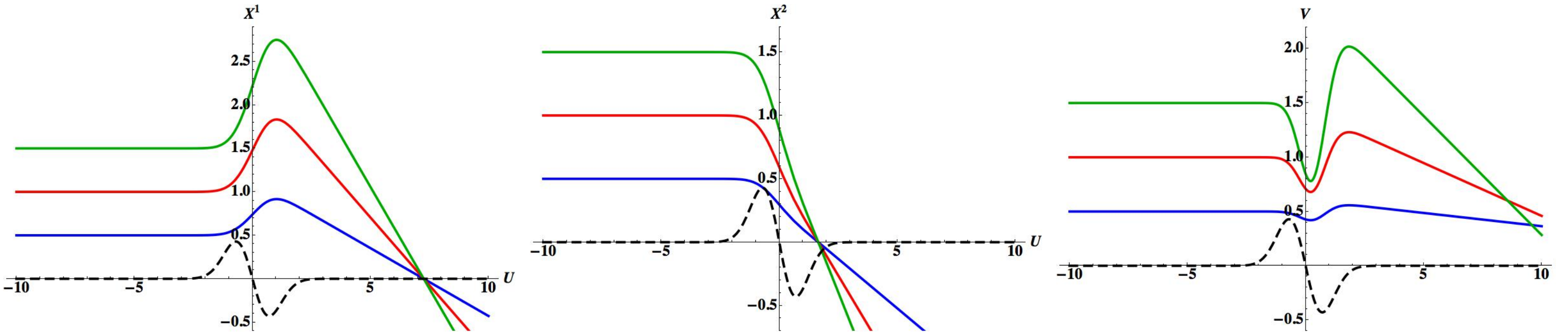
$\rightsquigarrow$ Sturm-Liouville.

DM for flyby

Toy model * : Profile for **flyby** proportional to *1st derivative* of Gaussian,

$$\mathcal{A}(U) = \frac{1}{2}\frac{d(e^{-U^2})}{dU}\,. \qquad (14)$$

No analytic solution. Numerically found geodesics :

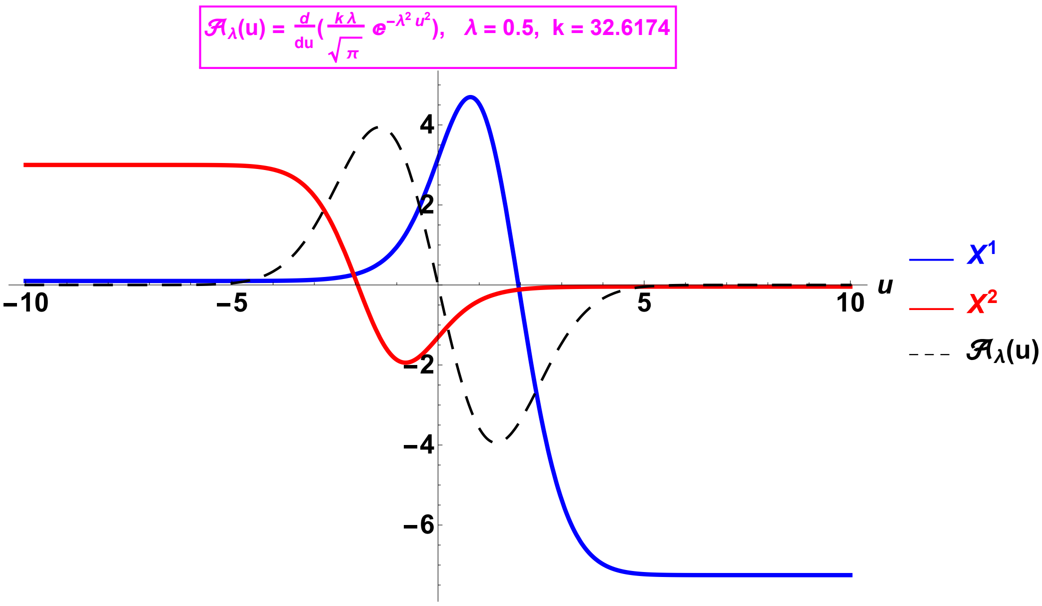


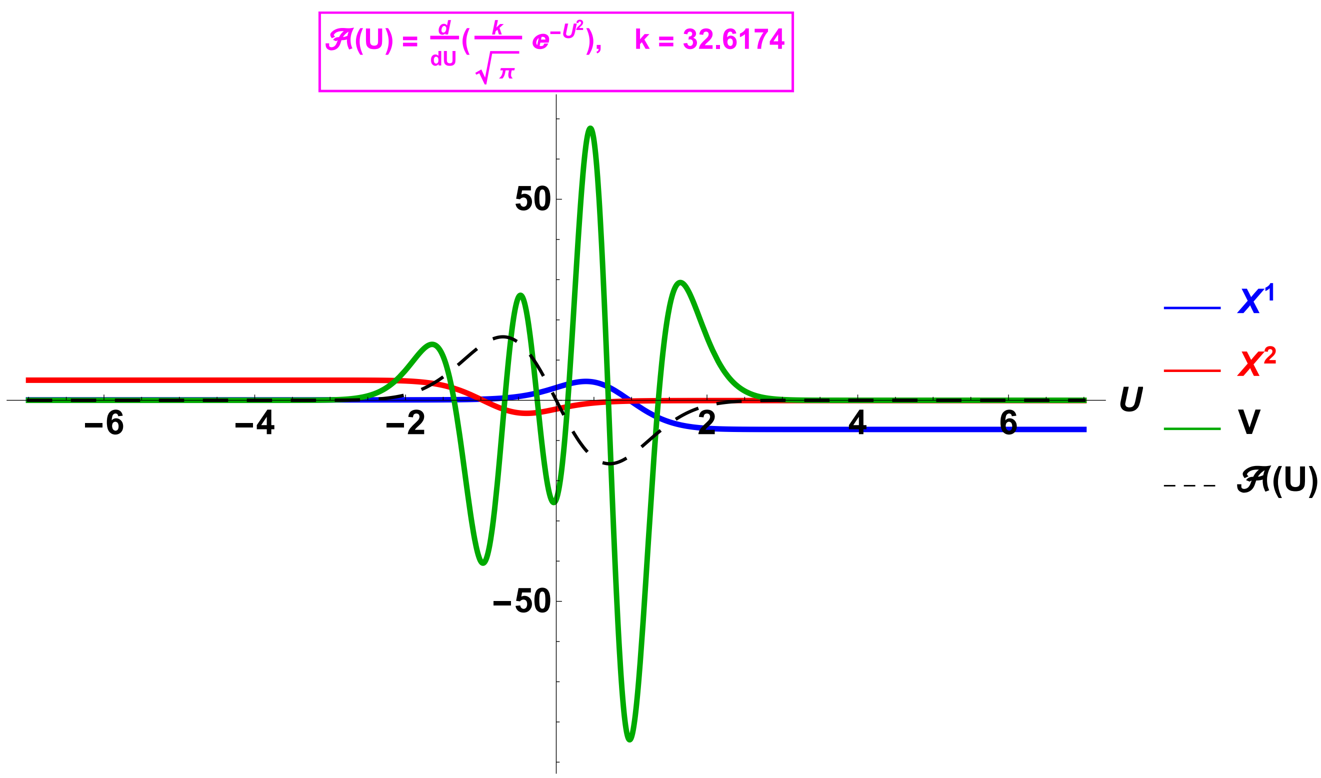
Geodesics for flyby profile (14) with amplitude $k = 1$.

Can VM become **DM** ???

*P. M. Zhang, C. Duval, G. W. Gibbons and P. A. Horvathy, *"Soft gravitons and the memory effect for plane gravitational waves,"* Phys. Rev. D **96** (2017) no.6, 064013[arXiv:1705.01378 [gr-qc]].

Miracle ! for (numerically found) "magic" parameters ⇝ DM for both components !





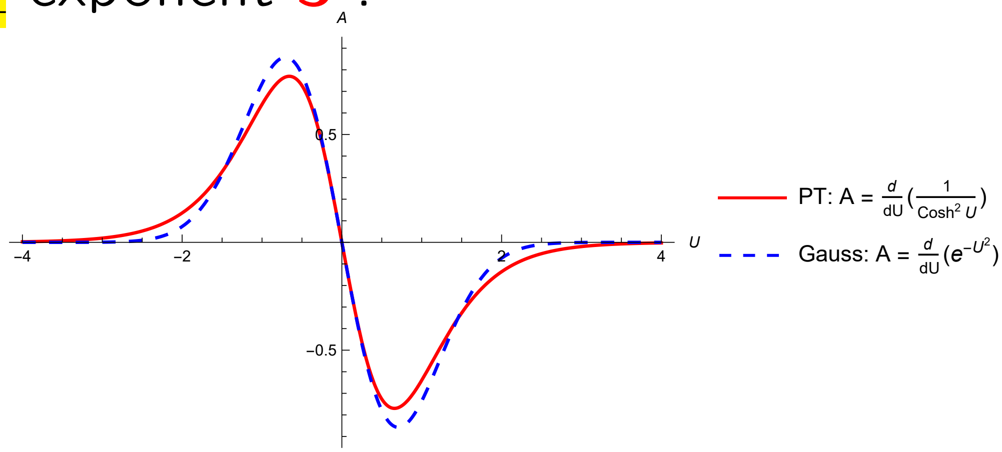
DM for both components !

Derived Pöschl-Teller (dPT) profile

$$K_{ij}(U)X^iX^j = \mathcal{A}^{dPT}\Big((X^1)^2 - (X^2)^2\Big), \qquad (15)$$

$$\mathcal{A}^{dPT}(U) = \frac{d}{dU}\left(\frac{g}{2\cosh^2 U}\right) = -g\frac{\sinh U}{\cosh^{\mathbf{3}} U}\,. \qquad (16)$$

N.B. exponent **3** !



Derivative of Pöschl-Teller potential is good approximation of d-Gaussian, (14)*.*

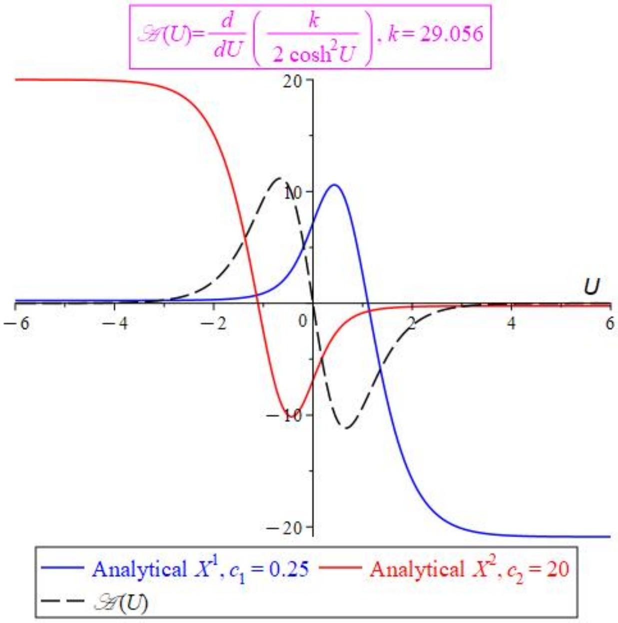
Approximate semi-analytical solution*

For dPT profile (16), putting $t = \tanh U$ eqns (13a)-(13b) become cf. (10)

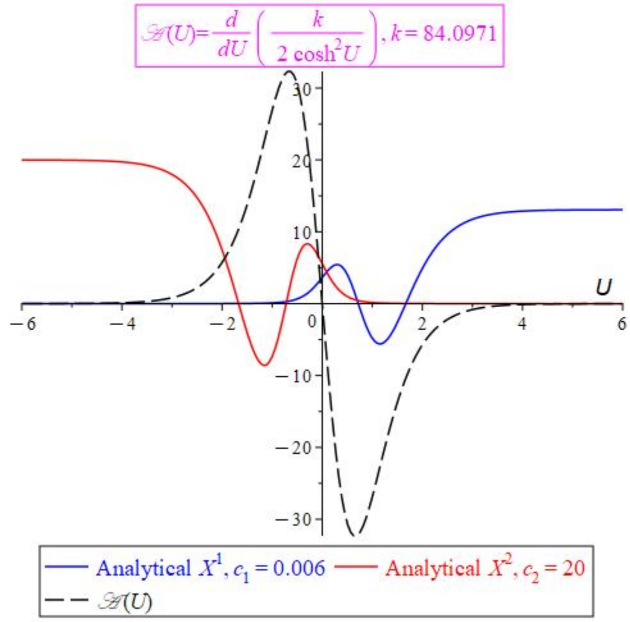
$$(1-t^2)\frac{\mathrm{d}^2X^\pm}{\mathrm{d}t^2} - 2t\frac{\mathrm{d}X^\pm}{\mathrm{d}t} \pm \frac{k}{2}tX^\pm = 0\,, \qquad (17)$$

where $\pm$ refers to upper/lower components.

*P. M. Zhang, Q. L. Zhao, J. Balog and P. A. Horvathy, *"Displacement memory for flyby,"* Annals Phys. **473** (2025), 169890 [arXiv:2407.10787 [gr-qc]].

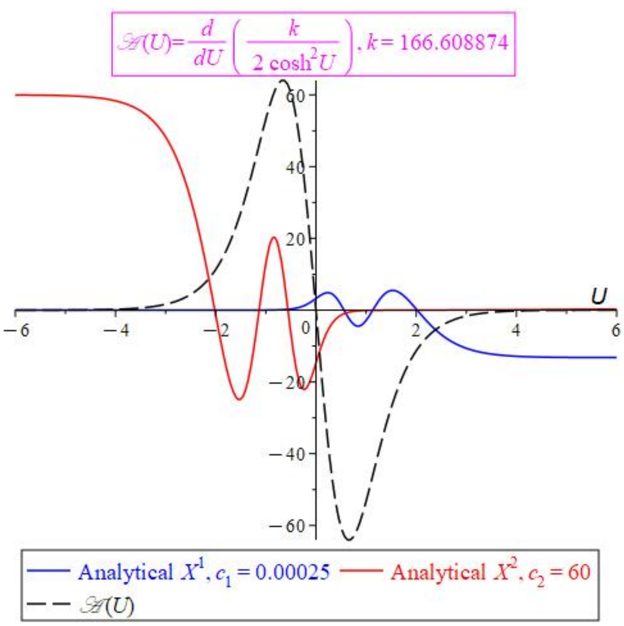


m = 1

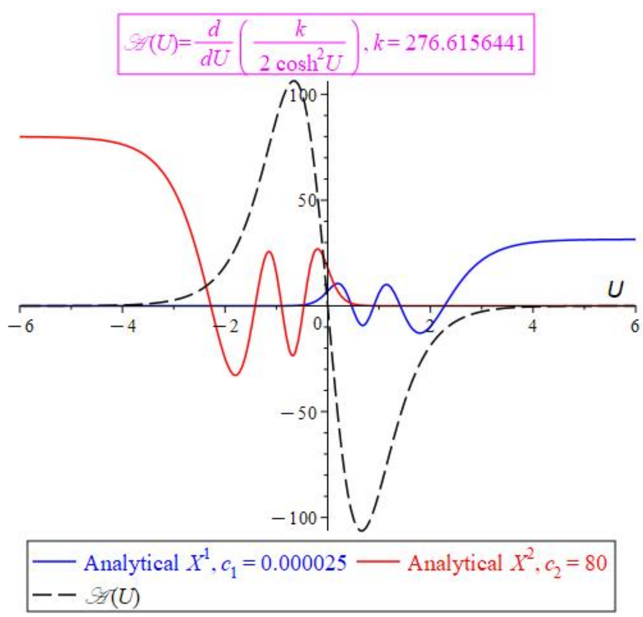


m = 2

For dPT-flyby profile (16), both numerical and (semi-) analytical trajectories exhibit **DM** *for both components.*



m = 3



m = 4

Analytic dPT trajectories for critical amplitude (i) $k_{crit} = 166....$ *& (ii)* $k_{crit} = 276....$ *in derivative-PT approx have half-wave numbers* $\mathbf{m=3}$ *and* $\mathbf{m=4}$.

Analytical solution for dPT*

Eqns (13a)-(13b) with dPT profile (16)

$$\frac{d^2}{dU^2}X^+(U) - \frac{1}{2}k\left(\tanh U - \tanh^3 U\right)X^+(U) = 0, \quad (18)$$

$$\frac{d^2}{dU^2}X^-(U) + \frac{1}{2}k\left(\tanh U - \tanh^3 U\right)X^-(U) = 0, \quad (19)$$

solved (computer software) $\rightsquigarrow$ Heun fcts,

$$X^+(U) = c_1 \mathrm{HeunC}\left(0, 0, 0, k, -\frac{k}{2}, \frac{\tanh U + 1}{2}\right) + c_2 \mathrm{HeunC}\left(0, 0, 0, k, -\frac{k}{2}, \frac{\tanh U + 1}{2}\right) \times \left(\int -\frac{1}{\mathrm{HeunC}^2\left(0, 0, 0, k, -\frac{k}{2}, \frac{\tanh U+1}{2}\right)} dU\right), \quad (20)$$

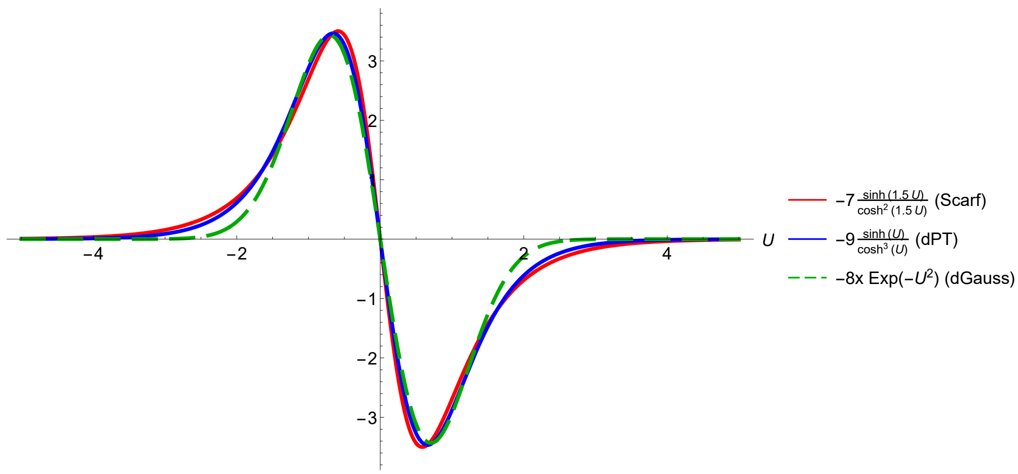
$$X^-(U) = c_3 \mathrm{HeunC}\left(0, 0, 0, -k, \frac{k}{2}, \frac{\tanh U + 1}{2}\right) + c_4 \mathrm{HeunC}\left(0, 0, 0, -k, \frac{k}{2}, \frac{\tanh U + 1}{2}\right) \times \left(\int -\frac{1}{\mathrm{HeunC}^2\left(0, 0, 0, -k, \frac{k}{2}, \frac{\tanh U+1}{2}\right)} dU\right) \quad (21)$$

* Zhao: analytic sol. for dPT (work in progress)

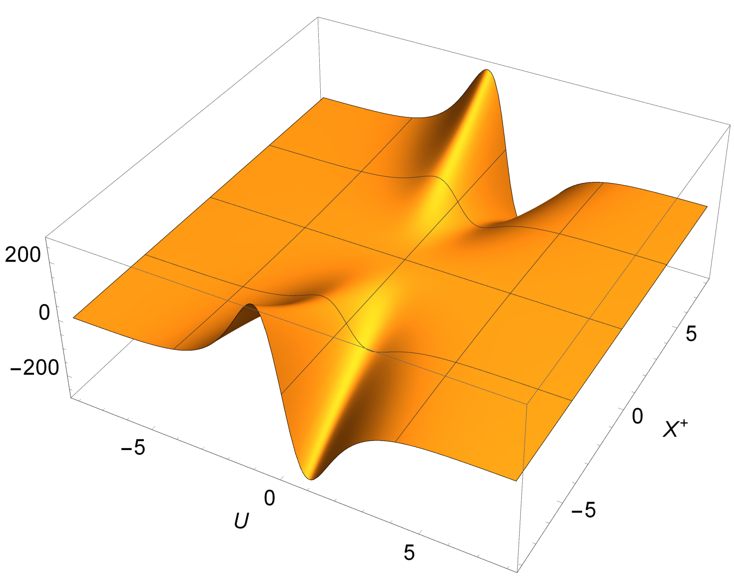
Scarf potential *

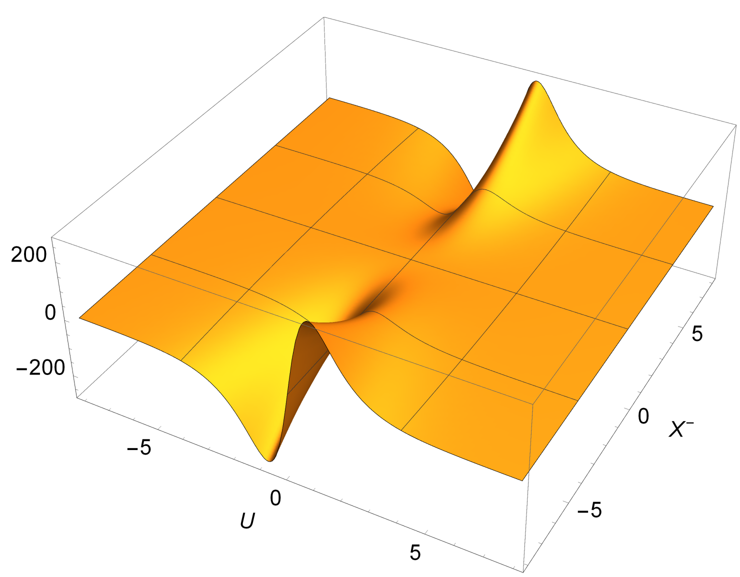
$$\mathcal{A}^{Scarf}(U) = 2g\frac{d}{dU}\left(\frac{1}{\cosh U}\right) = -2g\frac{\sinh U}{\cosh^{2} U}\,, \tag{22}$$

cf. dPT (16).



$D = 2$ GW with Scarf wave profile†





*F. L. Scarf, "New Soluble Energy Band Problem", Phys. Rev. **112**, 1137-1140 (1958)

†D. E. Alvarez-Castillo and M. Kirchbach, "The real exact solutions to the hyperbolic Scarf potential," Rev. Mex. Fis. E **53**, 143-154 (2007)

Sols for $\mathcal{A}^{Scarf}$: $t = \sinh U$ [cf. $t = \tanh U$ for dPT]. Geo ens. (13a)-(13b) :

$$(1+t^2)\frac{d^2X^\pm}{dt^2} + t\frac{dX^\pm}{dt} \boxed{\mp g\frac{t}{1+t^2}} X^\pm = 0\,. \qquad (23)$$

Eqns uncoupled $\Rightarrow$ solved independently. Coefficient t in front of linear-in-$X^\pm$ term whose sign change compensates $\mp$ sign-change, $t \to -t$, between components $\Rightarrow$ same g works for both components.

Eqn (23) is generalised hypergeometric type, solved analytically by Nikiforov-Uvarov algorithm* $\Rightarrow$

DM for quantized parameter

$$|g| = (2n+1)\sqrt{n(n+1)}\,, \quad n = 1, 2, \dots \qquad (24)$$

*P. M. Zhang, Z. K. Silagadze and P. A. Horvathy, "Flyby-induced displacement: analytic solution," Phys. Lett. **B** 868 (2015) 139687 [arXiv:2502.01326 [gr-qc]].

Trajectory

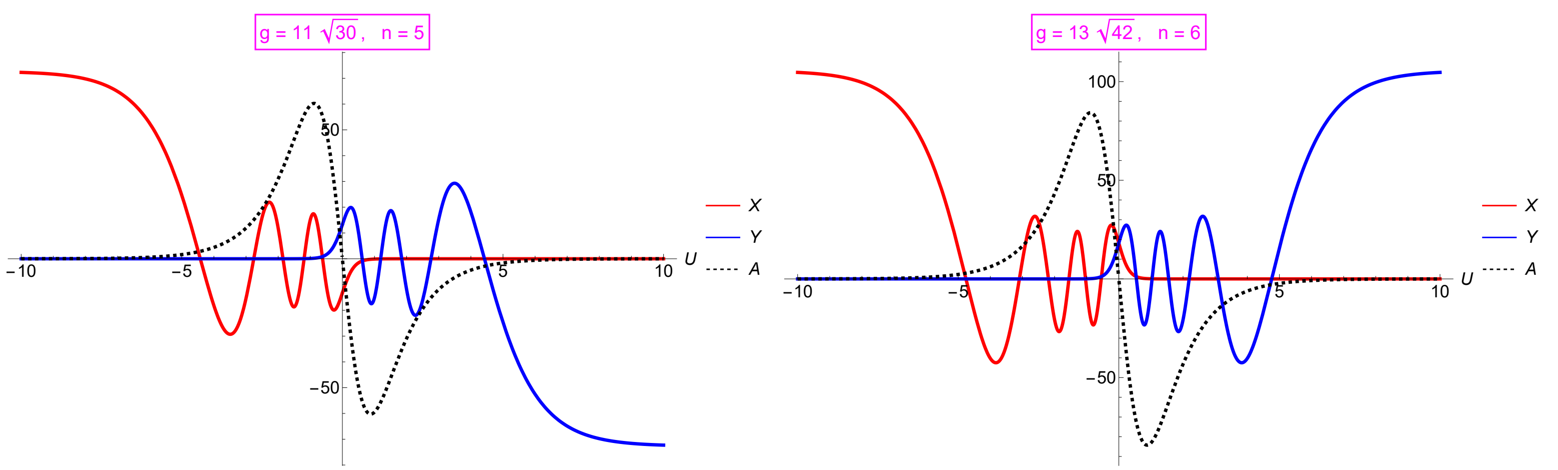
$$X_n(t) = (1+t^2)^{-\frac{n}{2}}\, e^{-\mathrm{sign}(g)\,\sqrt{n(n+1)}\,\arctan t}\, y_n(t)\,, \tag{25}$$

where $y_n(t) =$

$$\frac{B_n}{\rho(t)} \left[(1+t^2)^{-1/2} e^{-2\mathrm{sign}(g)\,\sqrt{n(n+1)}\,\arctan t} \right]^{(n)} .$$

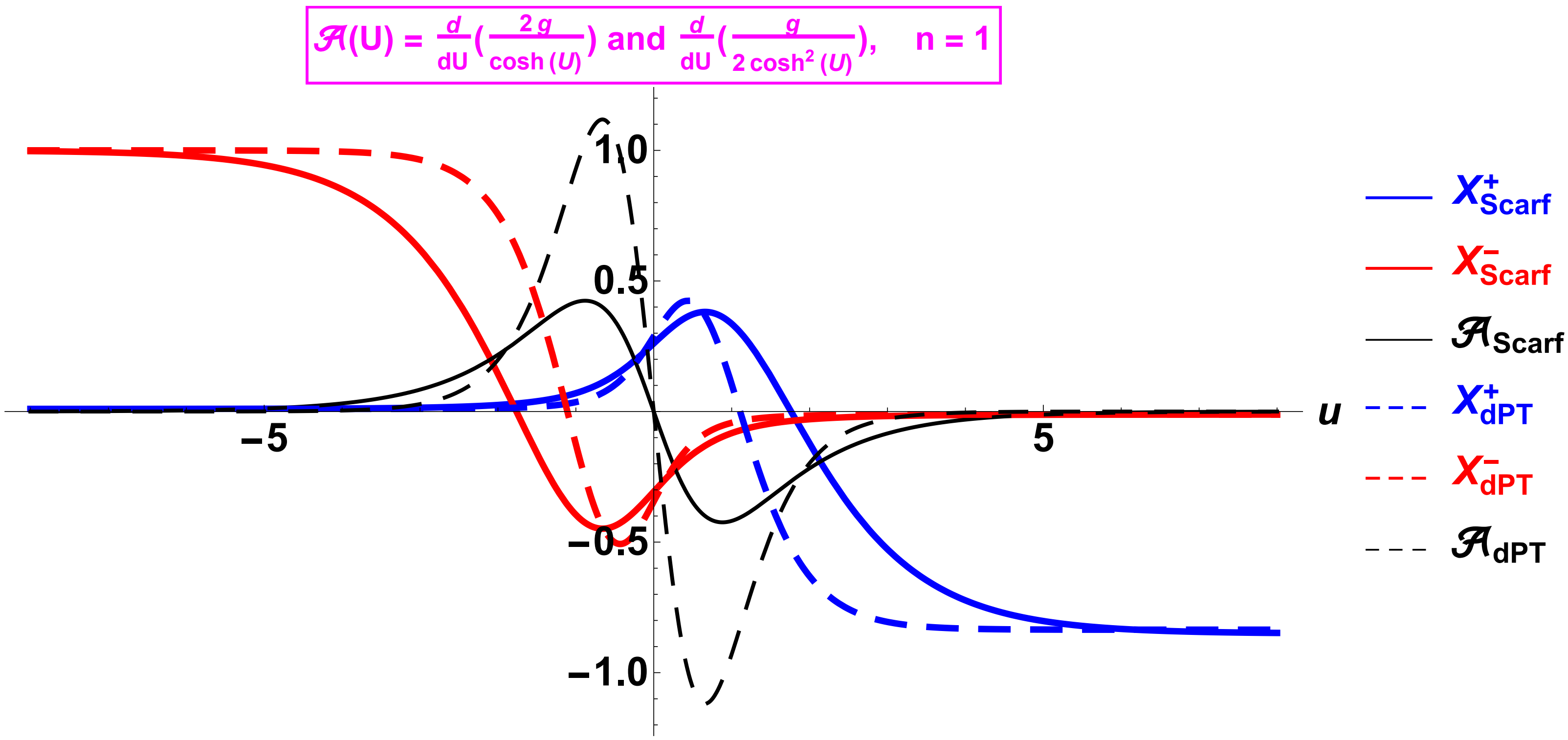
B_n const determined by initial conditions.

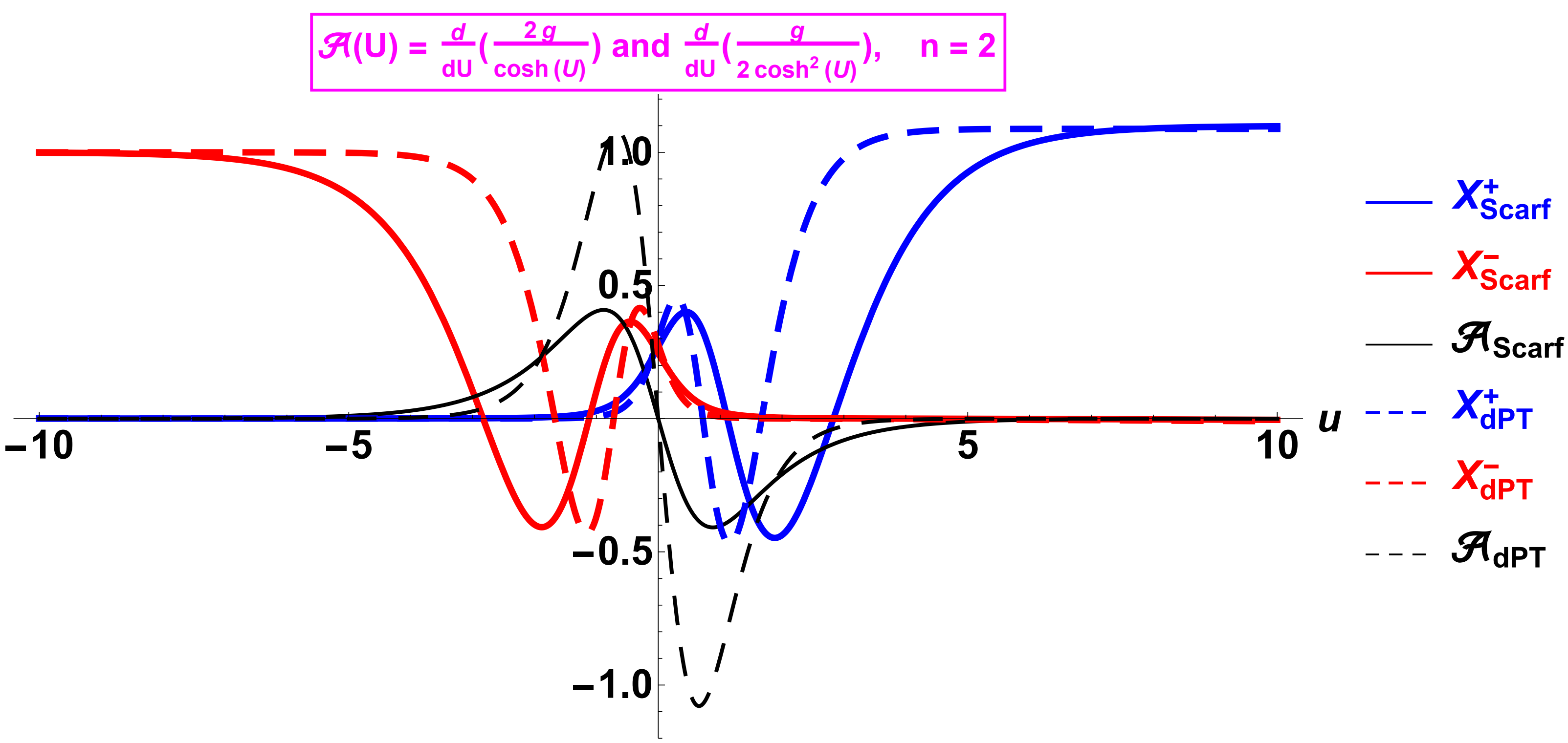
Precision also for high wave numbers



DM *geodesics for Scarf profile* (22) *for wave numbers* $\mathbf{n} = \mathbf{5}$ *and* $\mathbf{n} = \mathbf{6}$.

Scarf vs dPT (dashed)





Hodograph

$\equiv$ "Velocity diagram" in transverse space.

• $\mathbf{D = 1}$ transverse-space dim with coordinate Y,

$$W(U) = \frac{dY}{dU}\,. \tag{26}$$

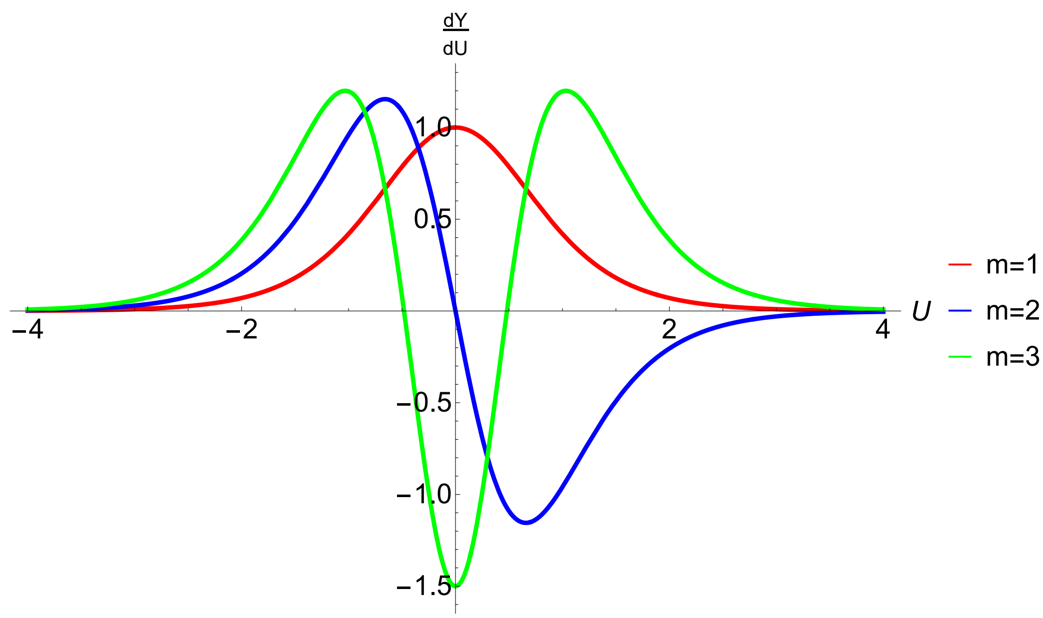
Particles in rest before sandwich wave arrives,

$$W(-\infty) = W_{-\infty} = 0\,. \tag{27}$$

Outgoing velocity constant by Newton's 1st law,

$$W(+\infty) = W_{\infty} = \text{const} \tag{28}$$

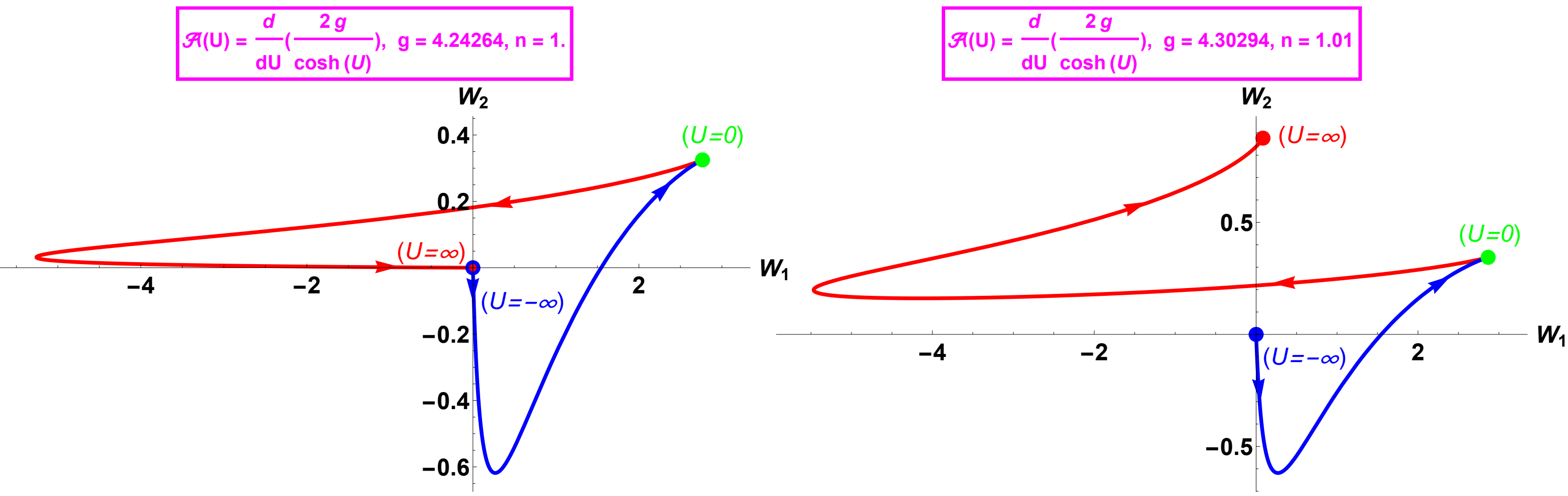
Generally, Velocity Effect, VM . **DM** , when $W_\infty = 0$



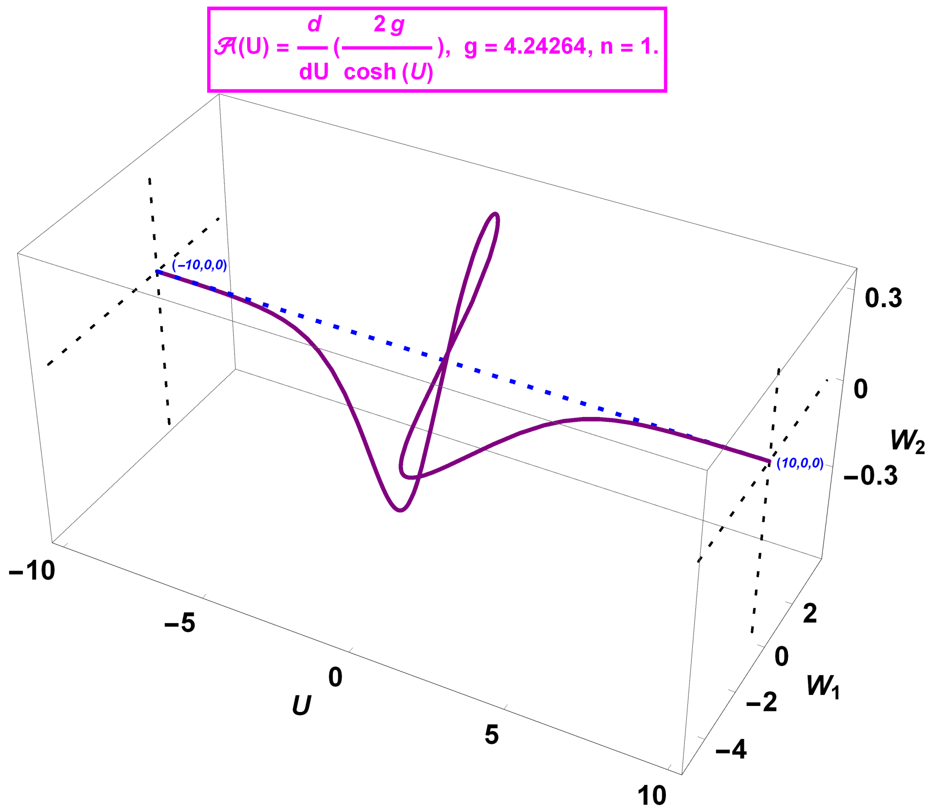
Unfolding hodograph : DM for quantized Pöschl-Teller amplitude $k_m = \, 4m(m+1)$, $\sim \mathbf{m = 1, 2, 3}$ *(half) waves.*

Similar behavior for dPT, Scarf.

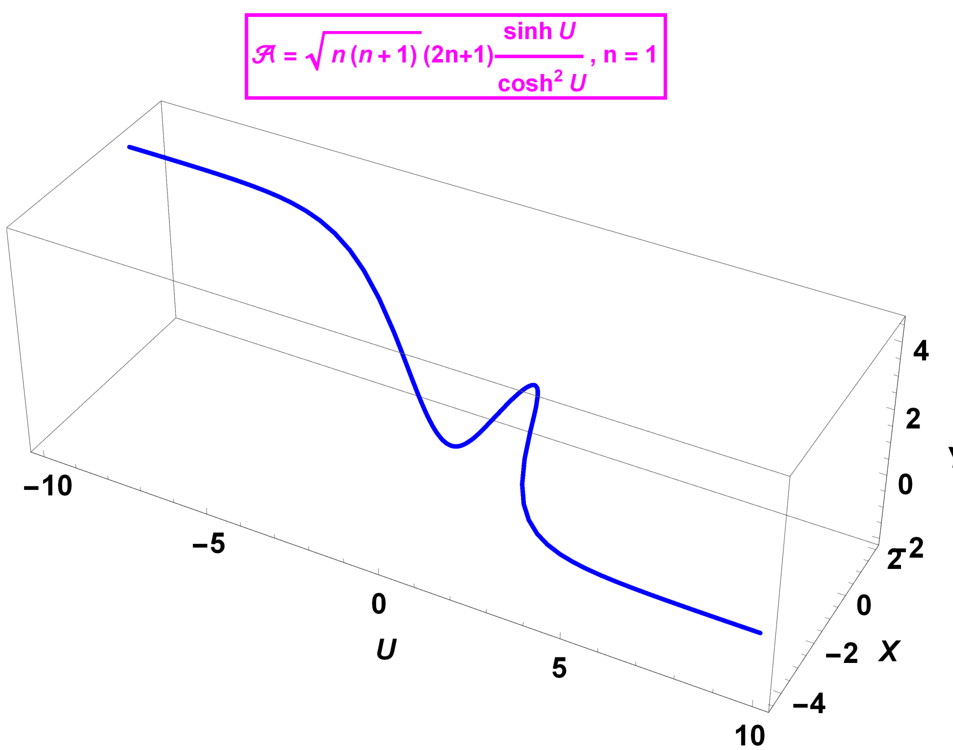
• $\mathbf{D=2}$ transverse dim. For critical amplitude get closed curve but shifting n to a non-integer we get VM, $\boldsymbol{W}(U=+\infty) \neq (0,0)$. Unfolding to $(2+1)D$ yields curve which starts, and for DM also ends, at origin,



Scarf hodograph in $\mathbf{D=2}$ *transverse dimension.* **DM** *with half wave number* $\mathbf{n=1}$ *obtained for amplitude* $g=g_1$ *in (37). Velocity starts from* $\boldsymbol{W}(U=-\infty)=(0,0)$*, follows* **blue** *curve to* $\boldsymbol{W}(U=0)$*, then returns along the* **red** *curve to origin. When* n *is not integer,* $\boldsymbol{W}(U=+\infty) \neq (0,0)$*, get VM.*



DM *hodograph unfolded to* $(2{+}1)D$*: curve starts and ends at* $\boldsymbol{W}(\pm\infty) = 0$*.*



DM *trajectory unfolded to* $(2+1)D$

Quantum Mechanics and Memory

1.) Quantum Mechanics. Consider time-indept Schrödinger eqn in $D = 1$ for PT potential

$$\left(-\frac{d^2}{dU^2} + V(U)\right)\psi = E\psi \,, \qquad (29)$$

where

$$V(U) = \frac{k}{2\cosh^2 U} \,. \qquad (30)$$

Bounded sol obtained for discrete amplitude*

$$k = k_n = -2n(n+1), \quad V = -\frac{n(n+1)}{\cosh^2 U}, \qquad (31)$$

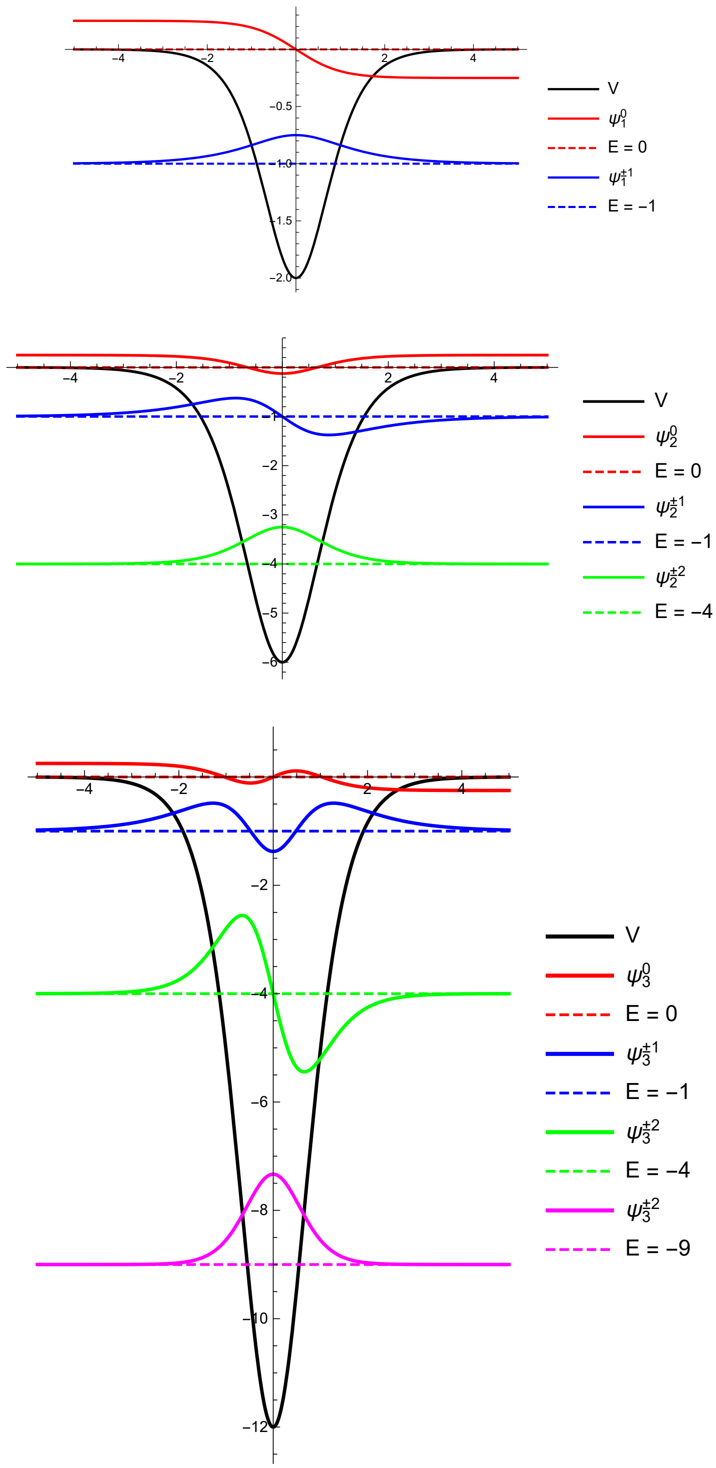
n=0, 1,2, . . .

Fixing n, bound state spectrum

$$E_m \propto -(n-m)^2, \quad m = 0, 1, \cdots, n. \qquad (32)$$

For $n = m$, **zero-energy non normalizable bound state**.

*Legendre polynomials cf. p.11.



Fixing the integer $n \geq 1$*, bounded energy eigenstates* ψ_n^m*,* $m = 1, \dots, n$ *are obtained for critical amplitude* $\mathbf{k_n = -2n(n+1)}$ *of the PT profile, shown for* $\mathbf{n = 1, 2, 3}$*. The highest, zero-energy eigenstate* ψ_n^n *is bounded but is* <u>*not*</u> *normalizable.*

2.) Geodesics. Fixing n, **DM** geodesics obtained as zero-energy solutions $X_n(U) \equiv \psi_n^n(U)$ of time-indept Schrödinger eqn (29) after reversing sign of potential,

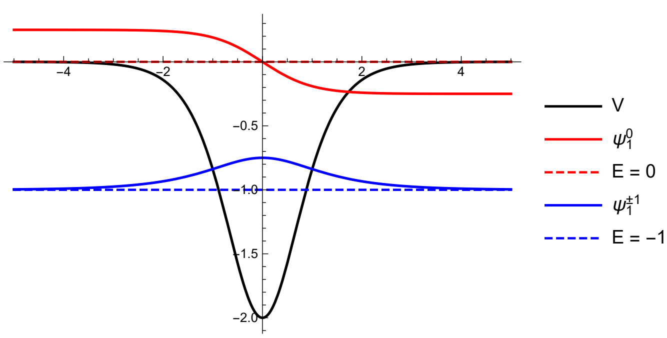
$$\left(\frac{d^2}{dU^2} + \frac{1}{2}\mathcal{A}(U)\right) X_n = 0, \qquad (33)$$

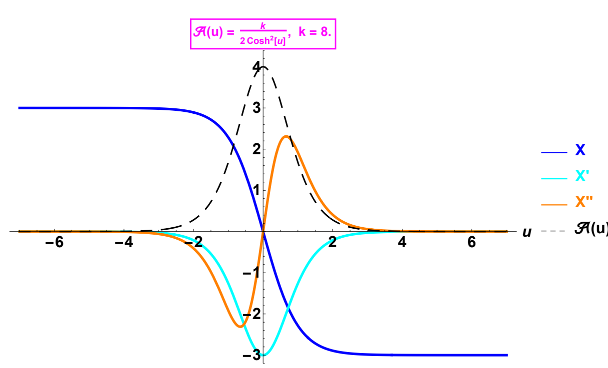
$$\boxed{\mathcal{A}(U) = \boxed{-}\, 2V(U)\,,} \qquad (34)$$

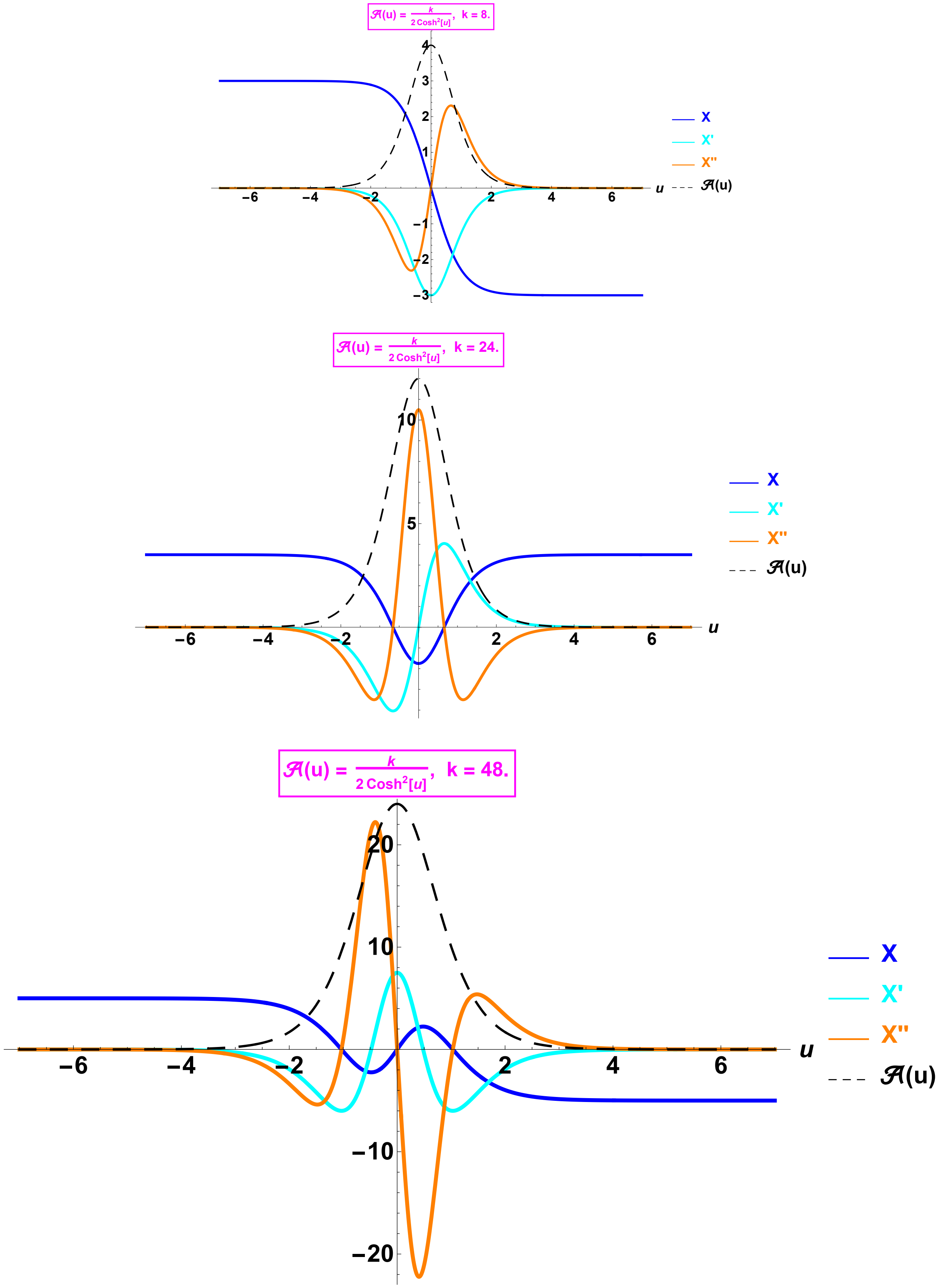
supplemented with asymptotic boundary cond

$$X(\pm\infty) = \text{const}\,. \qquad (35)$$

Attractive potential (**well**) of quantum pb has become for geodesics repulsive (**hill**) with sign reversed. Zero-energy non-normalizable highest **QM** bound state becomes **DM** .





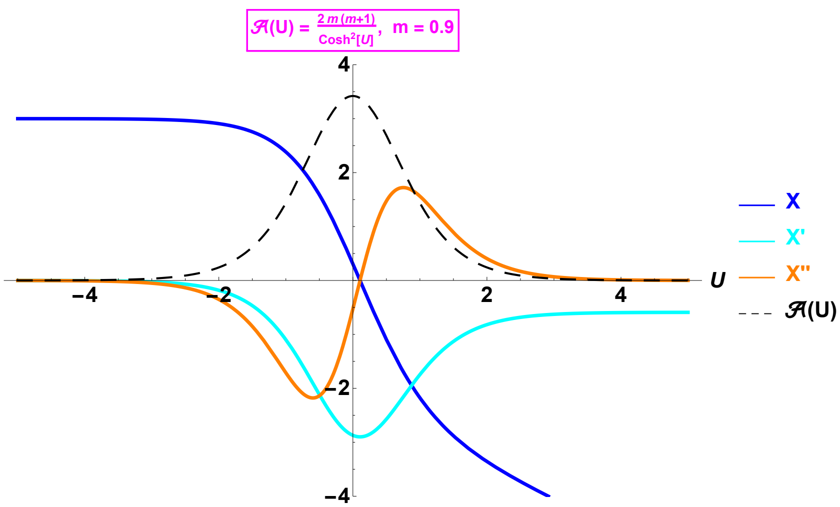


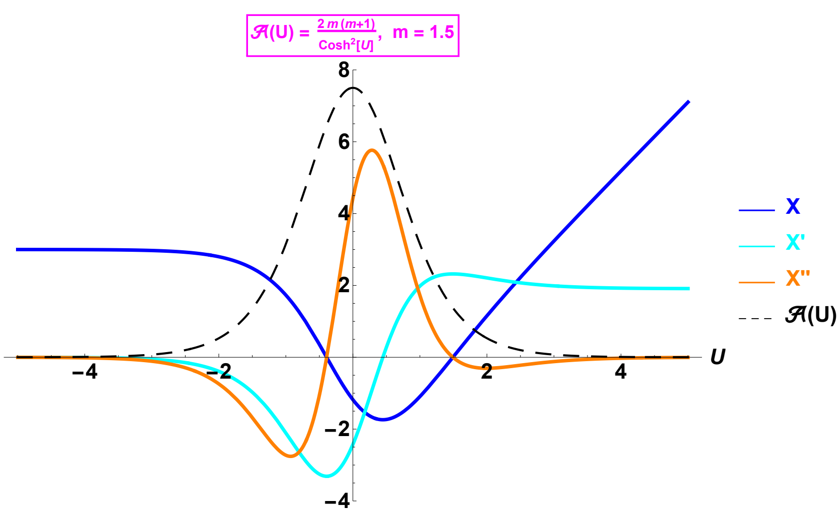
DM *geodesics for PT with half-wave numbers* $\mathbf{n = 1, 2, 3}$. *Displacement* $\mathbf{\Delta X_n} = X_n(+\infty) - X_n(+\infty) \neq 0$ *arises when* n *is* **odd**.

DM trajectory for each fixed $n \geq 1$,

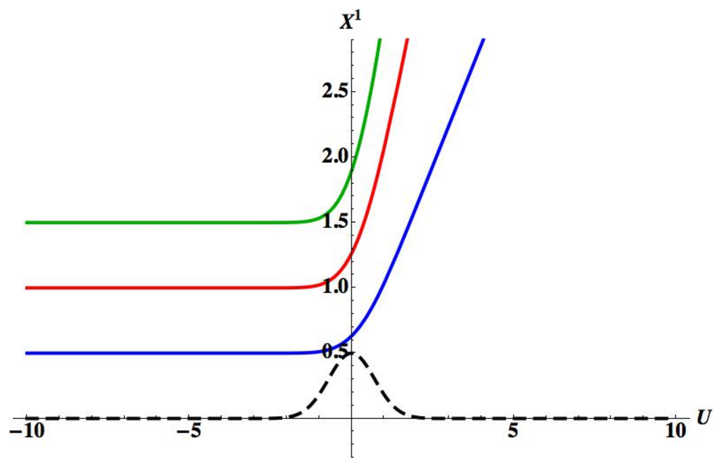
$$\begin{aligned} X_{n=1}(U) &= \tanh U & \Delta X_1 = 2, \\ X_{n=2}(U) &= 3\tanh^2 U - 1 & \Delta X_2 = 0, \\ X_{n=3}(U) &= 6\tanh U - 15\tanh U\,\mathrm{sech}^2 U & \Delta X_3 = 12 \end{aligned}$$

For $k \neq k_n$ one gets **VM**





The sign change in (34) implies that the **DM** boundary condition (35) can not be satisfied when the energy is positive: all geodesics with $\mathbf{E} > \mathbf{0}$ diverge.



For the inverted Pöschl-Teller potential (34) *all positive-energy geodesics diverge. No* **DM** *solutions are obtained.*

Similar investigations apply to **Scarf** profile (22) *,

$$\mathcal{A}^{Scarf}(U) = -2g\frac{\sinh U}{\cosh^2 U}\,. \qquad (36)$$

DM trajectories obtained for quantized parameter

$$|g_n| = (2n+1)\sqrt{n(n+1)}\,, \quad n = 1, 2, \ldots, \qquad (37)$$

for which geodesic eqn becomes (with positive g_n)

$$\frac{d^2X_n}{dU^2} - \Big((2n+1)\sqrt{n(n+1)}\,\operatorname{sech} U\,\tanh U\Big)X_n = 0 \qquad (38)$$

with boundary condition $X(\pm\infty) = \text{const}$. (35).

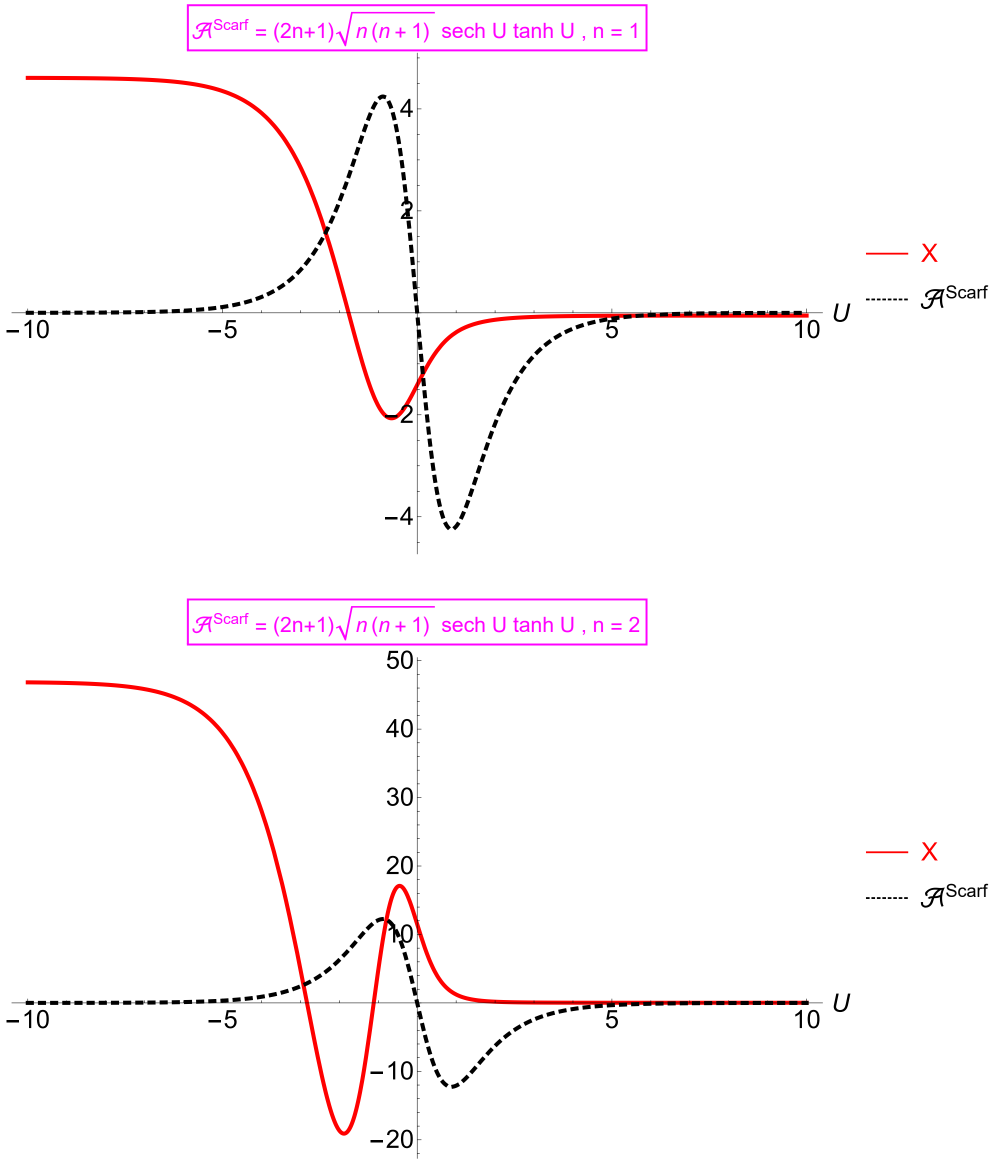
For **DM** for quantized parameter (37)

$$|g| = (2n+1)\sqrt{n(n+1)}\,, \quad n = 1, 2, \ldots \qquad (39)$$

Get analytic solutions (25)† :

*F. L. Scarf, "New Soluble Energy Band Problem", Phys. Rev. **112**, 1137-1140 (1958)

†P. M. Zhang, Z. K. Silagadze and P. A. Horvathy, "Flyby-induced displacement: analytic solution," Phys. Lett. **B** 868 (2015) 139687 [arXiv:2502.01326 [gr-qc]];
E. Catak, M. Elbistan and M. Mullahasanoglu, "Displacement memory effect from supersymmetry," Eur. Phys. J. Plus **140** (2025) no.6, 540 [arXiv:2504.05043 [gr-qc]].



Scarf trajectory with half wave numbers $\mathbf{m = 1, 2}$.

Geodesic eqns are Sturm-Liouville (SL) problems. To get DM, we require the velocity (of the transverse coordinates) to vanish for $U \pm \infty$, which implies Neumann boundary conditions. Such problems occur in quantum mechanics as, e.g., for PT and Scarf.

Our paper applies QM techniques to geodesics.

CONCLUSION

VM : Particles at rest hit by a burst of GWs fly apart, moving freely along straight lines. **DM** is possible for exceptional values of wave parameters which correspond to having integer # of half-waves in wave zone.

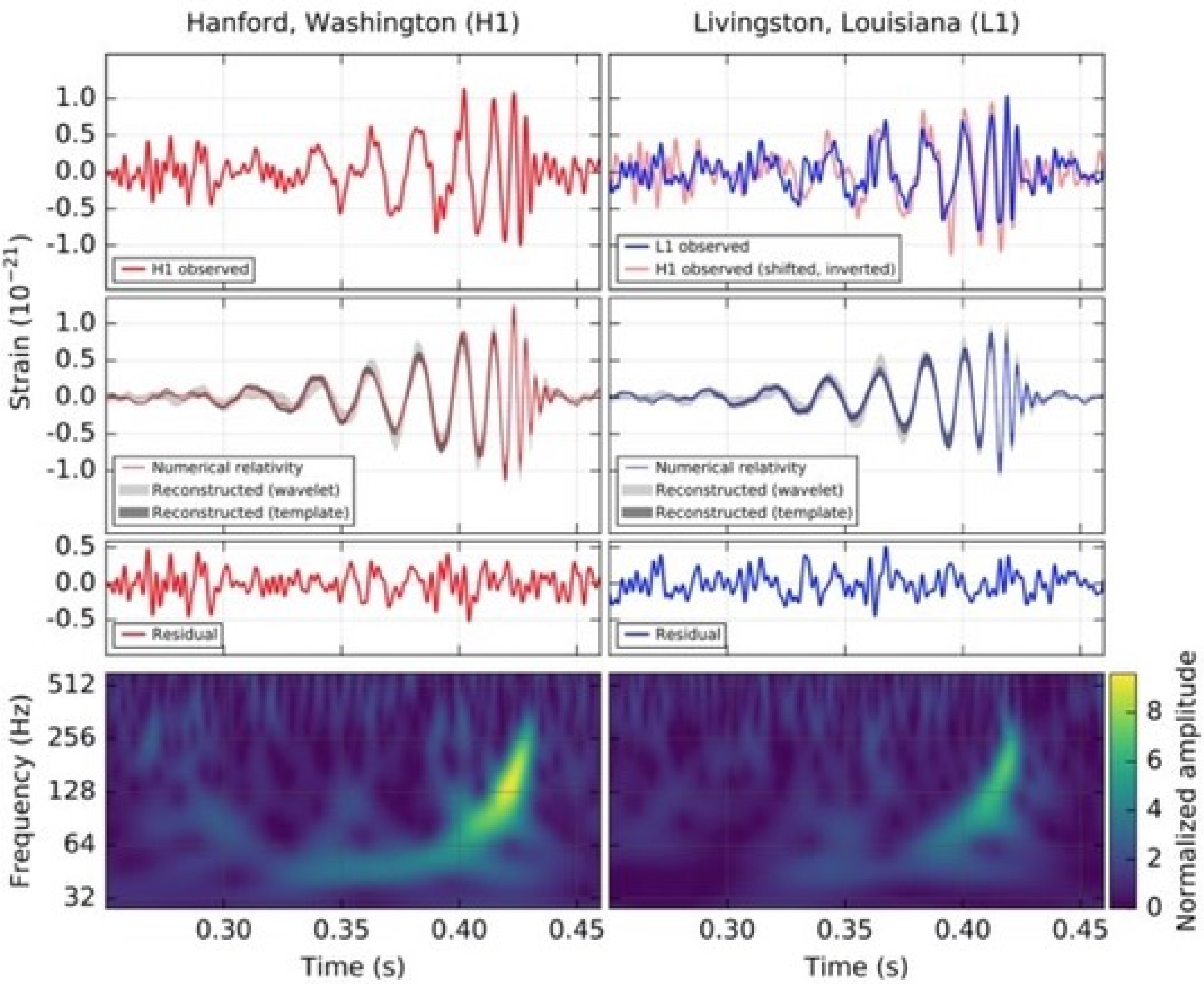
Clue: **DM** geodesics $\sim$ zero-energy sols of time-independent Sch eqn. (3a).

Analytic sols for PT, dPT, Scarf

TOY MODELS !

$\sim$ von Neumann's "dry water"

observed GW GW170814



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NO flyby-induced memory effect observed so far.