Null infinity as a weakly isolated horizon

Simone Speziale

Tours, 4 juillet 25

based on work with :

Abhay Ashtekar, *Null Infinity as a Weakly Isolated Horizon* (2402.17977)

Gloria Odak and Antoine Rignon-Bret, *General gravitational charges on null hypersurfaces* (2309.03854)

CPT Centre de Physique Théorique

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Motivations

Black holes and quasi-local horizons have a rich symmetry structure

… and a rich literature!

Geroch-Hansen ‘70, Hansen ’74,BCH ’73… Ashtekar-Beetle-Fairhurst-Krishnan-Lewandowski-etc ’00, Donnay-Giribert-Gonzales-Pino ‘15, Grumiller et al ‘19, Sheikh-Jabbari et al ‘20, Freidel- et al ’18, Chandrasekaran-Flanagan-Prabhu ’18, Ashtekar-Khera-Kolanowski-Lewandowski-etc ’22 Ciambelli-Freidel-Leigh ’23, Ruzziconi-Zwikel ’25, Agrawal-Charalambous-Donnay ’25 …

For a special type of quasi-local horizons called non-expanding horizons (NEHs) one finds a symmetry group that is very closely related to Scri: BMS + one global dilation

- Indeed, it is known that Scri can be described as a non-expanding horizon;
- but the physics of Scri is completely different from the equilibrium physics of a NEH!

So why the similarities, and at the same time the profound differences?

Plan of the talk

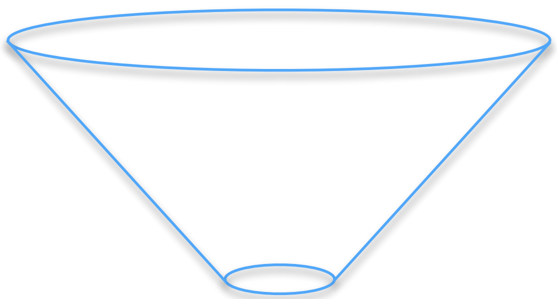
I - Horizons and null infinity

II - Symmetry groups of horizons and null infinity

III - Noether charges for the symmetry groups

IV - Weaker boundary conditions and larger symmetry groups

Geometry of null hypersurfaces

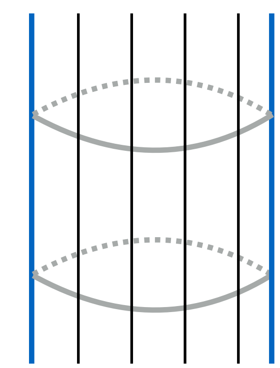


$$\mathcal{N} : \Phi = 0 \qquad l_\mu :\overset{\mathcal{N}}{=} -f\partial_\mu\Phi, \qquad l^2 \overset{\mathcal{N}}{=} 0$$

$$q_{\mu\nu} = g_{\mu\nu} \qquad \Rightarrow \qquad l^\mu \nabla_\mu l_\nu \overset{\mathcal{N}}{=} k l_\nu$$

k: inaffinity (tang. acc.)

Induced metric is degenerate: $q_{\mu\nu} = \underleftarrow{g_{\mu\nu}}$ $\quad q = (0,+,+) \quad q_{\mu\nu} l^\mu = 0$



$\mathcal{N}$

shear and expansion: $\pounds_l q_{\mu\nu} = \sigma_{\mu\nu} + \frac{\theta}{2} q_{\mu\nu}$

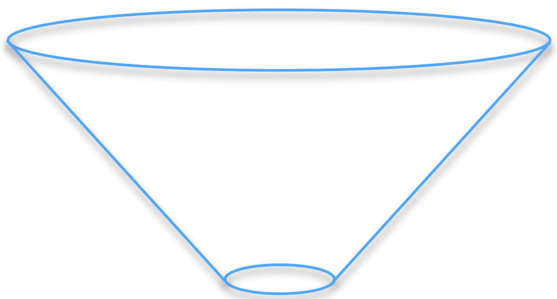
→ no projector, no induced connection

$$l_\mu X^\nu \nabla_\nu Y^\mu = -X^\nu Y^\mu \nabla_\nu l_\mu = -X^\nu Y^\mu (\sigma_{\mu\nu} + \frac{\theta}{2}\gamma_{\mu\nu})$$

(*X*,*Y* tangent vectors)

- null hypersurfaces with vanishing shear and expansion admit a unique connection
- in general case, possible to consider rigging/Carrolian connections

Geometry of null hypersurfaces



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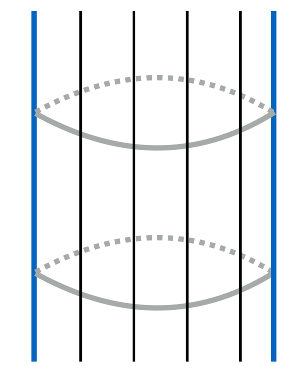

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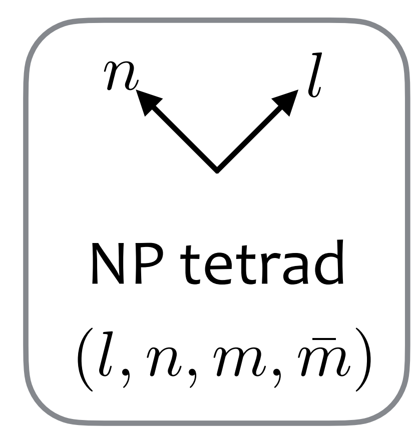
following Newman-Penrose (NP), useful to introduce an auxiliary null vector

$n^2 = 0, \qquad n^\mu l_\mu = -1$ aka (null) rigging vector

induces a local projector on the 2d spacelike planes

$$\gamma_{\mu\nu} := g_{\mu\nu} + 2 l_{(\mu} n_{\nu)} = 2 m_{(\mu} \bar{m}_{\nu)} \qquad \underleftarrow{\gamma_{\mu\nu}} = q_{\mu\nu}$$



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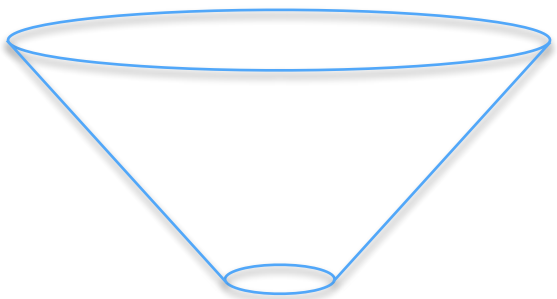

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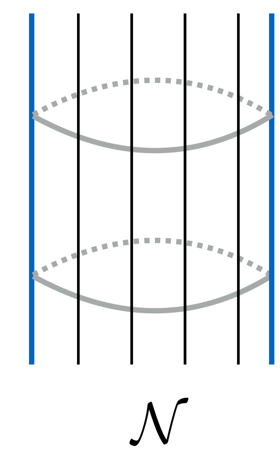


Extrinsic geometry via the Weingarten map:

$$W_\mu{}^\nu := \nabla_{\underleftarrow{\mu}} l^\nu \overset{\mathcal{N}}{=} -(\eta_\mu + \underbrace{k n_\mu) l^\nu}_{} + \gamma^{\nu\rho}(\sigma_{\mu\rho}+\frac{1}{2}\gamma_{\mu\nu}\theta)$$

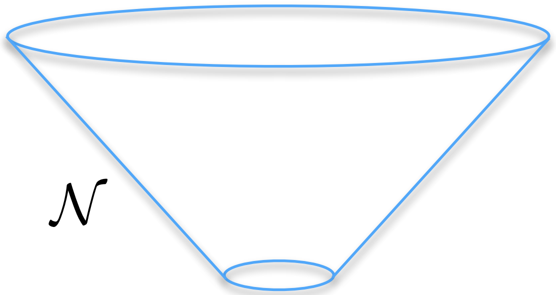
rotation 1-form in IH literature

Hajicek 1-form, or twist

Caveat : *(σ,ϑ,η,k) are not purely geometric quantities, but depend on kinematical choices*

They are class I and III - dependent, in the classification of Chandra's book

Non-expanding horizons (NEHs)



- topology $S^2 \times \mathbb{R}$
- vanishing expansion $\theta = 0$
- vanishing shear* $R_{\mu\nu} l^\nu = \alpha l_\mu \quad \Rightarrow \quad \sigma = 0$

(*in vacuum, same thing as shear and expansion free, but a subclass thereof in presence of matter)

→ NEHs admit a unique connection $D_\mu = \nabla_{\underleftarrow{\mu}}$

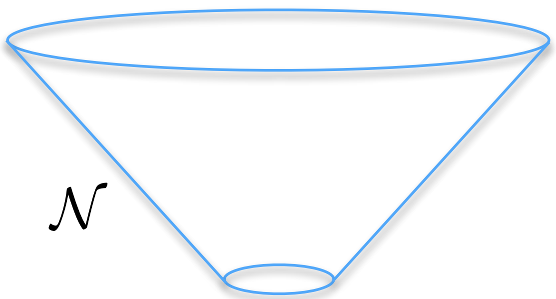
Convenient to limit the kinematical freedom (the rescaling *f*):

- choose affinely parametrized normals : $k = 0 \quad \Rightarrow \quad \pounds_l \eta_\mu = 0$ time-independent twist
- choose divergence-free twist : $D_\mu \eta^\mu = 0$

Only remaining arbitrariness: **constant** rescaling *f*

Notation: In the catalogue of Ashtekar-Beetle-Fairhurst-Krishnan-Lewandowski-etc, a NEH equipped with this restricted equivalence class of choices of normal is called *extremal weakly isolated horizon*

Non-expanding horizons are *not* stationary



At first sight, NEHs appear to be very close to equilibrium

$$\sigma = \theta = 0 \qquad \pounds_l \eta = 0$$

They are however not stationary, because the induced connection is in general time-dependent :

$$[\pounds_l, D_\mu] n_\nu = D_\mu \eta_\nu + \eta_\mu \eta_\nu + C_{\rho\mu\nu\sigma} l^\rho n^\sigma + \frac{1}{2}(S_{\mu\nu} + (\alpha - \frac{R}{6}) q_{\mu\nu})$$

Using Einstein's equations: **time-independent** **vanishes**

➩ geometry entirely determined by cod-2 free data η_μ, Ψ_2

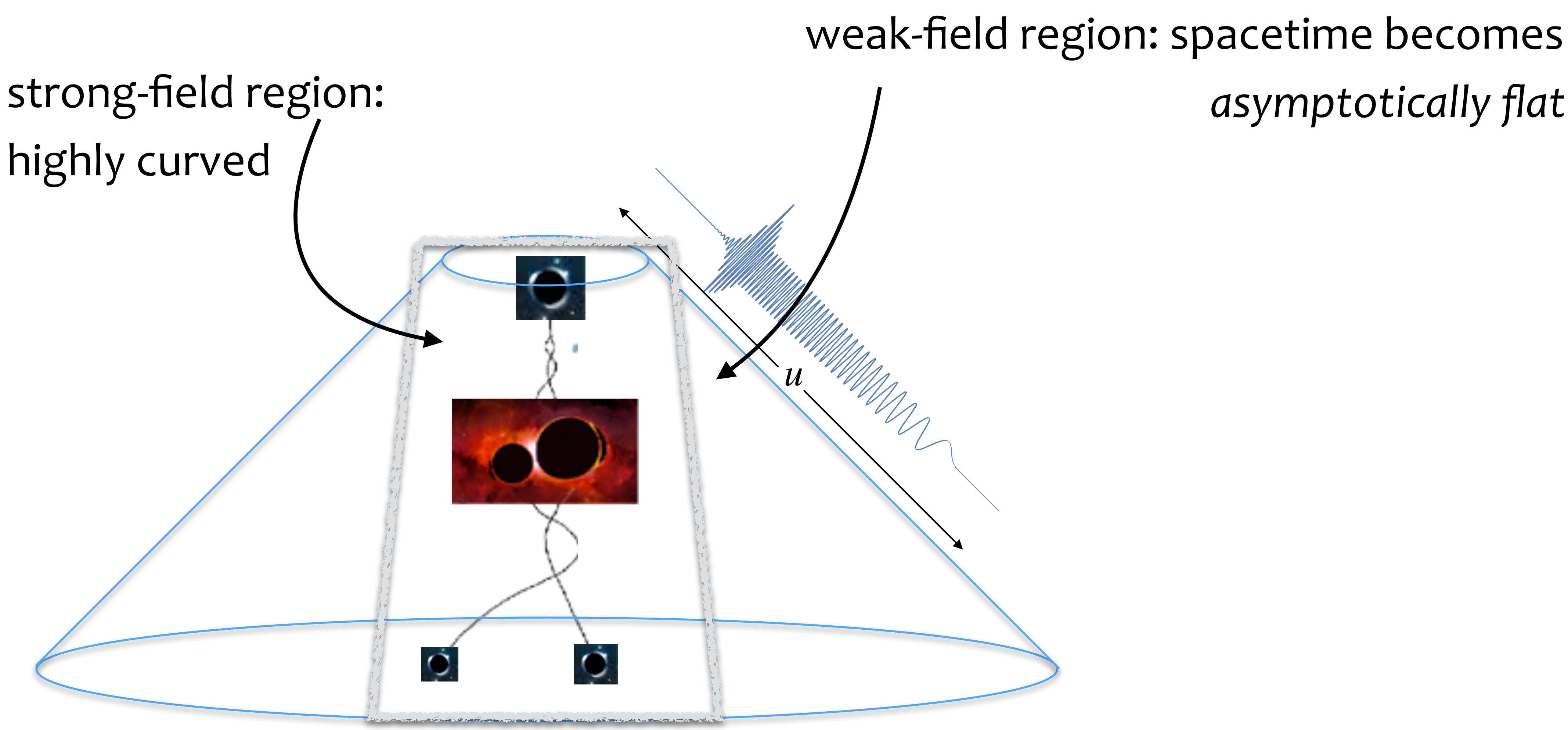
\`in equilibrium', although in a weaker sense than for a Killing horizon

Special cases

- isolated horizons $[\pounds_l, D_\mu] n_\nu = 0$
- Killing horizons $[\pounds_l, \nabla_\mu] n_\nu = 0$

(Future) null infinity

Idealization useful to model isolated gravitational systems



- Coordinate-dependent definition (Bondi-Sachs) based on null foliations
- Coordinate-independent definition (Penrose-Geroch) based on conformal compactifications

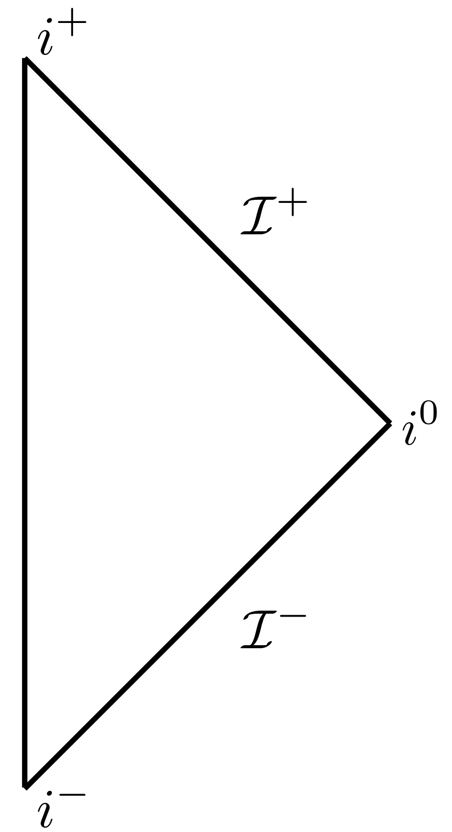
Penrose's conformal compactification

Warm-up example: Minkowski

$$ds^2 = -du^2 - 2dudr + r^2 q_{AB} dx^A dx^B$$

- retarded time: $u = t - r$ metric ill-defined at $r \to \infty$
- change coord.: $\Omega = \frac{1}{r}$ $ds^2 = -du^2 + \Omega^{-2}(2dud\Omega + q_{AB}dx^A dx^B)$
- define `unphysical metric': $\hat{\eta}_{\mu\nu} = \Omega^2 \eta_{\mu\nu}$

$$d\hat{s}^2 = \underline{2dud\Omega + q_{AB}dx^A dx^B} - \Omega^2 du^2$$



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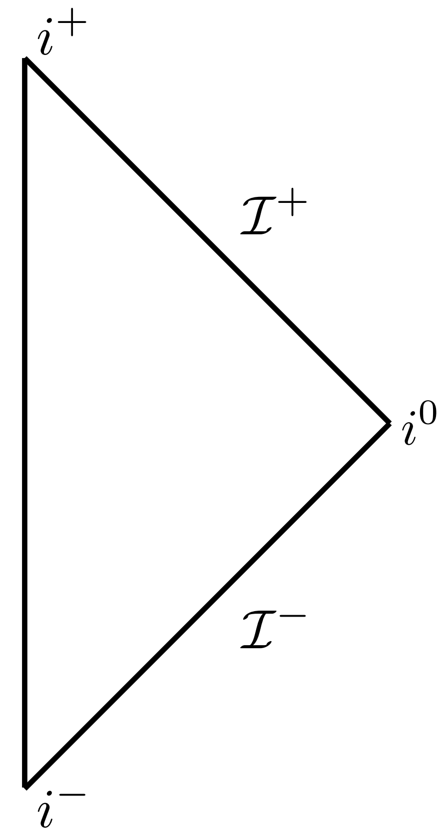

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Formal definition (Penrose '64): a spacetime is AF if it admits a conformal completion $(\hat{M}, \hat{g})$ such that :

- it has a boundary at $\Omega = 0$, with $\partial_\mu \Omega \neq 0$
- the boundary has topology $S^2 \times \mathbb{R}$
- the Einstein's equations are satisfied at the boundary with $T_{\mu\nu} = O(\Omega^2)$

Null infinity as a NEH

Geroch '77

Einstein's equation for the conformally rescaled metric

$$\hat{S}_{\mu\nu} := \hat{R}_{\mu\nu} - \frac{1}{6}\hat{g}_{\mu\nu}\hat{R} \mathrel{\hat{=}} -2\Omega^{-1}\hat{\nabla}_\mu n_\nu + \Omega^{-2}n^2\hat{g}_{\mu\nu} + O(\Omega^2)$$

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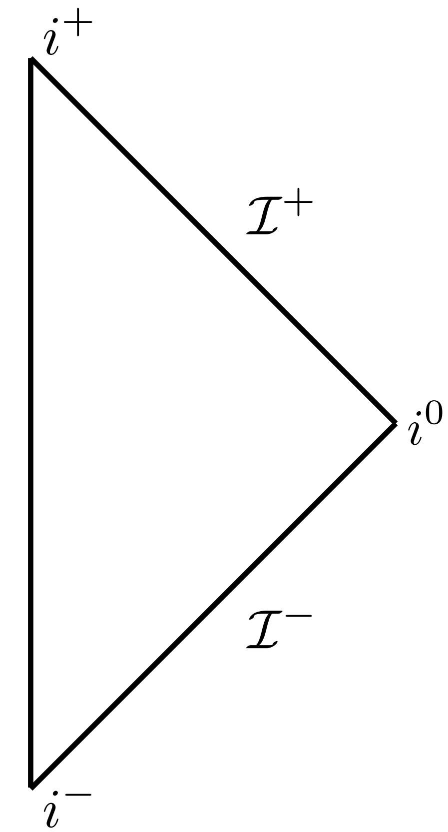

Smoothness of Scri then implies:

$$n^2 \overset{\mathscr{I}}{=} 0, \qquad \hat{\nabla}_{\langle a}n_{b\rangle} \overset{\mathscr{I}}{=} 0, \qquad \hat{\nabla}_\mu n^\mu = \pounds_n \ln\sqrt{-\hat{g}} \overset{\mathscr{I}}{=} \theta + 2k \mathrel{\hat{=}} 2\theta$$

⇨ Scri is shear-free, but not necessarily expansion-free

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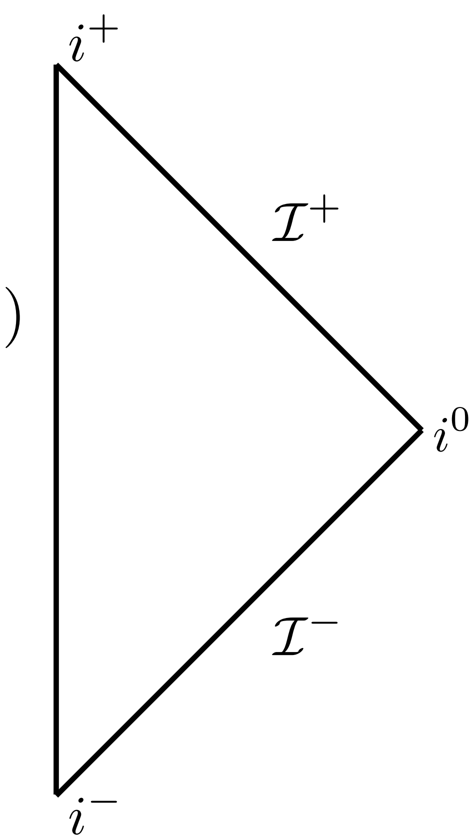

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However, **always** possible to use freedom in changing to conformal factor in order to pick $\theta = 0$

$$\Omega \to \omega\Omega \qquad \hat{\nabla}_\mu n^\mu \to \omega^{-1}(\hat{\nabla}_\mu + 4\pounds_n \ln\omega) \qquad \Rightarrow \qquad \hat{\nabla}_\mu n^\mu \overset{!}{=} 0$$

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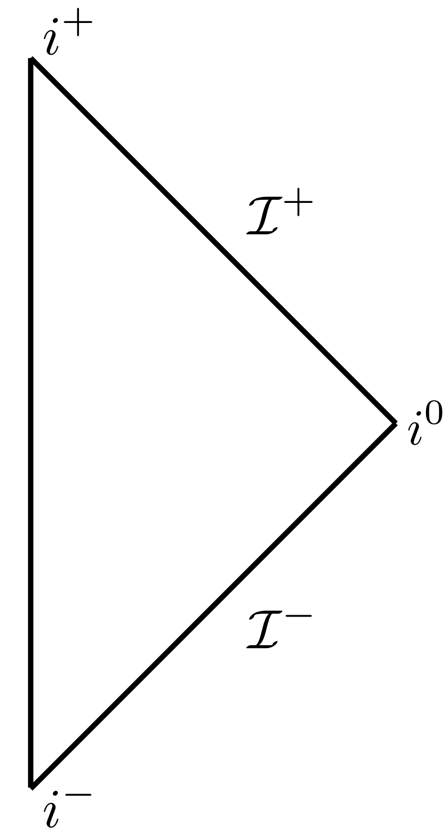

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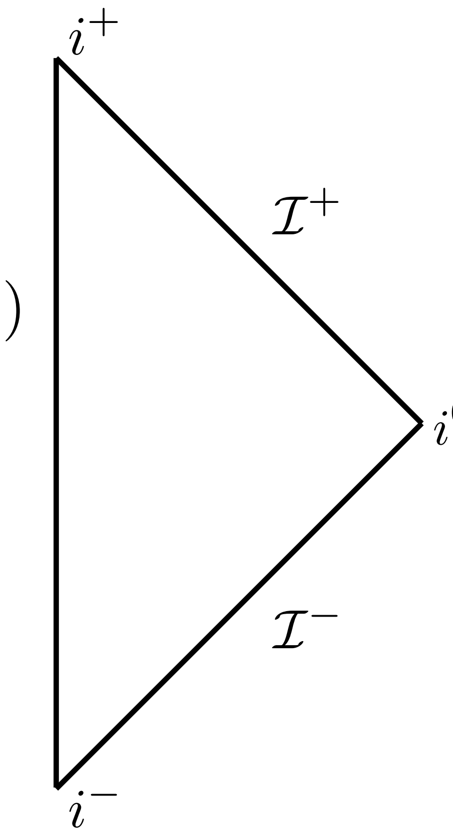

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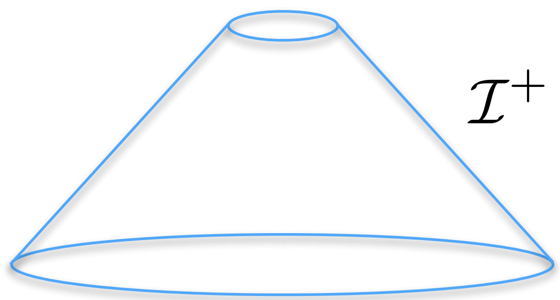
Then also $\hat{\nabla}_\mu n_\nu = 0$, so in a div-free frame, Scri has $\sigma = \theta = k = \eta = 0$ → *extremal WIH*

From a geometric viewpoint, a *weakly isolated horizon* is the same thing as a non-expanding horizon; the only difference is a piece of additional information in the kinematical structure used

Null infinity as a WIH



Geroch '77

In a div-free frame, Scri is a WIH with vanishing twist in $(\hat{M}, \hat{g})$

$$\sigma = \theta = k = \eta = 0$$

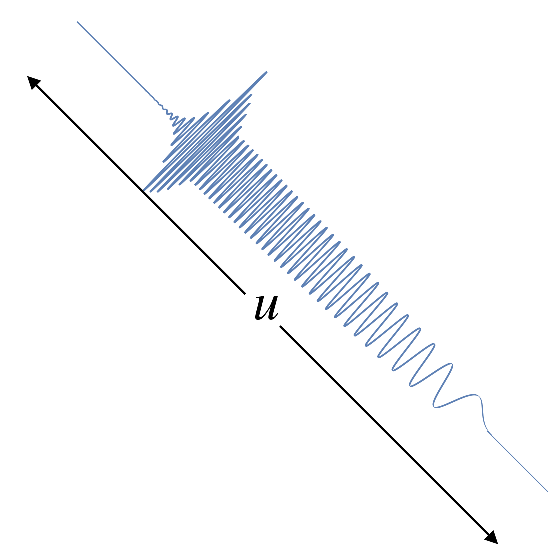
We can apply the previous general equation of WIH, but with completely different dynamics now!

$$[\pounds_l, D_\mu] n_\nu = \underbrace{D_\mu \eta_\nu + \eta_\mu \eta_\nu + C_{\rho\mu\nu\sigma} l^\rho n^\sigma}_{\text{vanishes}} + \frac{1}{2} \underbrace{(S_{\mu\nu} + (\alpha - \frac{R}{6}) q_{\mu\nu})}_{\text{time-dep, carries radiation}}$$

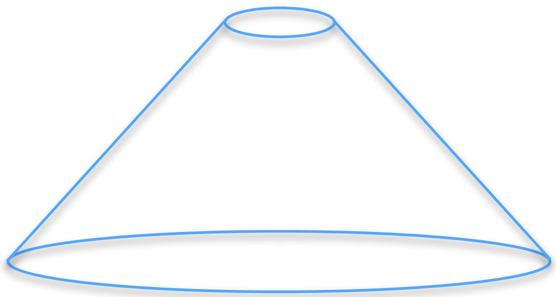
Using conformal Einstein's eqs: **vanishes** **time-dep, carries radiation**

Bondi news

(Caveat: conformal invariance)



Null infinity vs horizons



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Einstein's eqs
at a physical horizon
select this part

conformal Einstein's eqs
at null infinity
select this part

II - Symmetry groups

General relativity and symmetries

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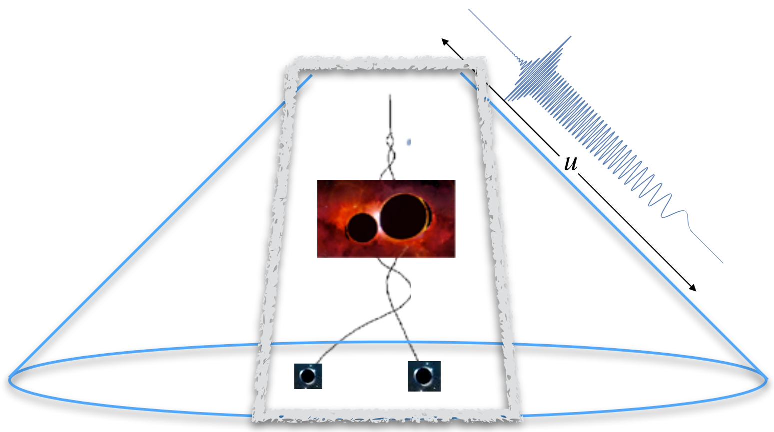

When modelling isolated systems,
we assume a flat background metric but only asymptotically far away from the sources
⇨ ***asymptotic symmetries***

E.g. Poincare at spatial infinity, BMS at null infinity: Asymptotic symmetries that arise from the isometries of the asymptotic boundary conditions

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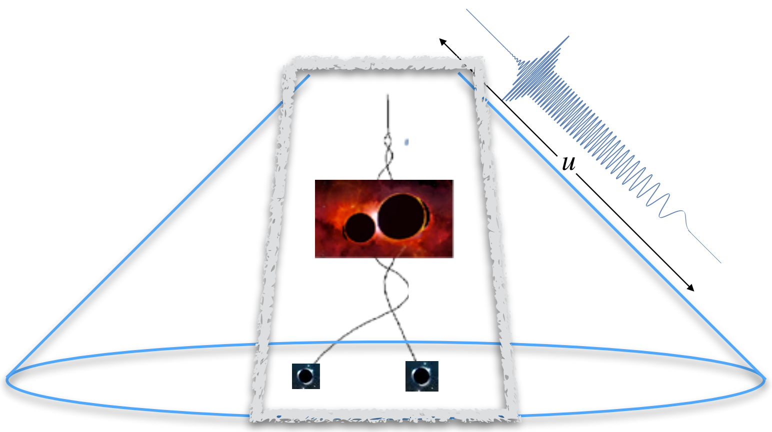

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More in general, we can define **boundary symmetries**, as residual diffeomorphisms preserving the boundary conditions

Symmetries of BH horizons, and of null hypersurfaces in general, arise as residual diffeos preserving the boundary conditions defining the horizons and null hypersurfaces

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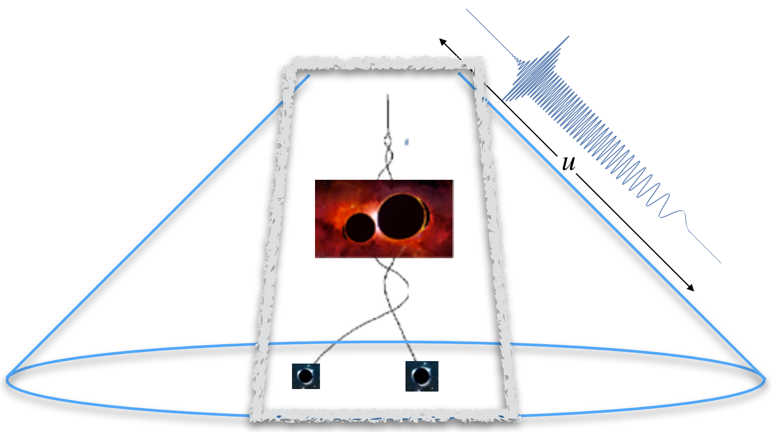

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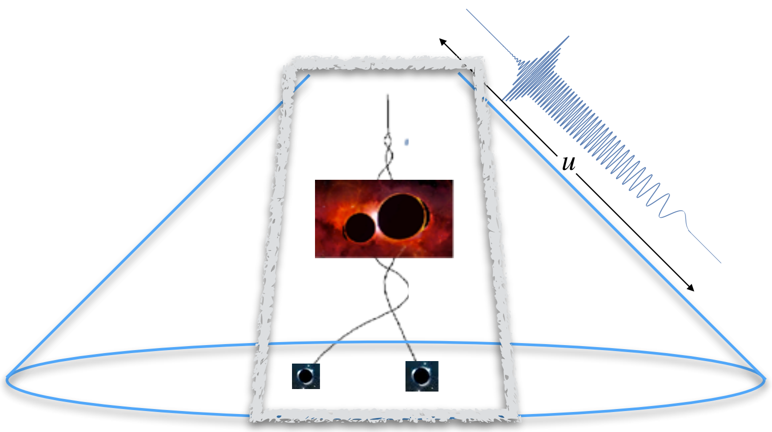

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Since Scri is a NEH in the unphysical spacetime then, should the symmetry groups match?

Almost! Scri: $G^{\rm BMS} = SO(3,1) \ltimes ST$ Physical NEH: $G^{\rm NEH} = SO(3,1) \ltimes ST \ltimes D$

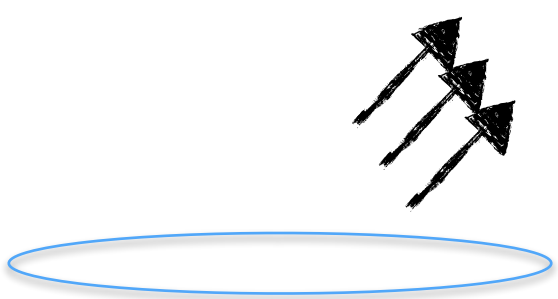
Asymptotic symmetries at spatial and null infinity

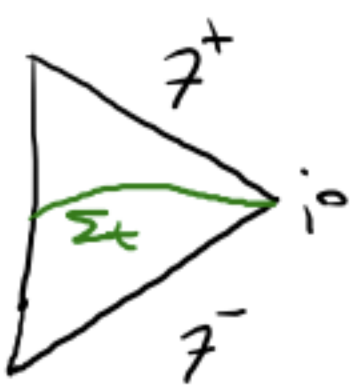
At fixed time:

(t, r, θ, ϕ)

$r \mapsto \infty$ Minkowski metric

misses all radiation





Spatial infinity

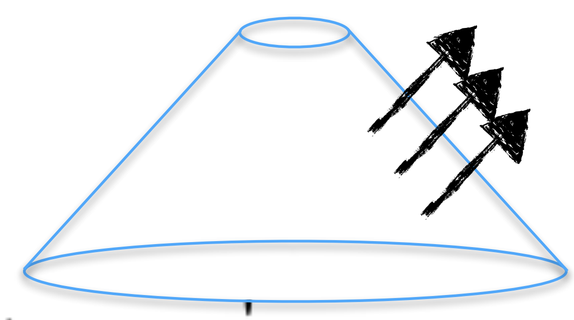
Poincare group $P^4 = SO(3,1) \ltimes T^4$

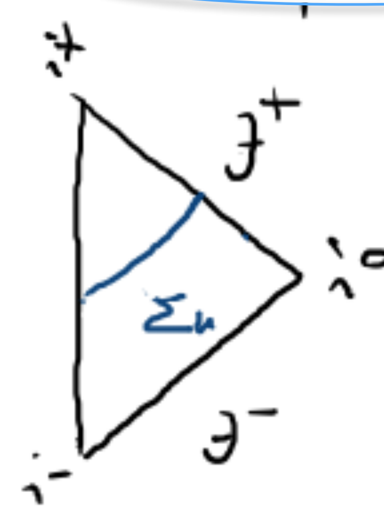
At fixed retarded time:

(u, r, θ, ϕ) $u := t - r$

$r \mapsto \infty$ Minkowski metric

captures all radiation





Null infinity

BMS group $G^{\mathrm{BMS}} = SO(3,1) \ltimes ST$

Why super-translations?

Very easy to gain intuition using Penrose's conformal picture :

$$d\hat{s}^2 = 2dud\Omega + q_{AB}dx^A dx^B - \Omega^2 du^2$$

Global symmetries: $\pounds_\xi \eta = 0 \quad \Rightarrow \quad \pounds_\xi \hat{\eta}_{\mu\nu} - 2\alpha_\xi \hat{\eta}_{\mu\nu} = 0$

conformal isometries of full unphysical flat metric

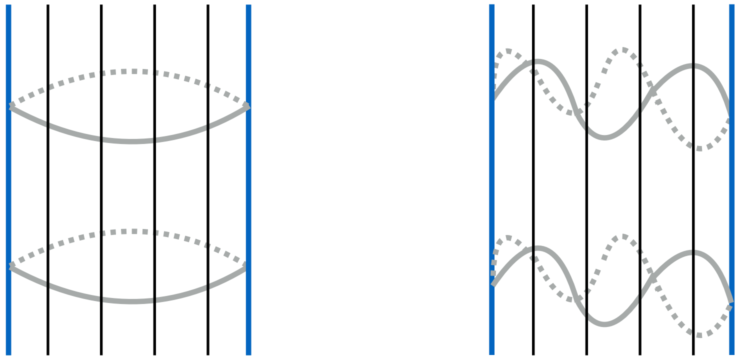
$$P^4 = SO(3,1) \ltimes T^4$$

Asymptotic symmetries: $\pounds_\xi \hat{\eta}_{\mu\nu} - 2\alpha_\xi \hat{\eta}_{\mu\nu} = O(\Omega)$

conformal isometries of *leading order* unphysical flat metric

$$G^{\text{BMS}} = SO(3,1) \ltimes ST$$

$\mathcal{I}$:



Why super-translations?

Very easy to gain intuition using Penrose's conformal picture :

$$d\hat{s}^2 = \underline{2dud\Omega + q_{AB}dx^A dx^B} - \Omega^2 du^2$$

Global symmetries: $\pounds_\xi \eta = 0 \quad \Rightarrow \quad \pounds_\xi \hat{\eta}_{\mu\nu} - 2\alpha_\xi \hat{\eta}_{\mu\nu} = 0$ Only valid for Minkowski

conformal isometries of full unphysical flat metric

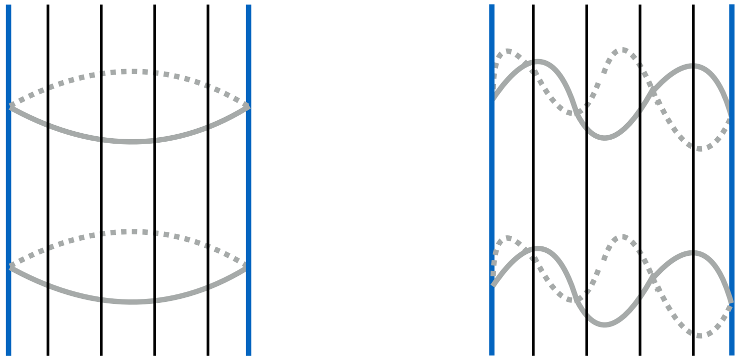
$$P^4 = SO(3,1) \ltimes T^4$$

Asymptotic symmetries: $\pounds_\xi \hat{\eta}_{\mu\nu} - 2\alpha_\xi \hat{\eta}_{\mu\nu} = O(\Omega)$ **Property of any AF spacetime**

conformal isometries of *leading order* unphysical flat metric

$$G^{\rm BMS} = SO(3,1) \ltimes ST$$

$\mathcal{I}$:



AF metrics match the flat metric at Scri,
with departures at subleading orders

Null infinity symmetry: BMS

For the general derivation, there are two equivalent approaches in the literature:

- Coordinate-dependent definition (**Bondi-Sachs**) based on null foliations
  - ➪ Asymptotic symmetries as residual diffeomorphisms preserving the boundary conditions
- Coordinate-independent definition (**Penrose-Geroch**) based on conformal compactifications
  - ➪ Asymptotic symmetries as isometries of the universal structure shared by AF metrics

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  - ➪ Asymptotic symmetries as isometries of the universal structure shared by AF metrics

Universal structure at Scri:

The induced metric at Scri is just Minkowski, hence universal;

$$\begin{cases} q_{\mu\nu} = \underleftarrow{\hat{g}_{\mu\nu}} = \Omega^2 \underleftarrow{g_{\mu\nu}} \\ n = d\Omega \end{cases}$$

however, there is freedom in choosing the conformal factor:

$$\Omega \to \omega\Omega \qquad ➪ \qquad \{(q_{\mu\nu}, n^\mu) \sim (\omega^2 q_{\mu\nu}, \omega^{-1} n^\mu)\}$$

$\pounds_n \omega = 0$ in order to preserve the Bondi condition (Scri as a WIH)

Isometries of the universal structure:

$$\pounds_\xi q_{\mu\nu} = 2\alpha_\xi q_{\mu\nu}, \quad \pounds_\xi n^\mu = -\alpha_\xi n^\mu$$

BMS group

$$\pounds_\xi q_{\mu\nu} = 2\alpha_\xi q_{\mu\nu}, \quad \pounds_\xi n^\mu = -\alpha_\xi n^\mu$$

$$G^{\rm BMS} = SO(3,1) \ltimes ST$$

Infinite dimensional extension of the Poincare group with structure constants

$$[\xi,\chi] = (T_\xi \dot{f}_\chi + Y_\xi[f_\chi] - (\xi \leftrightarrow \chi))\partial_u + [Y_\xi, Y_\chi]^A \partial_A$$

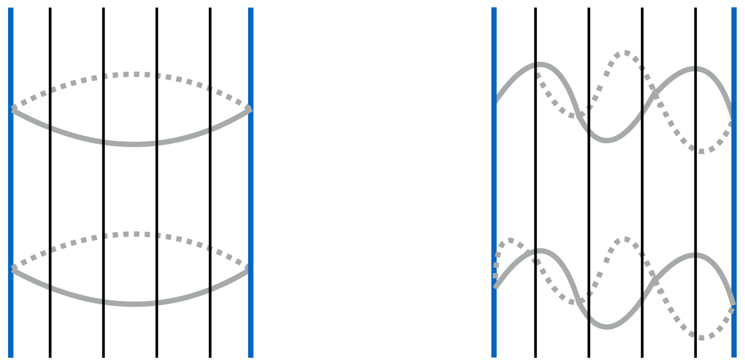
In particular, super-translations don't commute with rotations and boosts:

$$[\xi_Y, \chi_T] = \xi_{T'}, \qquad T' = Y^A\partial_A T - \dot{f} T$$

⇨ Infinitely many Lorentz subgroups, one per `cut'

Only ideal subalgebra is the global translations

$\mathcal{I}$:



Physical WIH symmetry: BMS+Dilation

Universal structure at a physical WIH: $\sigma = \theta = 0 \quad \Rightarrow \quad \pounds_n q_{\mu\nu} = 0$

Since the induced metric is time-independent, it is the pull-back of a 2d space-like metric

Any 2d space-like metric can be written as a round sphere metric up to a conformal transformation:

$$q = \varphi_*(\psi^2 \mathring{q})$$

The existence of a round sphere metric is thus a universal property of any metric admitting a WIH

But there is a 3 parameter family of round spheres: Lorentz boosts

⇨ the universal structure is really an *equivalence class* of round spheres $\{\mathring{q} \sim \omega^2 \mathring{q}\}$

How about the normal vector? recall that the multipole moments are contained in the Hajicek

$D_\mu l^\nu = \eta_\mu l^\nu$ ⇨ useful to rescale also l so that multipole moments can be defined wrt $\mathring{q}$

$$\{(\mathring{q}_{\mu\nu}, [\mathring{l}^\mu]) \sim (\omega^2 \mathring{q}_{\mu\nu}, \omega^{-1}[\mathring{l}^\mu])\}$$

Isometries of the universal structure:

$$\pounds_\xi q_{\mu\nu} = 2\alpha_\xi q_{\mu\nu}, \quad \pounds_\xi n^\mu = -(\alpha_\xi + W_\xi)n^\mu$$

The reason why in the physical case we have on more generator is because of the different universal structures. **Remark:** extra generator related to the area as Noether charge (Wald '94)

Symmetries of physical WIH vs null infinity

Null infinity:

$$\{(q_{\mu\nu}, n^\mu) \sim (\omega^2 q_{\mu\nu}, \omega^{-1} n^\mu)\}$$

$$\pounds_\xi q_{\mu\nu} = 2\alpha_\xi q_{\mu\nu}, \quad \pounds_\xi n^\mu = -\alpha_\xi n^\mu$$

$$G^{\text{BMS}} = SO(3,1) \ltimes ST$$

Physical WIH:

$$\{(\mathring{q}_{\mu\nu}, [\mathring{l}^\mu]) \sim (\omega^2\, \mathring{q}_{\mu\nu}, \omega^{-1}[\mathring{l}^\mu])\}$$

$$\pounds_\xi q_{\mu\nu} = 2\alpha_\xi q_{\mu\nu}, \quad \pounds_\xi n^\mu = -(\alpha_\xi + W_\xi) n^\mu$$

$$G^{\text{NEH}} = SO(3,1) \ltimes ST \ltimes D$$

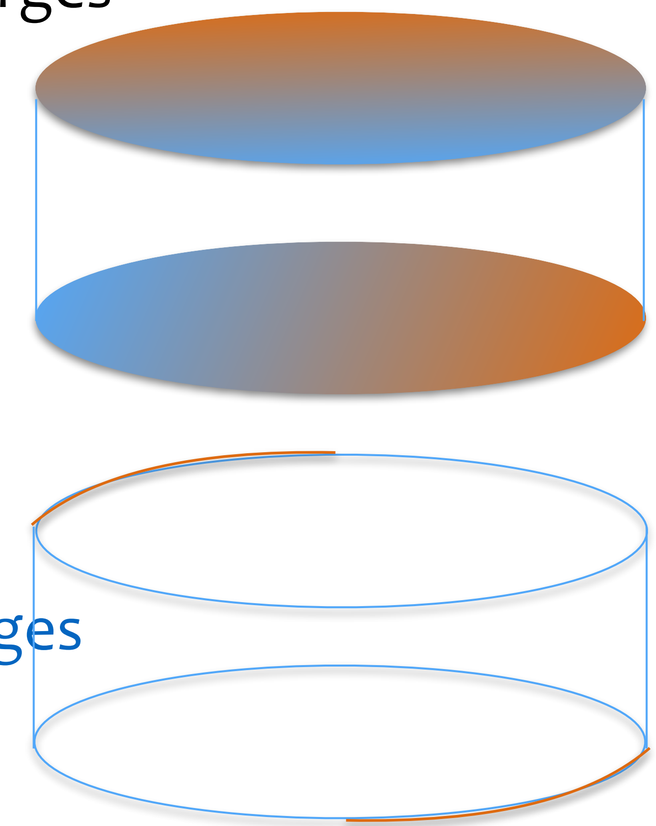
The reason why in the physical case we have on more generator is because of the different universal structures

Remark: extra generator related to the area as Noether charge (Wald '94)

III - Noether charges for horizon symmetries

Noether theorem in gravity

If we have symmetries at the horizon, we can study their Noether charges and gain intuition on the physical interpretation of the symmetries



Even though Noether's theorem was motivated precisely by GR, its application to GR is very subtle, and often badly used

Noether charges for residual gauge transformations are surface charges
⇨ completely ambiguous (see e.g. problems of Komar expression)

In some conditions the ambiguities can be fixed looking at the canonical generators, but in the presence of radiation, some of the boundary symmetries are **not** Hamiltonian vector fields ⇨ no canonical generators

Noether theorem in gravity

Because of these difficulties, conservation/flux-balance laws for the asymptotic symmetries were first identified using purely Einstein's equations and physical requirements (ADM '60, Regge-Teitelboim '74, Geroch '77, Ashtekar-Streubel '79, Dray-Streubel '84)

and only later identified as canonical generators (Iyer-Wald '94, Wald-Zoupas '99)
and as improved Noether charges (Harlow-Wu '19, Odak-Rignon-Bret-SSp '22)

The Wald-Zoupas prescription was then applied to general null hypersurfaces and NEHs in Chandrasekaran-Flanagan-Prabhu '18, Ashtekar-Khera-Kolanowski-Lewandowski-etc '22
Odak-Rignon-Bret-SSp '23

The Wald-Zoupas prescription solves the ambiguities by imposing criteria on the symplectic potential

BMS charges

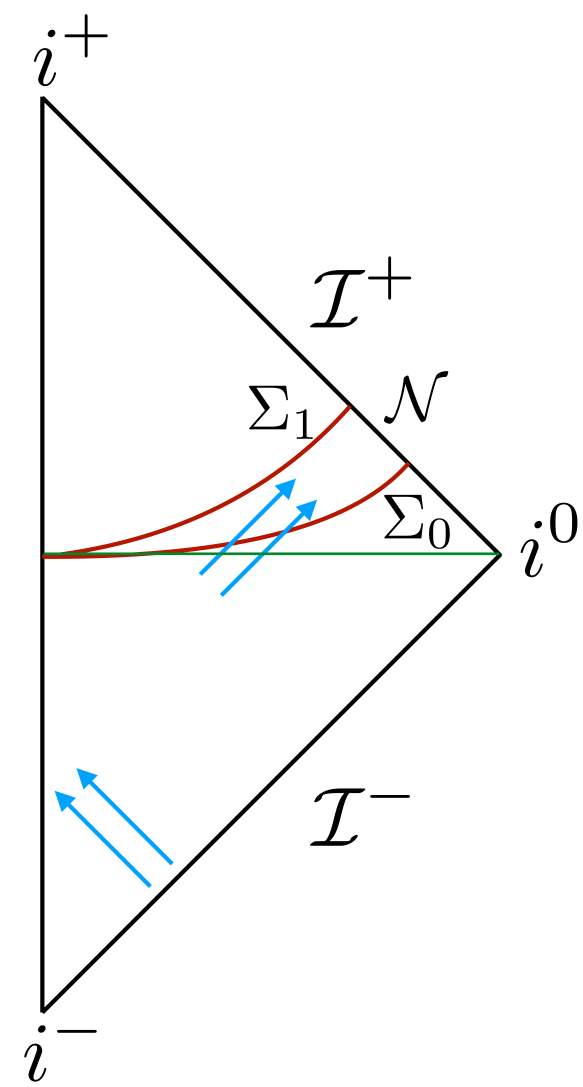
$$G^{\text{BMS}} = SO(3,1) \ltimes ST$$

$$\xi = fn + Y \qquad f = T + \frac{u}{2} D_a Y^a$$

$$Q_\xi^{\text{BMS}} = \frac{1}{8\pi G} \oint_S (2f M_\rho + Y^a J_a)\epsilon_S$$

$$M = -\left(\psi_2 + \sigma\dot{\bar{\sigma}} + \frac{1}{2}\left(\eth^2\bar{\sigma} - cc\right)\right) = -\text{Re}(\psi_2 + \sigma\dot{\bar{\sigma}}),$$

$$m^a J_a = -\left(\psi_1 + \sigma\eth\bar{\sigma} + \frac{1}{2}\eth(\sigma\bar{\sigma})\right).$$



The Newman-Penrose tetrad is fixed requiring
n to be tangent to Scri and *l* tangent to the cut

This is the unique set of charges that satisfies the 3 criteria of Wald-Zoupas

0. Being associated to the standard Einstein-Hilbert symplectic 2-form
1. Being covariant
2. Being conserved when the Bondi news vanish

Horizon charges

For a general null hypersurface, with CFP boundary conditions:

$$G^{\text{CFP}} = \text{Diff}(S) \ltimes ST \ltimes W \qquad\qquad Q^c_\xi = \oint \Big(T\theta_\lambda + Y^\mu\eta_\mu + W(1-\lambda\theta_\lambda)\Big)\epsilon_S$$

• Satisfy flux-balance laws such that they are conserved when the null hypersurface is shear free and expansion free

• Special case of NEH: $G^{\text{NEH}} = SO(3,1) \ltimes ST \ltimes D$

• Weyl charge used as dynamical entropy (Hollands-Wald-Zhang '24)

Odak-Rignon-Bret-SSp '23:

There is a one-parameter family of charges,

correspondent to different choices of polarizations in the phase space

$$Q^c_\xi = \oint \Big(\frac{c}{2}T\theta_\lambda + Y^\mu\eta_\mu + W(1-\frac{c}{2}\lambda\theta_\lambda)\Big)\epsilon_S$$

• Dirichlet polarization: c=2

• York-type conformal polarization: c=1

This alternative polarization leads to a positive-definite flux and a notion of dynamical entropy that increases only during collapse (Rignon-Bret '23)

Charges covariance

The charges so constructed provide a realization of the symmetry in the phase space

$$\delta_\chi Q_\xi - \oint i_\chi j_\xi \hat{=} Q_{[\xi,\chi]}$$

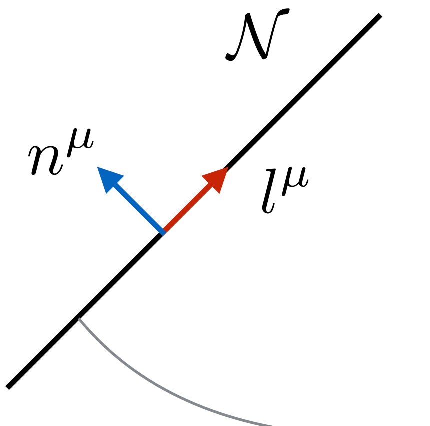
This property is *guaranteed* by Wald-Zoupas covariance (Rignon-Bret-SS '24)

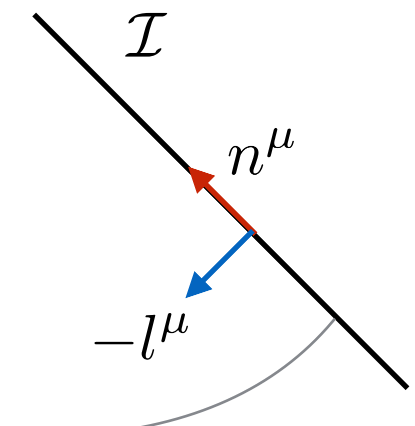
For instance the famous field-dependent 2-cocycle found for the BMS charge algebra in Barnich-Troessaert '11 is entirely due to the use of non covariant charges, and can be removed correcting the prescription

IV - Weaker boundary conditions and larger symmetry groups

Boundary conditions and residual diffeomorphisms

finite distance $l_\mu \overset{\mathcal{N}}{=} -N\partial_\mu\Phi$	residual diffeos	null infinity $n_\mu = \partial_\mu\Omega$
$\delta\Phi = 0$	$\mathrm{Diff}(\mathcal{N})$	$\delta\Omega = 0$
$\delta l_\mu = 0$	$\mathrm{Diff}(\mathcal{N})$	$\delta n_\mu = 0$
$\delta l^\mu = 0$	$\mathrm{Diff}_l(\mathcal{N}) := \mathrm{Diff}(S) \ltimes \mathrm{Diff}(\mathbb{R})^S$	$\delta n^\mu = 0$
$l^\nu \delta g_{\mu\nu} = 0$		$n^\nu \delta g_{\mu\nu} = 0$
CFP $\delta k = 0$	$(\mathrm{Diff}(S) \ltimes \mathbb{R}^S_W) \ltimes \mathbb{R}^S_T$	$\delta k = 0$ BMSW
iff $\mathcal{N}$ is a NEH: $\delta \overset{\circ}{q}_{\mu\nu} = 0$	$\mathrm{Diff}(S) \ltimes \mathbb{R}^S_T$	$\delta\sqrt{q} = 0$ gBMS
$SO(3,1) \ltimes \mathbb{R}^S_T \ltimes D$	$SO(3,1) \ltimes \mathbb{R}^S_T$	$\delta q_{\mu\nu} = 0$ BMS





Donnay et al ‘15, Adami et al ’21, Grumiller et al ‘21,
Chandrasekaran-Flanagan-Prabhu ’18
Ashtekar-Khera-Kolanowski-Lewandowski ’21
Chandrasekaran-Flanagan ’23…

Campiglia-Laddha ‘15
Compere-Fiorucci-Ruzziconi ’18
Freidel-Oliveri-Pranzetti-SSp ’21
…

Conclusions

NEHs share many geometric features with Scri, but a completely different physics

- The shared features explain why the symmetry groups are so closely related
- The different physics is encoded in the dynamics

Symmetries of black hole horizons, of quasi-local horizons, and null hypersurfaces in general, can be studied using the same techniques of asymptotic symmetries like ADM and BMS: as residual diffeos compatible with boundary conditions, or equivalently as isometries of the universal structure

(The second method is coordinate-independent and tends to give easier geometric formulas)

The number of symmetries and the resulting symmetry groups depends on the boundary cond.
Relaxing too much the b.c. leads to charges that are never conserved ⇨ probably not useful

Ambiguities in the Noether charges can be removed insisting on (a generalization of) the Wald-Zoupas prescription

The same results can be obtained without selecting a symplectic potential, but instead equipping the phase space with a norm (Ashtekar-SSp '24)

Can this prescription be applied to celestial symmetries?