# Killing horizon data

Marc Mars

University of Salamanca

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Black holes and their symmetries

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A spacetime $(\mathcal{M}, g)$ admits a Killing horizon $\mathcal{H}$ provided:

(i) $\mathcal{H}$ is a null hypersurface.

(ii) $(\mathcal{M}, g)$ admits a Killing vector $\eta$.

(iv) $\eta$ is null, tangent and non-zero at $\mathcal{H}$.

- Killing generator: $\bar{\eta} := \eta|_{\mathcal{H}}$
- First fundamental form: $\gamma$

General properties:

$X, W \in \mathfrak{X}(\mathcal{H})$

- $\gamma$ is degenerate $\gamma(\bar{\eta}, \cdot) = 0$,
- $\mathcal{L}_{\bar{\eta}}\gamma = 0$,
- $\nabla_X \eta \overset{\mathcal{H}}{=} \boldsymbol{\sigma}(X)\eta$, $\boldsymbol{\sigma}$ connection one-form,
- $\mathcal{L}_{\bar{\eta}}\boldsymbol{\sigma} = 0$,
- $\nabla_X W$ is tangent to $\mathcal{H}$.

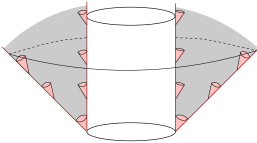
Surface gravity: $\kappa = \boldsymbol{\sigma}(\eta)$.

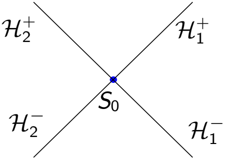
- $\kappa$ is constant on $\mathcal{H}$ iff $\mathrm{Ric}_g(\eta, X) \overset{\mathcal{H}}{=} 0$, $\forall X$ tangent to the horizon.

# Characteristic initial value for bifurcate Killing horizons

Characteristic initial value problem:

- Data prescribed on two intersecting null hypersurfaces,
- Data does not involve transverse derivatives,
- Existence of spacetime from such data: [Rendall, '90]






Bifurcate Killing horizon:

- Four Killing horizons of $\eta$ emanating from a spacelike codimension two surface $S_0$.

**Theorem (Characteristic IVP Bifurcate KH [Rácz 07'], [Chruściel, Paetz, '13])**

*Assume:* • $(\mathcal{S}, h)$: *Riemannian manifold of dimension* $n \geq 2$, • $\zeta \in \mathfrak{X}^\star(\mathcal{S})$.

*Then:* $\exists$ *a* $\lambda$*-vacuum spacetime* $(\mathcal{M}^{n+2}, g)$ *with boundary admiting a Killing vector* $\eta$.

- $\partial\mathcal{M} = \mathcal{H}_1^+ \cup \mathcal{H}_2^+ \cup \mathcal{S}_0$
- $\mathcal{H}_1^+$, $\mathcal{H}_2^+$ *Killing horizons of* $\eta$.
- $\overline{\mathcal{H}_1^+} \cap \overline{\mathcal{H}_2^+} = \mathcal{S}_0$
- $(\mathcal{S}_0, g|_{\mathcal{S}_0})$ *isometric to* $(\mathcal{S}, h)$
- $\zeta$: *torsion one-form of* $\mathcal{S}_0$.

Recall: $\zeta(X) := g(\nabla_X \nu^-, \nu^+)$, $\nu^\pm$ null normals, $X \in \mathfrak{X}(\mathcal{S}_0)$.

# Existence outside the horizon?

Characteristic IVP: existence is only to the future/past. Existence in the exterior?

**Theorem (Analyticity [Chruściel '05])**

*Assume:* $(\mathcal{M}, g)$ *$\lambda$-vacuum with a Killing vector $\eta$ defining a bifurcate Killing horizon.*

*If $\eta$ is hypersurface orthogonal, then outside the bifurcate Killing horizon the metric $g$ is analytic up to the boundary*

**Theorem (Staticity [M. & Chruściel '23])**

*Assume:* $(\mathcal{M}, g)$ *$\lambda$-vacuum with a Killing vector $\eta$ defining a bifurcate Killing horizon.*

*If the torsion one-form $\zeta$ of the bifurcation surface is closed, then $\eta$ is hyp. orthogonal.*

- If data $(S, h, \zeta)$ is not analytic and $d\zeta = 0 \rightsquigarrow$ Exterior region cannot be $\lambda$-vacuum.
- Expected to be true in the stationary case as well.

In the analytic case, there is an existence result that only needs a non-degenerate Killing horizon

- No bifurcate Killing horizon, just one of the four branches.

# Existence and uniqueness from non-degenerate Killing horizons

[Moncrief '82] In dimension four, an analytic Ricci flat spacetime admitting a non-degenerate Killing horizon $\mathcal{H} = \mathbb{R} \times \mathbb{T}^2$ can be constructed uniquely from six (coordinate dependent) functions on $\mathcal{H}$.

- Uniqueness part reformulated in geometric terms and extended to the smooth case:

[O. Petersen & K. Kroencke '23] Let $(\mathcal{M}, g)$ be Ricci flat and admit a non-degenerate Killing horizon $\mathcal{H}$. The asymptotic expansion of $g$ at $\mathcal{H}$ is determined by

$$\{\mathcal{H}, \overline{g}, \mathcal{V}\}$$

where $(\mathcal{H}, \overline{g})$ is Riemannian space admitting a unit Killing vector $\mathcal{V}$.

- Again, data does not involve transverse derivatives.

Aim of this talk:

- Identify the free data on a Killing horizon
- Prove existence of a spacetime in the $\lambda$-vacuum case
- Work intrisically on the null hypersurface (detached from spacetime)
- Allow for any topology on $\mathcal{H}$ and zeroes of the KV.

# Null manifold

Null manifold $(\mathcal{H},\gamma)$: manifold $\mathcal{H}$ with a symmetric tensor $\gamma_{ab}$ of signature $\{0,+,\cdots,+\}$.

- $T_p\mathcal{H}$ has a privileged one-dimensional subspace: The radical of $\gamma$:

$$\text{Rad}_\gamma|_p \subset T_p\mathcal{H}, \qquad \text{Rad}_\gamma|_p := \{n \in T_p\mathcal{H}, \gamma(n,\cdot) = 0\}.$$

$(\mathcal{H},\gamma)$ is called ruled if there exists a nowhere zero vector field $n$ satisfying $n|_p \in \text{Rad}_p$.

Some basic facts:

- Any $(\mathcal{H},\gamma)$ admits a double cover which is ruled.

$$\exists f : \widetilde{\mathcal{H}} \longrightarrow \mathcal{H} \quad \text{double cover, such that} \quad (\widetilde{\mathcal{H}}, \widetilde{\gamma} := f^\star(\gamma)) \quad \text{is ruled.}$$

- The geometry of the double cover $(\widetilde{\mathcal{H}},\widetilde{\gamma})$ is simpler.
- Many of the results can be transferred to $(\mathcal{H},\gamma)$.

A geometry on $(\mathcal{H},\gamma)$ can be developed in full generality. Here we assume:

- $(\mathcal{H},\gamma)$ is ruled.
- $\mathcal{L}_n\gamma = 0$ where $n$ is a generator of the radical.

# Null metric hypersurface data

To study $(\mathcal{H}, \gamma)$: useful to enlarge the definition and introduce an equivalence relation.

Null metric hypersurface data: $\mathcal{D} = \{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ where $\boldsymbol{\ell} \in \mathfrak{X}^{\star}(\mathcal{H})$ and $\ell^{(2)} \in \mathcal{F}(\mathcal{H})$, provided $\boldsymbol{\ell}(n) \neq 0$ everywhere.

Equivalence relation:

- Group: $\mathcal{G} := \mathcal{F}^{\star}(\mathcal{H}) \times \mathfrak{X}(\mathcal{H})$ with the product

$$(u, W) \cdot (z, V) = (uz, V + z^{-1} W).$$

- Action of $(z, V) \in \mathcal{G}$ on $\mathcal{D}$: $\quad \mathcal{G}_{(z,V)}(\gamma) = \gamma,$

$$\mathcal{G}_{(z,V)}(\boldsymbol{\ell}) = z(\boldsymbol{\ell} + \gamma(V, \cdot)), \qquad \mathcal{G}_{(z,V)}(\ell^{(2)}) = z^2 \left(\ell^{(2)} + 2\boldsymbol{\ell}(V) + \gamma(V, V)\right).$$

Definition: $\quad \mathcal{D}_1 \sim \mathcal{D}_2$ (geometrically equivalent) provided $\mathcal{G}_{(z,V)}(\mathcal{D}_1) = \mathcal{D}_2$.

There is a one-to-one correspondence between ruled null manifold structures on $\mathcal{H}$ and equivalence classes of null metric hypersurface data in $\mathcal{H}$.

# The metric hypersurface data connection $\overset{\circ}{\nabla}$

- Given $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ fix n by $\boldsymbol{\ell}(n) = 1$  Consequence: $\mathcal{G}_{(z,V)}(n) = z^{-1} n$.

Recall we are assuming $\mathcal{L}_{\mathrm{n}}\gamma = 0$.

There exists precisely one torsion-free connection $\overset{\circ}{\nabla}$ defined by:

$$\overset{\circ}{\nabla}_a \gamma_{bc} = 0, \qquad \overset{\circ}{\nabla}_a \boldsymbol{\ell}_b + \overset{\circ}{\nabla}_b \boldsymbol{\ell}_a = 0.$$

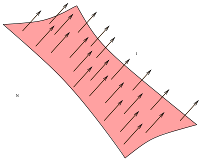
- $\overset{\circ}{\nabla}$ has good behaviour under the gauge group $\mathcal{G}$:
  - (i) $\left(\mathcal{G}_{(u,W)} \circ \mathcal{G}_{(z,V)}\right)(\overset{\circ}{\nabla}) = \mathcal{G}_{(u,W)\cdot(z,V)}(\overset{\circ}{\nabla})$.
  - (ii) $\mathcal{G}_{(z,V)}(\overset{\circ}{\nabla}) = \overset{\circ}{\nabla} + \dfrac{1}{2z} n \otimes \left(\mathcal{L}_{zV}\gamma + \boldsymbol{\ell} \otimes dz + dz \otimes \boldsymbol{\ell}\right)$

## Connection between the detached and the embedded pictures.

So far $(\mathcal{H}, \gamma)$ has been considered on its own.

- Need to connect with the geometry of submanifolds.

Link relies on the notion of rigging vector.



[Schouten '54]: A rigging $\xi$ is a vector field along a hypersurface $\mathcal{N}$ transverse everywhere to $\mathcal{N}$.

- Exists iff $\mathcal{N}$ is two-sided.

### Definition (Embedded metric hypersurface data)

Null metric hypersurface data $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is $(\Phi, \xi)$-embedded in $(\mathcal{M}, g)$ if $\exists$ an embedding $\Phi : \mathcal{H} \hookrightarrow \mathcal{M}$ and a rigging $\xi$ along $\Phi(\mathcal{H})$ such that,

$$\Phi^\star(g) = \gamma, \qquad \Phi^\star\left(g(\xi, \cdot)\right) = \boldsymbol{\ell}, \qquad g(\xi, \xi) = \ell^{(2)}.$$

- Gauge group $\mathcal{G}$ is the manifestation of non-uniqueness of the rigging.

$\mathcal{D} := \{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is $(\Phi, \xi)$-embedded in $(\mathcal{M}, g)$ iff $\mathcal{G}_{(z,V)}(\mathcal{D})$ is $(\Phi, \xi')$-embedded in $(\mathcal{M}, g)$ with $\xi' := z\xi + \Phi^\star(V)$.

# Extrinsic curvature and full hypersurface data

- $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ encodes intrinsic properties of the hypersurface.
- Extrinsic properties incorporated in the formalism with an additional tensor:

Null metric hypersurface data $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ together with a symmetric $(0,2)$-tensor $\mathrm{Y}$ defines null hypersurface data provided

$$\mathcal{G}_{(z,\zeta)}(\mathrm{Y}) := z\mathrm{Y} + \frac{1}{2}\left(\mathcal{L}_{z\zeta}\gamma + \boldsymbol{\ell} \otimes dz + dz \otimes \boldsymbol{\ell}\right).$$

- The connection $\overline{\nabla} := \overset{\circ}{\nabla} - n \otimes \mathrm{Y}$ is gauge invariant (relies crucially on $\mathcal{L}_n\gamma = 0$).

The notion of embeddedness gives geometric meaning to $\mathrm{Y}$.

Null metric hypersurface data is $(\Phi, \xi)$-embedded in $(\mathcal{M}, g)$ provided

- $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is $(\Phi, \xi)$-embedded in $(\mathcal{M}, g)$,
- $\mathrm{Y} = \frac{1}{2}\Phi^\star(\mathcal{L}_\xi g)$.

- Relationship between $\overset{\circ}{\nabla}$, $\overline{\nabla}$ and the Levi-Civita connection $\nabla$ of $(\mathcal{M}, g)$:

Fix $\nu$: normal to $\Phi(\mathcal{H})$ satisfying $g(\nu, \xi) = 1$. Then, along tangential directions to $\mathcal{H}$,

$$\nabla = \overline{\nabla} \qquad \text{and} \qquad \nabla = \overset{\circ}{\nabla} - \nu\mathrm{Y}.$$

- $\{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ encodes the spacetime metric $g$ along $\Phi(\mathcal{H})$.
- Hypersurface data encodes $g$, $\nabla$ and extrinsic curvature of the hypersurface.

What about ambient curvature?

$\Phi^{\star}(\mathrm{Ric}_g)$ computable from hypersurface data

- Introduce a tensor defined in terms of the data

**Definition (Constraint tensor [M. & M. Manzano '23])**

Let $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathrm{Y}\}$ be null hypersurface data. The constraint tensor $\mathcal{R}_{ab}$ is

$$\mathcal{R}_{ab} := -2\mathcal{L}_n \mathrm{Y}_{ab} - 2\varkappa \mathrm{Y}_{ab} - 2\overset{\circ}{\nabla}_{(a}\omega_{b)} - 2\omega_a\omega_b + \text{terms depending on } \{\gamma, \boldsymbol{\ell}, \ell^{(2)}\}$$

where $\omega_a := \frac{1}{2}\mathcal{L}_n \ell_a - \mathrm{Y}_{ab} n^b$ and $\varkappa := \omega(n)$.

- Gauge behaviour: $\mathcal{G}_{(z,V)}(\mathcal{R}) = \mathcal{R}$.
- By construction, embedded data satisfy $\Phi^{\star}(\mathrm{Ric}_g) = \mathcal{R}$.

# The Lie derivative of the connection

- The difference of connections is a tensor.
- The Lie derivative of a connection $\nabla$ is a tensor. Notation $\Sigma[\eta] := \mathcal{L}_\eta \nabla$

Basic properties:

- $\Sigma[\eta]$ is symmetric if $\nabla$ torsion-free,
- $\Sigma[f\eta] = f\Sigma[\eta] + \text{Hess}\, f \otimes \eta + 2\nabla f \otimes_s \nabla\eta$.
- $\mathcal{L}_\eta Ric_{ab} = \nabla_c \Sigma[\eta]^c{}_{ab} - \nabla_b \Sigma[\eta]^c{}_{ac}$

Application to the connection $\overline{\nabla}$: Define $\boldsymbol{s} = \frac{1}{2}\mathcal{L}_n \boldsymbol{\ell}$.

**[M. & M. Manzano '23]**

$\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$: metric hypersurface data satisfying $\mathcal{L}_n \gamma = 0$. The tensor $\mathcal{L}_n \overline{\nabla}$ is

$$(\mathcal{L}_n \overline{\nabla})^c_{ab} = n^c \left( \overset{\circ}{\nabla}_a \boldsymbol{s}_b + \nabla_b \boldsymbol{s}_a - \mathcal{L}_n \mathrm{Y}_{ab} \right)$$

# Abstract notion of horizon (I)

- A spacetime $(\mathcal{M}, g)$ with a horizon $\mathcal{N}$ admits a privileged vector field $\eta$.
  - At $\mathcal{N}$, $\eta$ is null and tangent.

Detached level: assume $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ admits a privileged vector $\bar{\eta}$ satisfying $\gamma(\bar{\eta}, \cdot) = 0$.

- Define $\alpha \in \mathcal{F}(\mathcal{H})$ by $\bar{\eta} = \alpha n$,
- $\mathcal{G}_{(z,V)}(\bar{\eta}) = \bar{\eta}$,
- We allow fixed points (vanishing $\alpha$),
- $\mathcal{G}_{(z,V)}(\alpha) = z\alpha$.

If $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is enlarged to $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathrm{Y}\}$: Two key notions associated to $\bar{\eta}$.

Surface gravity: $\kappa := \alpha\omega(n) + n(\alpha)$ (recall $\omega = s - \mathrm{Y}(n, \cdot)$)

$\mathcal{L}_{\bar{\eta}}\overline{\nabla} := n \otimes \Pi^{(\eta)}$: $\Pi^{(\eta)}_{ab} = \mathring{\nabla}_a \mathring{\nabla}_b \alpha + 2 s_{(a}\nabla_{b)}\alpha + 2\alpha \mathring{\nabla}_{(a} s_{b)} + n(\alpha)\mathrm{Y}_{ab} - \mathcal{L}_{\alpha n}\mathrm{Y}_{ab}$

Gauge properties: $\mathcal{G}_{(z,V)}(\kappa) = \kappa$, $\mathcal{G}_{(z,V)}(\Pi^{(\eta)}) = z\Pi^{(\eta)}$,

Embedded interpretation:

Assume $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathrm{Y}\}$ (satisfying $\mathcal{L}_n \gamma = 0$) to be $(\Phi, \xi)$-embedded. Let $\eta$ be any extension of $\Phi_\star(\bar{\eta})$. Then

$$\Pi^{(\eta)} = -\tfrac{1}{2}\Phi^\star\left(\mathcal{L}_\xi \mathcal{L}_\eta g + \mathcal{L}_W g\right), \qquad W^\alpha := \xi_\beta \mathcal{L}_\eta g^{\alpha\beta}$$

$\Pi^{(\eta)}$ vanishes for a Killing horizon.

# Abstract notion of horizon (II)

- Combining with constrant tensor $\rightsquigarrow$ algebraic expression for Y:

**Master identity** [M. & M. Manzano]

$$\overset{\circ}{\nabla}_a \overset{\circ}{\nabla}_b \alpha + 2\omega_{(a} \overset{\circ}{\nabla}_{b)} \alpha + \alpha \left( \nabla_{(a}\omega_{b)} + \omega_a \omega_b + \frac{1}{2}\mathcal{R}_{ab} - \frac{1}{2} \overset{\circ}{\mathrm{Ric}}_{(ab)} + \frac{1}{2} \overset{\circ}{\nabla}_{(a} \boldsymbol{s}_{b)} - \frac{1}{2} \boldsymbol{s}_a \boldsymbol{s}_b \right)$$
$$+ \kappa \mathrm{Y}_{ab} - \Pi^{(\eta)}_{ab} = 0.$$

By comparison with the characteristic IVP:

- Data should involve metric hypersurface data + a one-form
- We cannot prescribe the full extrinsic tensor Y.

Idea: Use the master identity to trade information on $\mathcal{R}$ with information on Y.

Requires non-degenerate case: $\kappa \neq 0$ everywhere.

- Degenerate case $\kappa = 0$: Master identity already restricts the data $\{\mathcal{H}, \gamma, \alpha, \omega\}$.

Generalizes the Near Horizon Geometry equation in three directions

(i) No topological assumption $\mathcal{H} = \mathbb{R} \times \mathcal{S}$,

(ii) Allows for fixed points: $\alpha = 0$,

(iii) Applies to general $\Pi^{(\eta)}$: (e.g. conformal or homothetic horizons).

Contraction of $\Pi^{(\eta)}$ and the master identity with $n$ yields the following two identities:

(a) $\Pi^{(\eta)}(n,\cdot) = \mathcal{L}_{\alpha n}\omega - d(\mathcal{L}_n(\alpha))$ (b) $\Pi^{(\eta)}(n,\cdot) = \alpha\mathcal{R}(n,\cdot) + d\kappa$

- We defined $\omega$ in terms of Y, but Y cannot be prescribed.
  - View $\omega$ as an a priori independent object.
- View $\Pi^{(\eta)}$ and $\mathcal{R}$ also as prescribed quantities (e.g. $\Pi^{(\eta)} = 0$ and $\mathcal{R} = \lambda\gamma$).

Prescribed data $\{\mathcal{H}, \gamma, \omega, \alpha, \Pi^{(\eta)}, \mathcal{R}\}$ cannot be given arbitrarily:

Certainly one must satisfy (a), (b) and, in the degenerate case, the master identity.

Basic existence question: Given data $\{\mathcal{H}, \gamma, \alpha, \omega, \Pi^{(\eta)}, \mathcal{R}\}$ with $\kappa \neq 0$

Can the data be completed with $Y$ so that it can be embedded in a spacetime fulfilling the prescribed quantities?

# Basic existence result

- The expression of Y is dictacted by the master identity:

$$\mathrm{Y}_{ab} = \frac{1}{\kappa}\left(\Pi^{(\eta)}_{ab} - \mathring{\nabla}_a\mathring{\nabla}_b\alpha - 2\omega_{(a}\mathring{\nabla}_{b)}\alpha - \alpha\left(\nabla_{(a}\omega_{b)} + \omega_a\omega_b + \tfrac{1}{2}\mathcal{R}_{ab} - \tfrac{1}{2}\,\mathring{\mathrm{Ric}}_{(ab)} + \tfrac{1}{2}\mathring{\nabla}_{(a}\boldsymbol{s}_{b)} - \tfrac{1}{2}\boldsymbol{s}_a\boldsymbol{s}_b\right)\right)$$

First step: show existence of $(\mathcal{M}, g)$ where $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \mathrm{Y}\}$ can be embedded.

- Tasks:
  - (i) Prove $(\mathcal{L}_{\bar{\eta}}\nabla)(X, W) = \nu\,\Pi^{(\eta)}(X, W), \qquad X, W \in \mathfrak{X}(\mathcal{H})$
  - (ii) Prove $\mathrm{Y}(n, \cdot) = \boldsymbol{s} - \omega$
  - (iii) Prove $\Phi^\star(\mathrm{Ric}_g) = \mathcal{R}$

(i) is consequence of $\Phi(\mathcal{H})$ being totally geodesic (so, $\nabla_X Y$ is intrinsic to $\Phi(\mathcal{H})$).

(ii) is consequence of the data compatibility conditions

(a) $\Pi^{(\eta)}(n, \cdot) = \mathcal{L}_{\alpha n}\omega - d(\mathcal{L}_n(\alpha))$ (b) $\Pi^{(\eta)}(n, \cdot) = \alpha\mathcal{R}(n, \cdot) + d\kappa$

(iii) not true in general. Necessary and sufficient conditions can be found to ensure it.

- Consequence of the following general identity:

$$(\mathcal{L}_n + \kappa)\,\Pi^{(\eta)}_{ab} - \mathring{\nabla}_{(a}\Pi^{(\eta)}_{b)c}n^c - \Pi^{(\eta)}(n, n)\mathrm{Y}_{ab} - \tfrac{1}{2}\mathcal{L}_n\mathcal{R}_{ab} - \tfrac{1}{2}\kappa(\Phi^\star(\mathrm{Ric}_g) - \mathcal{R})_{ab} = 0$$

# Necessary and sufficient condition and definiton of Killing Horizon Data

To ensure $\Phi_\star(\mathrm{Ric}_g) = \mathcal{R}$, the prescribed data needs to satisfy an extra condition

$$(\mathcal{L}_n + \kappa)\,\Pi^{(\eta)}_{ab} - \overset{\circ}{\nabla}_{(a}\Pi^{(\eta)}_{b)c}n^c - \Pi^{(\eta)}(n,n)\mathrm{Y}_{ab} - \frac{1}{2}\mathcal{L}_n\mathcal{R}_{ab} = 0$$

- Automatically satisfied for Killing horizon ($\Pi^{(\eta)} = 0$)+ $\lambda$-vacuum ($\mathcal{R} = \lambda\gamma$).

## Killing horizon data

$\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \alpha, \omega\}$ defines (abstract or detached) Killing horizon data (KHD) provided

(i) $\{\mathcal{H}, \gamma, \boldsymbol{\ell}, \ell^{(2)}\}$ is null metric hypersurface data with $\mathcal{L}_n\gamma = 0$

(ii) $\mathcal{G}_{z,V}(\alpha) = z\alpha, \quad \mathcal{G}_{z,V}(\omega) = \omega - z^{-1}dz$

(iii) $\mathcal{L}_{\alpha n}\omega - d(\mathcal{L}_n(\alpha)) = 0$

(iv) $\kappa := \alpha\omega(n) + \mathcal{L}_n\alpha$ is constant.

(v) If $\kappa = 0$, master equation holds:

$$\overset{\circ}{\nabla}_a\overset{\circ}{\nabla}_b\alpha + 2\omega_{(a}\overset{\circ}{\nabla}_{b)}\alpha + \alpha\left(\nabla_{(a}\omega_{b)} + \omega_a\omega_b + \tfrac{1}{2}\lambda\gamma_{ab} - \tfrac{1}{2}\overset{\circ}{\mathrm{Ric}}_{(ab)} + \tfrac{1}{2}\overset{\circ}{\nabla}_{(a}\boldsymbol{s}_{b)} - \tfrac{1}{2}\boldsymbol{s}_a\boldsymbol{s}_b\right) = 0$$

- KHD can be embedded in $(\mathcal{M}, g)$ satisfying $\mathcal{L}_{\bar{\eta}}\nabla|_{\Phi(\mathcal{H})} = 0, \quad (\mathrm{Ric}_g - \lambda g)|_{\Phi(\mathcal{H})} = 0$.

Going to all higher orders requires substantial additional work: [M. & Sánchez-Pérez]

# Existence and asymptotic uniqueness from KHD data

- KHD non-degenerate: $\kappa$ is non-zero.

**Theorem (Existence [M. & G. Sánchez-Pérez '25])**

*Let $\mathcal{K} = \{\mathcal{H}^n, \gamma, \boldsymbol{\ell}, \ell^{(2)}, \alpha, \omega\}$ be non-degenerate KHD. Then, there exists a spacetime $(\mathcal{M}^{n+1}, g)$ and a smooth extension $\eta$ of $\overline{\eta} = \alpha n$ off $\Phi(\mathcal{H})$ such that:*

- *$\mathcal{K}$ is $(\Phi, \xi)$-embedded in $(\mathcal{M}, g)$,*
- *$\eta$ is a Killing vector of $g$,*
- *$(\mathcal{M}, g)$ satisfies the $\Lambda$-vacuum equations to all orders on $\Phi(\mathcal{H})$.*

Two KHD $\mathcal{K}$, $\mathcal{K}'$ are isometric when $\exists$ diffeomorphism $\psi : \mathcal{H} \to \mathcal{H}'$ and $(z, V) \in \mathcal{G}$ s.t.

$$\psi^\star(\{\gamma', \boldsymbol{\ell}', \ell^{(2)\prime}, \alpha', \omega'\}) = \mathcal{G}_{(z,V)}(\{\gamma, \boldsymbol{\ell}, \ell^{(2)}, \alpha, \omega\}).$$

**Theorem (Asymptotic uniqueness [M. & G. Sánchez-Pérez '24])**

*Assume: $\mathcal{K}$ and $\mathcal{K}'$ are isometric, non-degenerate KHD embedded in spacetimes $(\mathcal{M}, g)$ and $(\mathcal{M}', g')$ whose existence is established in the previous theorem.*

*Then: $\exists$ neighbourhoods $\mathcal{U} \supset \Phi(\mathcal{H})$ and $\mathcal{U}' \supset \Phi'(\mathcal{H}')$ and a diffeo $\Psi : \mathcal{U} \to \mathcal{U}'$ such that*

$$\Psi^\star\left(\mathcal{L}^{(k)}_{\xi'} g'\right) = \left(\mathcal{L}^{(k)}_{\xi} g\right) \quad k \in \mathbb{N} \cup \{0\}, \qquad \Psi_\star(\eta) = \eta'.$$

## Example: Schwarzschild-de Sitter/Nariai black hole in $d$ dimensions

- The SdS metric in advanced Eddington-Finkelstein coordinates:

$$g_{\mathrm{SdS}} = -\left(1 - \left(\frac{r_0}{r}\right)^{d-3} - \frac{\lambda}{d-3}\left(1 - \left(\frac{r_0}{r}\right)^{d-1}\right)\right) dv^2 + 2dvdr + r^2 \gamma_{\mathbb{S}^{d-2}}$$

Horizon at $r = r_0$ is:

- Black hole horizon if $\lambda r_0^2 < d-3$
- Cosmological horizon if $\lambda r_0^2 > d-3$
- Degenerate horizon if $\lambda r_0^2 = d-3$.

- The Nariai metric has a horizon of radius $r_0^2 = \frac{d-3}{\lambda}$ (degenerate & non-degenerate).
- Form adapted to non-degenerate KV: $g_{\mathrm{Nariai}} = u\,(1 + \lambda u)\,dv^2 + 2dvdu + \dfrac{d-3}{\lambda}\gamma_{\mathbb{S}^{d-2}}$

**Theorem** ([M. & Sánchez-Pérez '24])

*Let $(\mathcal{M}, g)$ be a $d$-dimensional spacetime satistfying $Ric(g) = \lambda g$, $\lambda > 0$.*

*Assume:*

(i) *$(\mathcal{M}, g)$ admits a non-degenerate Killing horizon $\mathcal{H}$ with spherical sections of radius $r_0$.*

(ii) *The corresponding KHD data satisfies $\boldsymbol{\omega} \wedge d\boldsymbol{\omega} = 0$.*

*Then:* *Near $\mathcal{H}$, $(\mathcal{M}, g)$ is isometric to SdS if $\lambda r_0^2 \neq d-3$ and to Nariai if $\lambda r_0^2 = d-3$.*

Thank you for your attention