Equivalence Principle and generalised accelerating black holes from binary systems

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Introduction & Motivations

- Accelerating black holes with NUT parameter are puzzling, while the rotating case (Accelerating Kerr) belongs to the Plebanski-Demianski class the non rotating metric (accelerating Taub-NUT) seems not to belong to Petrov type D.
- Solution generating techniques in GR can generate all axisymmetric and stationary solutions of the theory. In this context Einstein (-Maxwell) and Ernst equations are equivalent. Symmetries of the Ernst equations allow us to generate new non-trivial solutions stating from old ones.
- The Ehlers transformation of Ernst equations is able to add the gravitomagnetic mass to a given axisymmetric and stationary seed. Which is its action when applied to accelerating black holes?

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Introduction: Theory and Field Equations

Theory under consideration: General Relativity coupled with Maxwell electromagnetism

$$I[g_{\mu\nu}, A_\mu] := \frac{1}{16\pi G} \int d^4x \sqrt{-g} \left[\mathrm{R} - \frac{G}{\mu_0} \; F_{\mu\nu} F^{\mu\nu} \right] \quad .$$

Field equations for the metric $g_{\mu\nu}$ and electromagnetic vector potential A_μ

$$\mathrm{R}_{\mu\nu} - \frac{\mathrm{R}}{2} g_{\mu\nu} = \frac{2G}{\mu_0} \left(F_{\mu\rho} F_\nu{}^\rho - \frac{1}{4} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma} \right) \quad ,$$
$$\partial_\mu (\sqrt{-g} F^{\mu\nu}) = 0 \quad .$$

The Faraday tensor $F_{\mu\nu}$ is defined from the gauge potential, $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

The most generic axisymmetric and stationary spacetime, containing two commuting killing vectors ∂_t and ∂_φ, can be written, for this theory, in the Lewis-Weyl-Papapetrou (LWP) form as

$$ds^2 = -f\,(dt - \omega d\varphi)^2 + f^{-1} \left[\rho^2 d\varphi^2 + e^{2\gamma} \left(d\rho^2 + dz^2 \right) \right] \; .$$

All the three structure functions appearing in the metric f, ω and γ depends only on the non-Killing coordinates (ρ, z).

A generic electromagnetic potential compatible with the spacetime symmetries, and the circularity of the LWP metric, is given by $A = A_t(\rho, z) dt + A_\varphi(\rho, z) d\varphi$.

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Ernst (Phys. Rev. 1968) discovered that, when the Einstein field equations are restricted to the axisymmetric and stationary LWP ansatz, they reduce to a couple of complex vectorial differential equations

Ernst Equations

$$\left(\text{Re } \mathcal{E} + |\mathbf{\Phi}|^2\right) \nabla^2 \mathcal{E} = \left(\vec{\nabla}\mathcal{E} + 2\, \mathbf{\Phi}^* \vec{\nabla}\mathbf{\Phi}\right) \cdot \vec{\nabla}\mathcal{E} \quad ,$$
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The complex Ernst potential are defined as

$$\mathbf{\Phi} := A_t + i\tilde{A}_\varphi \quad , \quad \mathcal{E} := f - |\mathbf{\Phi}\mathbf{\Phi}^*| + ih \; ,$$

where $\tilde{A}_\varphi$ and h can be obtained from

$$\vec{\nabla}\tilde{A}_\varphi := -fr^{-1}\vec{e}_\varphi \times (\vec{\nabla}A_\varphi - \omega\vec{\nabla}A_t) \; ,$$
$$\vec{\nabla}h := -f^2 r^{-1}\vec{e}_\varphi \times \vec{\nabla}\omega - 2\,\text{Im}(\mathbf{\Phi}^*\vec{\nabla}\mathbf{\Phi}) \; .$$

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The remaining unknown function $\gamma(\rho, z)$, remains decoupled from the previous ones and can be obtained by quadratures:

$$
\begin{aligned}
\partial_\rho \gamma(\rho, z) &= \frac{\rho}{4[Re(\mathcal{E}) + \mathbf{\Phi}\mathbf{\Phi}^*]^2} \Big[\Big(\partial_\rho \mathcal{E} + 2\mathbf{\Phi}^* \partial_\rho \mathbf{\Phi} \Big) \Big(\partial_\rho \mathcal{E}^* + 2\mathbf{\Phi} \partial_\rho \mathbf{\Phi}^* \Big) \\
&\qquad - \Big(\partial_z \mathcal{E} + 2\mathbf{\Phi}^* \partial_z \mathbf{\Phi} \Big) \Big(\partial_z \mathcal{E}^* + 2\mathbf{\Phi} \partial_z \mathbf{\Phi}^* \Big) \Big] \\
&- \frac{\rho}{Re(\mathcal{E}) + \Phi\mathbf{\Phi}^*} \Big(\partial_\rho \mathbf{\Phi} \partial_\rho \mathbf{\Phi}^* - \partial_z \mathbf{\Phi} \partial_z \mathbf{\Phi}^* \Big) \quad , \qquad (1) \\
\partial_z \gamma(\rho, z) &= \frac{\rho}{4[Re(\mathcal{E}) + \mathbf{\Phi}\mathbf{\Phi}^*]^2} \Big[\Big(\partial_\rho \mathcal{E} + 2\mathbf{\Phi}^* \partial_\rho \mathbf{\Phi} \Big) \Big(\partial_z \mathcal{E}^* + 2\mathbf{\Phi} \partial_z \mathbf{\Phi}^* \Big) \\
&\qquad + \Big(\partial_z \mathcal{E} + 2\mathbf{\Phi}^* \partial_z \mathbf{\Phi} \Big) \Big(\partial_\rho \mathcal{E}^* + 2\mathbf{\Phi} \partial_\rho \mathbf{\Phi}^* \Big) \Big] \\
&- \frac{\rho}{Re(\mathcal{E}) + \mathbf{\Phi}\mathbf{\Phi}^*} \Big(\partial_\rho \mathbf{\Phi} \partial_z \mathbf{\Phi}^* + \partial_z \mathbf{\Phi} \partial_\rho \mathbf{\Phi}^* \Big) \quad .
\end{aligned}
$$

Introduction: Symmetries of Ernst Equations

Ernst equations can be derived by an effective action

$$I(\mathcal{E},\boldsymbol{\Phi}) = \int dz \int d\rho \left[\frac{\left(\vec{\nabla}\mathcal{E} + 2\boldsymbol{\Phi}^*\vec{\nabla}\boldsymbol{\Phi}\right)\left(\vec{\nabla}\mathcal{E}^* + 2\boldsymbol{\Phi}\vec{\nabla}\boldsymbol{\Phi}^*\right)}{\left(\mathcal{E}+\mathcal{E}^*+2\boldsymbol{\Phi}\boldsymbol{\Phi}^*\right)^2} - \frac{\vec{\nabla}\boldsymbol{\Phi}\vec{\nabla}\boldsymbol{\Phi}^*}{\mathcal{E}+\mathcal{E}^*+2\boldsymbol{\Phi}\boldsymbol{\Phi}^*} \right]$$

This action has a set of Lie point symmetries which form the $SU(2,1)$ group. These symmetries can be written as a set of five independent transformation:

Ernst Equations Symmetries (Lie point)

$$\begin{array}{lll}
(I) & \mathcal{E} \longrightarrow \mathcal{E}' = \lambda\lambda^*\mathcal{E} \quad , & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \lambda\boldsymbol{\Phi} \quad , \\
(II) & \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} + i\, b \quad , & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \boldsymbol{\Phi} \quad , \\
(III) & \mathcal{E} \longrightarrow \mathcal{E}' = \dfrac{\mathcal{E}}{1+ic\mathcal{E}} \quad , & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \dfrac{\boldsymbol{\Phi}}{1+ic\mathcal{E}} \quad , \\
(IV) & \mathcal{E} \longrightarrow \mathcal{E}' = \mathcal{E} - 2\beta^*\boldsymbol{\Phi} - \beta\beta^* \quad , & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \boldsymbol{\Phi} + \beta \quad , \\
(V) & \mathcal{E} \longrightarrow \mathcal{E}' = \dfrac{\mathcal{E}}{1-2\alpha^*\boldsymbol{\Phi}-\alpha\alpha^*\mathcal{E}} \quad , & \boldsymbol{\Phi} \longrightarrow \boldsymbol{\Phi}' = \dfrac{\boldsymbol{\Phi}+\alpha\mathcal{E}}{1-2\alpha^*\boldsymbol{\Phi}-\alpha\alpha^*\mathcal{E}}
\end{array}$$

where $b, c \in \mathbb{R}$ and $\alpha, \lambda, \beta \in \mathbb{C}$.

Some of these transformation are just gauge symmetries and can be reabsorbed by a coordinate transformation, while others actually have non-trivial physical effects.

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- (V) [Kerr] = Kerr-Newman
- (V) [C-metric] = **?** $\neq$ Accelerating-RN (type-D)
  - RN black hole for acceleration $\to 0$
  - Rindler for $m \to 0$ and $\alpha = s \to 0$
- (V) [Minkowski] = Minkowski

So the Harrison transformational do not just add electric charge to the seed

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Action of the Harrison transformation

General fall off for an asymptotically flat axisymmetric and stationary solution

$$
\begin{aligned}
\mathcal{E} \quad &\sim \quad 1-\frac{2\,(M-iB)}{r}+\frac{(z_*+2iJ)x+const}{r^2}+O\left(\frac{1}{r^3}\right) \quad ,\\
\mathbf{\Phi} \quad &\sim \quad \frac{Q_e+iQ_m}{r}+\frac{(D_e+iD_m)x+const}{r^2}+O\left(\frac{1}{r^3}\right) \quad ,
\end{aligned}
$$

After a Harrison transformation we get

$$
\begin{aligned}
\bar{\mathcal{E}} \quad &\sim \quad 1-\frac{2\,(\bar{M}-i\bar{B})}{r}+\frac{(\bar{z}_*+2i\bar{J})x+\overline{const}}{r^2}+O\left(\frac{1}{r^3}\right) \quad ,\\
\bar{\mathbf{\Phi}} \quad &\sim \quad \frac{\bar{Q}_e+i\bar{Q}_m}{r}+\frac{(\bar{D}_e+i\bar{D}_m)x+\overline{const}}{r^2}+O\left(\frac{1}{r^3}\right) \quad ,
\end{aligned}
$$

$$
\begin{aligned}
\bar{M} &= M\sqrt{1+4|\alpha|^2}-2[Q_e Re(\alpha)+Q_m Im(\alpha)]\ ,\\
\bar{B} &= B\sqrt{1+4|\alpha|^2}-2[Q_e Im(\alpha)-Q_m Re(\alpha)],\\
\bar{J} &= J\sqrt{1+4|\alpha|^2}+2[D_e Im(\alpha)-D_m\ Re(\alpha)]\ ,\\
\bar{Q}_e &= Q_e\sqrt{1+4|\alpha|^2}-2M Re(\alpha)-2B\ Im(\alpha)\ ,\\
\bar{Q}_m &= Q_m\sqrt{1+4|\alpha|^2}-2M Im(\alpha)+2B\ Re(\alpha)\ ,\\
\bar{D}_e &= D_e\sqrt{1+4|\alpha|^2}+z^* Re(\alpha)-2J\ Im(\alpha)\ ,\\
\bar{D}_m &= D_m\sqrt{1+4|\alpha|^2}+z^* Im(\alpha)+2J Re(\alpha)
\end{aligned}
$$

Action of the Harrison transformation

General fall off for an asymptotically flat axisymmetric and stationary solution

$$\mathcal{E} \sim 1 - \frac{2\,(M - iB)}{r} + \frac{(z_* + 2iJ)x + const}{r^2} + O\left(\frac{1}{r^3}\right) \quad ,$$

$$\mathbf{\Phi} \sim \frac{Qe + iQm}{r} + \frac{(De + iDm)x + const}{r^2} + O\left(\frac{1}{r^3}\right) \quad ,$$

After a Harrison transformation we get

$$\bar{\mathcal{E}} \sim 1 - \frac{2\,(\bar{M} - i\bar{B})}{r} + \frac{(\bar{z}_* + 2i\bar{J})x + c\bar{o}nst}{r^2} + O\left(\frac{1}{r^3}\right) \quad ,$$

$$\bar{\mathbf{\Phi}} \sim \frac{\bar{Q}e + i\bar{Q}m}{r} + \frac{(\bar{D}e + i\bar{D}m)x + c\bar{o}nst}{r^2} + O\left(\frac{1}{r^3}\right) \quad ,$$

$$\bar{M} = M\sqrt{1+4|\alpha|^2} - 2[Q_e Re(\alpha) + Q_m Im(\alpha)] \ ,$$

$$\bar{B} = B\sqrt{1+4|\alpha|^2} - 2[Q_e Im(\alpha) - Q_m Re(\alpha)],$$

$$\bar{J} = J\sqrt{1+4|\alpha|^2} + 2[D_e Im(\alpha) - D_m \ Re(\alpha)] \ ,$$

$$\bar{Q}e = Qe\sqrt{1+4|\alpha|^2} - 2MRe(\alpha) - 2B \ Im(\alpha) \ ,$$

$$\bar{Q}m = Qm\sqrt{1+4|\alpha|^2} - 2MIm(\alpha) + 2B \ Re(\alpha) \ ,$$

$$\bar{D}e = De\sqrt{1+4|\alpha|^2} + z^* Re(\alpha) - 2J \ Im(\alpha) \ ,$$

$$\bar{D}m = Dm\sqrt{1+4|\alpha|^2} + z^* Im(\alpha) + 2JRe(\alpha)$$

The C-metric

Accelerating Schwarzschild metric in spherical-like coordinates $(t, r, x = \cos\theta, \varphi)$

$$ds^2 = \frac{\left(1 - \frac{2m}{r}\right)(A^2r^2 - 1)dt^2 + \frac{dr^2}{\left(1 - \frac{2m}{r}\right)(1 - A^2r^2)} + \frac{r^2\ dx^2}{w(x)} + r^2 w(x)\ d\varphi^2}{(1 + Arx)^2}$$

where $\qquad w(x) := (1 + 2mAx)(1 - x^2)$

or in Weyl cylindrical coordinates (t, ρ, z, φ)

$$ds^2 = -\frac{\mu_1\mu_3}{\mu_2}\ dt^2 + \frac{16\ C_f\ \mu_1^3\mu_2^3\mu_3^3\ (d\rho^2 + dz^2)}{\mu_{12}\mu_{23}W_{13}^2 W_{11} W_{22}} + \rho^2 \frac{\mu_2}{\mu_1\mu_3}\ d\varphi^2 \ ,$$

with $\quad \mu_i = w_i - z + \sqrt{\rho^2 + (z - w_i)^2} \ , \qquad \mu_{ij} = (\mu_i - \mu_j)^2 \ , \qquad W_{ij} = \rho^2 + \mu_i\mu_j \ .$

$$\rho \quad \to \quad \frac{\sqrt{(r^2 - 2mr)(1 - A^2r^2)(1 + 2mAx)(1 - x^2)}}{(1 + Arx)^2} \ ,$$

$$z \quad \to \quad \frac{(Ar + x)[r - m(1 - Arx)]}{(1 + Arx)^2} \ .$$

$$w_1 = -m \ , \qquad w_2 = m \ , \qquad w_3 = \frac{1}{2A} \ , \qquad C_f = \frac{m^2}{A^3} \ . \qquad (2)$$

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$$\rho \quad \rightarrow \quad \frac{\sqrt{(r^2 - 2mr)(1 - A^2r^2)(1 + 2mAx)(1 - x^2)}}{(1 + Arx)^2}\ ,$$

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C-metric - rod representation

Accelerating Schwarzschild metric in Weyl coordinates

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C-metric - rod representation

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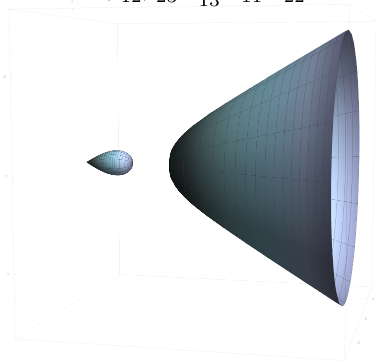
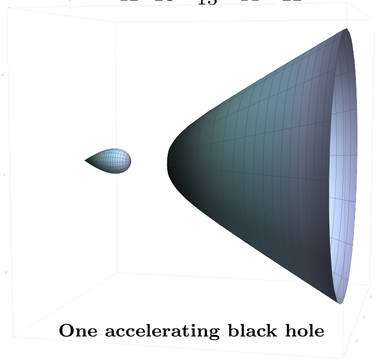


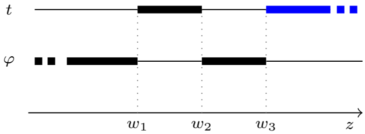
C-metric - rod representation

Accelerating Schwarzschild metric in Weyl coordinates

$$ds^2 = -\frac{\mu_1\mu_3}{\mu_2}\, dt^2 + \frac{16\, C_f\, \mu_1^3\mu_2^3\mu_3^3\, (d\rho^2 + dz^2)}{\mu_{12}\mu_{23}W_{13}^2 W_{11} W_{22}} + \rho^2 \frac{\mu_2}{\mu_1\mu_3}\, d\varphi^2 \ ,$$



One accelerating black hole



Accelerating Schwarzschild metric in external gravitational multipolar expansion

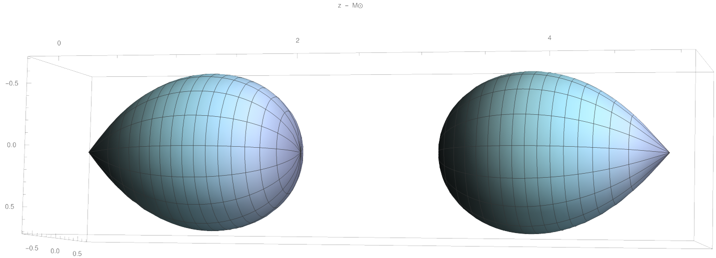
$$
\begin{aligned}
ds^2 &= -\frac{\mu_1\mu_3}{\mu_2} e^{2b_1 z + 2b_2 z^2 - b_2\rho^2}\, dt^2 + \frac{16\ C_f\ \mu_1^3\mu_2^3\mu_3^3\ k(\rho,z)\ (d\rho^2 + dz^2)}{\mu_{12}\mu_{23}W_{13}^2 W_{11} W_{22}} + \\
&+ \frac{\rho^2\mu_2}{\mu_1\mu_3} e^{-2b_1 z - 2b_2 z^2 + b_2\rho^2}\, d\varphi^2 \ ,
\end{aligned}
$$

$$
\begin{aligned}
k(\rho,z) &= exp\Big\{ -b_1^2\rho^2 + 2b_1(z - 2b_2 z\rho^2 + \mu_1 - \mu_2 + \mu_3) \\
&+ \frac{b_2}{2}[b_2\rho^4 + z^2(4 - 8b_2\rho^2) + 2(\rho^2 + 4w_1 - \mu_1^2 - 4w_2\mu_2 + \mu_2^2 + 4w_3\mu_3 - \mu_3^2)]\Big\} \ .
\end{aligned}
$$

[arXiv:2106.02058]

Double Schwarzschild metric (Bach-Weyl 1922)

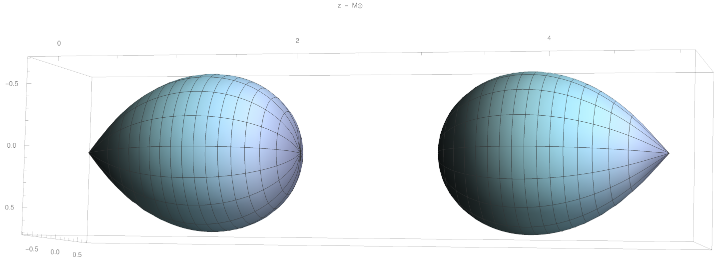
$$ds^2 = -\frac{\mu_1\mu_3}{\mu_2\mu_4}\ d\tilde{t}^2 + \frac{16\ \tilde{C}_f\ \mu_1^3\mu_2^5\mu_3^3\mu_4^5\ (d\rho^2 + dz^2)}{\mu_{12}\mu_{14}\mu_{23}\mu_{34}W_{13}^2W_{24}^2W_{11}W_{22}W_{33}W_{44}} + \rho^2\frac{\mu_2\mu_4}{\mu_1\mu_3}\ d\tilde{\varphi}^2\ ,$$



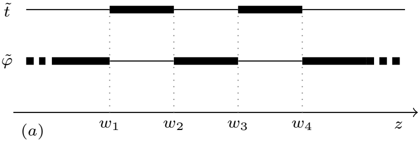
Binary systems

Double Schwarzschild metric (Bach-Weyl 1922)

$$ds^2 = -\frac{\mu_1\mu_3}{\mu_2\mu_4}\; d\tilde{t}^2 + \frac{16\ \tilde{C}_f\ \mu_1^3\mu_2^5\mu_3^3\mu_4^5\ (d\rho^2 + dz^2)}{\mu_{12}\mu_{14}\mu_{23}\mu_{34}W_{13}^2W_{24}^2W_{11}W_{22}W_{33}W_{44}} + \rho^2\frac{\mu_2\mu_4}{\mu_1\mu_3}\; d\tilde{\varphi}^2 \ ,$$



Two black holes

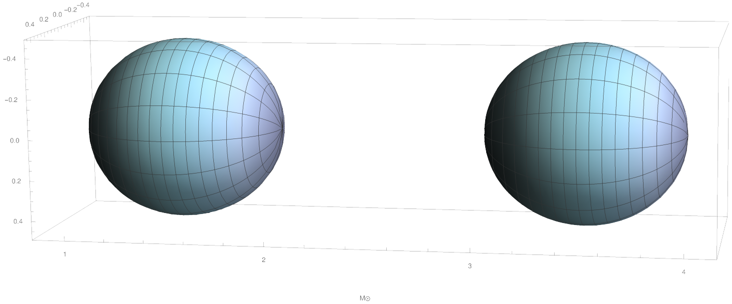


Introduction: Binary systems - Regular

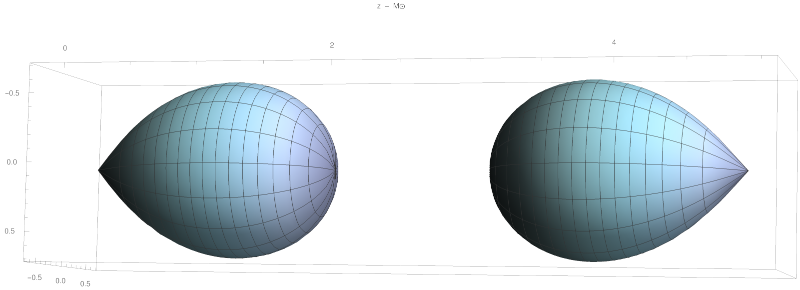
Double Schwarzschild metric in external gravitational field [arXiv:2104.07686]

$$ds^2 = -\frac{\mu_1\mu_3}{\mu_2\mu_4} e^{2b_1 z + 2b_2 z^2 - b_2\rho^2} \, d\tilde{t}^2 + \frac{16\ \tilde{C}_f\ \mu_1^3\mu_2^5\mu_3^3\mu_4^5\ \tilde{k}(\rho,z)\ (d\rho^2 + dz^2)}{\mu_{12}\mu_{14}\mu_{23}\mu_{34}W_{13}^2W_{24}^2W_{11}W_{22}W_{33}W_{44}}$$
$$+ \rho^2 \frac{\mu_2\mu_4}{\mu_1\mu_3} e^{-2b_1 z - 2b_2 z^2 + b_2\rho^2} \, d\tilde{\varphi}^2,$$

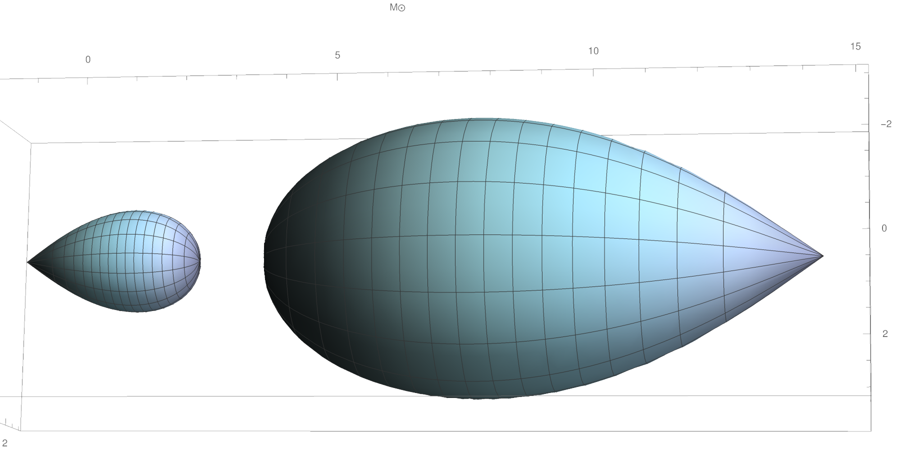
$$\tilde{k}(\rho,z) := exp\Big\{ - b_1^2\rho^2 - 2b_1(z + 2b_2 z\rho^2 - \mu_1 + \mu_2 - \mu_3 + \mu_4) + b_2[\frac{b_2}{2}\rho^4 -$$
$$- 2z^2(1 + 2b_2\rho^2) + 4w_1\mu_1 - \mu_1^2 - 4w_2\mu_2 + \mu_2^2 + 4w_3\mu_3 - \mu_3^2 - 4w_4\mu_4 + \mu_4^2]\Big\} .$$



Consider the binary black hole system

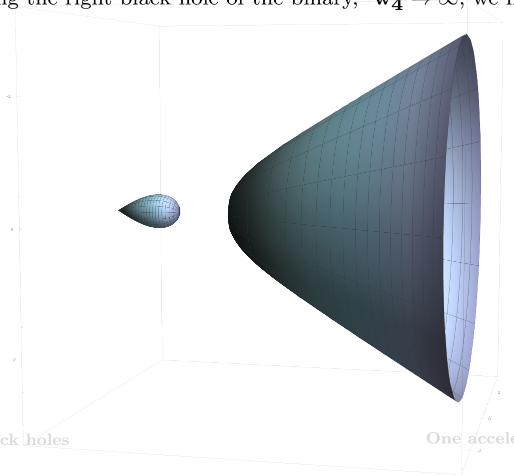


Let's enlarge the right black hole of the binary while keeping the left and the distance fixed



Introduction: C-metric and binary systems

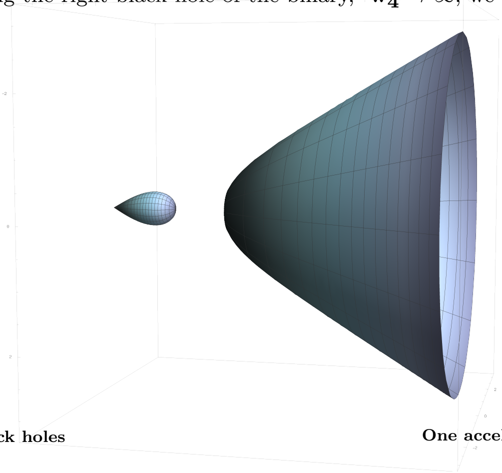
Further enlarging the right black hole of the binary, $\mathbf{w_4} \to \infty$, we have



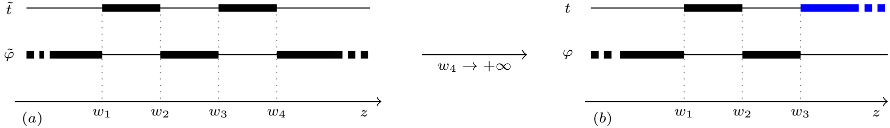
Two black holes

One accelerating black hole

Further enlarging the right black hole of the binary, $\mathbf{w_4} \to \infty$, we have



Two black holes **One accelerating black hole**



Charged Binary

The Harrison transformed Bach-Weyl metric gives a charged binary

$$
\begin{aligned}
ds^2 &= -\frac{\mu_1\mu_2\mu_3\mu_4}{(\mu_2\mu_4 - \hat{\alpha}^2\mu_1\mu_3)^2}\, d\hat{t}^2 + \rho^2 \frac{(\mu_2\mu_4 - \hat{\alpha}^2\mu_1\mu_3)^2}{\mu_1\mu_2\mu_3\mu_4} d\hat{\varphi}^2 , \\
&+ \frac{16 C_f \mu_1^3\mu_2^3\mu_3^3\mu_4^3(\mu_2\mu_4 - \hat{\alpha}^2\mu_1\mu_3)^2}{\mu_{12}\mu_{14}\mu_{23}\mu_{34}W_{13}^2 W_{24}^2 W_{11}W_{22}W_{33}W_{44}}\,(d\rho^2 + dz^2) ,
\end{aligned}
$$

$$
A_\mu = \left(A_{t_0} + \frac{\hat{\alpha}\mu_1\mu_3}{\mu_2\mu_4 - \hat{\alpha}^2\mu_1\mu_3}, 0, 0, 0 \right) .
$$

At extremality this solution become the Majumbdar-Papapetrou metric. Then if you change the coordinates as follows

$$
\rho \to \frac{\sqrt{r(r-2m)(1-A^2r^2)(1+2Amx)(1-x^2)}}{(1+Arx)^2} , \qquad \hat{t} \to \sqrt{\frac{A}{2w_4}}\, t ,
$$

$$
z \to z_1 + \frac{(Ar+x)[r-m(1-Arx)]}{(1+Arx)^2} , \qquad \hat{\varphi} \to \sqrt{\frac{2w_4}{A}}\, \varphi .
$$

with $w_1 = z_1 - m$, $w_2 = z_1 + m$, $z_3 = z_1 + \frac{1}{2A}$, $C_f = \frac{2w_4m^2}{A^3}$ and $\hat{\alpha} = \sqrt{A}s$

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Charged Binary

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&+ \frac{16 C_f \mu_1^3\mu_2^3\mu_3^3\mu_4^3(\mu_2\mu_4-\hat{\alpha}^2\mu_1\mu_3)^2}{\mu_{12}\mu_{14}\mu_{23}\mu_{34}W_{13}^2W_{24}^2W_{11}W_{22}W_{33}W_{44}}\,(d\rho^2+dz^2)\ ,
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$$

$$
z \rightarrow z_1 + \frac{(Ar+x)[r-m(1-Arx)]}{(1+Arx)^2}\ , \qquad \hat{\varphi} \rightarrow \sqrt{\frac{2w_4}{A}}\ \varphi\ .
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Reissner-Nordstrom in charged accelerating background

... and take the limit for $w_4 \to \infty$, you get the Reissner-Nordstrom spacetime in an accelerating and electric background (Type I : Harrison[C-metric])

$$ds^2 = -f(r,x)dt^2 + \frac{1}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] \ ,$$

where

$$f(r,x) := \frac{r^2\Delta_r\Omega^2}{(s^2\Delta_r - r^2\Omega^2)^2} \qquad \Omega(r,x) := 1 + Arx \ ,$$

$$\gamma(r,x) := \frac{1}{2}\log\left(\frac{\Delta_r}{\Omega^4}\right) \ , \qquad \Delta_r(r) := (1 - A^2r^2)(r^2 - 2mr) \ ,$$

$$\rho(r,x) := \frac{\sqrt{\Delta_r}\sqrt{\Delta_x}}{\Omega^2} \ , \qquad \Delta_x(x) := (1 - x^2)(1 + 2mAx) \ ,$$

$$A_\mu = \left(\frac{s\Delta_r}{r^2\Omega^2 - s^2\Delta_r}0, 0, 0\right) \ .$$

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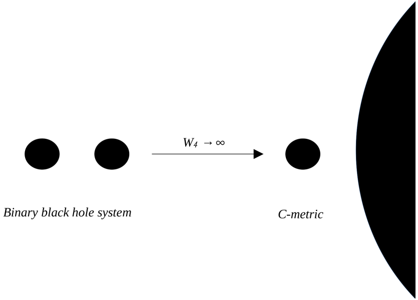
$$\rho(r,x) := \frac{\sqrt{\Delta_r}\sqrt{\Delta_x}}{\Omega^2} , \qquad \Delta_x(x) := (1 - x^2)(1 + 2mAx) ,$$



Removing the black hole (vanishing its parameters m, ...) we have

$$
\begin{aligned}
ds^2 &= -\hat{f}(r,x)\left[dt - \left(\frac{2Acr^2(1-x^2)}{(1+Arx)^2} + \omega_0\right)d\varphi\right]^2 \\
&+ \frac{1}{\hat{f}(r,x)}\left\{\frac{1}{(1+Arx)^4}\left[dr^2 + (r^2 - A^2r^4)\left(\frac{dx^2}{1-x^2} + (1-x^2)d\varphi^2\right)\right]\right\}
\end{aligned}
$$

with

$$
\hat{f}(r,x) := \frac{(1-A^2r^2)(1+Arx)^2}{c^2(1-A^2r^2)^2 + [(1+Arx)^2 - s^2(1-A^2r^2)]^2} \ .
$$

$$
A_\mu = \hat{f}\left\{s - s^3\frac{1-A^2r^2}{(1+Arx)^2},\ 0,\ 0,\ \frac{2cs[1+Ar(Ar+2x)][(1-A^2r^2)s^2 - (1+Arx)^2]}{A(1+Arx)^4}\right\} .
$$

This can be obtained as the near-horizon limit of the RN-NUT solution ($w_2 \to \infty$)

Accelerating electric and nutty background

Removing the black hole (vanishing its parameters m, ...) we have

$$
\begin{aligned}
ds^2 &= -\hat{f}(r,x)\left[dt - \left(\frac{2Acr^2(1-x^2)}{(1+Arx)^2} + \omega_0\right)d\varphi\right]^2 \\
&+ \frac{1}{\hat{f}(r,x)}\left\{\frac{1}{(1+Arx)^4}\left[dr^2 + (r^2 - A^2r^4)\left(\frac{dx^2}{1-x^2} + (1-x^2)d\varphi^2\right)\right]\right\}
\end{aligned}
$$

with

$$
\hat{f}(r,x) := \frac{(1-A^2r^2)(1+Arx)^2}{c^2(1-A^2r^2)^2 + [(1+Arx)^2 - s^2(1-A^2r^2)]^2} \ .
$$

$$
A_\mu = \hat{f}\left\{s - s^3\frac{1-A^2r^2}{(1+Arx)^2},\ 0,\ 0,\ \frac{2cs[1+Ar(Ar+2x)][(1-A^2r^2)s^2-(1+Arx)^2]}{A(1+Arx)^4}\right\} .
$$

This can be obtained as the near-horizon limit of the RN-NUT solution ($w_2 \to \infty$)

$(c=0)$ In fact, in Weyl coordinates

$$\begin{aligned} \rho &= \sqrt{r^2-2mr-q^2}\ \sqrt{1-x^2}\ , \\ z &= z_1+(r-m)\ x\ . \end{aligned}$$

the Reissner-Nordstrom black hole reads

$$ds^2 = -\frac{(R_+ + R_-)^2 - 4(m^2-q^2)}{(2m+R_+R_-)^2}dt^2 + \frac{(2m+R_+ + R_-)^2}{4R_+R_-}(d\rho^2+dz^2) + \frac{\rho^2(2m+R_+R_-)^2\ d\varphi^2}{(R_+ + R_-)^2-4(m^2-q^2)}$$

$$A_\mu = \left[\frac{q^2}{(2m+R_+R_-)^2},0,0,0\right]\ , \quad R_\pm = \sqrt{\rho^2+\left[\pm(z-z_1)+\sqrt{m^2-q^2}\right]^2}\ .$$

Taking $w_2 \to \infty$ the accelerating electric background can be obtained

$$m \to \frac{1}{2}(w_2-w_1)(1+2w_2\delta^2)\ , \qquad q \to (w_1-w_2)\sqrt{2w_2}\delta\ , \qquad z_1 \to \frac{1}{2}(w_2+w_1)(1-2w_2\delta^2)\ , \tag{3}$$

$$\rho \to \rho(1-2w_2\delta^2)\ , \qquad z \to z(1-2w_2\delta^2)\ , \qquad t \to \frac{\sqrt{A}}{\delta^2\sqrt{2w_2}}t\ , \qquad \varphi \to \frac{\sqrt{A}}{\delta^2\sqrt{2w_2}}\varphi\ .$$

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Ehlers transformation

The Ehlers transformation acts on a solution with the following asymptotic fall-off

$$\begin{aligned}\mathcal{E} &\sim 1 - \frac{2\,(M - iB)}{r} + O\left(\frac{1}{r^2}\right) \quad , \\ \mathbf{\Phi} &\sim \frac{Q_e + iQ_m}{r} + O\left(\frac{1}{r^2}\right) \quad ,\end{aligned}$$

in the following way

$$\begin{aligned}\bar{\mathcal{E}} &\sim 1 - \frac{2(\bar{M} - i\bar{B})}{r} + O\left(\frac{1}{r^2}\right) = 1 - \frac{2(1-c^2)M - 4Bc}{(1+c^2)r} + i\,\frac{2(1-c^2)B + 4Mc}{(1+c^2)r} + O\left(\frac{1}{r^2}\right) \\ \bar{\mathbf{\Phi}} &\sim \frac{Q_e + iQ_m}{r} + O\left(\frac{1}{r^2}\right) \quad .\end{aligned}$$

so basically is a rotation on between the mass and gravitomagnetic mass

$$\begin{pmatrix}\bar{M}\\ \bar{B}\end{pmatrix} = \begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix}_{\psi\to 2\arctan c} \begin{pmatrix}M\\ B\end{pmatrix} = \begin{pmatrix}\frac{(1-c^2)M-2Bc}{1+c^2}\\ \frac{(1-c^2)B+2Mc}{1+c^2}\end{pmatrix} \quad .$$

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- Ehlers [Schwarzschild] = Taub-NUT
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$$
\begin{aligned}
ds^2 &= -\frac{4\mu_1\mu_2\mu_3}{4\mu_2 + c^2\mu_1^2\mu_3^2}[dt - c(2z + \mu_1 - \mu_2 + \mu_3 + w_0)d\varphi]^2 \\
&+ \frac{4\mu_2 + c^2\mu_1^2\mu_3^2}{4\mu_1\mu_2\mu_3}\left[\frac{16C_f\mu_1^4\mu_2^2\mu_3^4(d\rho^2 + dz^2)}{\mu_{12}\mu_{23}W_{11}W_{22}W_{33}W_{13}^2} + \rho^2 d\varphi^2\right].
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Plebanski-Demianski seed

Plebanski-Demianski seed in terms of the metric and electromagnetic potential:

$$ds^2 = -f(r,x)\left[dt - \omega(r,x)d\varphi\right]^2 + \frac{1}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] .$$

$$f(r,x) := \frac{\hat\omega^2\Delta_x - \Delta_r}{\hat\omega\Omega^2\mathcal{R}^2} \; , \qquad \gamma(r,x) := \frac{1}{2}\log\left(\frac{\Delta_r - \hat\omega^2\Delta_r}{\Omega^4}\right) \; ,$$

$$\omega(r,x) := \frac{\hat\omega(r^2\Delta_x + x^2\Delta_r)}{\Delta_r - \hat\omega^2\Delta_x} \; , \qquad \rho(r,x) := \frac{\sqrt{\Delta_r}\sqrt{\Delta_x}}{\hat\omega\Omega^2} \; ,$$

$$\mathcal{R}(r,x) := \sqrt{r^2 + \hat\omega^2 x^2} \qquad \Omega(r,x) := 1 - \alpha r x \; ,$$

$$\Delta_r(r) := -\hat\omega(e^2 + p^2 + k\hat\omega^2) + 2m\hat\omega r - \epsilon\hat\omega r^2 + 2\hat n\alpha r^3 + k\alpha^2\hat\omega r^4 \; ,$$

$$\Delta_x(x) := -k\hat\omega - 2\hat n x + \epsilon\hat\omega x^2 - 2m\alpha\hat\omega x^3 + \alpha^2\hat\omega(e^2 + p^2 + k\hat\omega^2)x^4 \; ,$$

$$A_\mu(r,x) = \left(-\frac{er + \hat\omega p x}{\mathcal{R}^2} \; , \; 0 \; , \; 0 \; , \; \frac{e\hat\omega r x^2 - p x r^2}{\mathcal{R}^2}\right) .$$

P-D seed in terms of Ernst complex gravitational and electromagnetic potentials

$$\mathcal{E}(r,x) = \frac{r\hat\omega\Delta_x + i\{\Omega^2\hat\omega[-ikr\hat\omega + x(e^2+p^2+k\hat\omega^2)] + x\Delta_y\}}{rx\Omega^2\hat\omega(-ir + \hat\omega x)} \; ,$$

$$\Phi(r,x) = -\frac{e + ip}{r + i\hat\omega x} \; .$$

Plebanski-Demianski seed in terms of the metric and electromagnetic potential:

$$ds^2 = -f(r,x)\,[dt - \omega(r,x)d\varphi]^2 + \frac{1}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] .$$

$$f(r,x) := \frac{\hat{\omega}^2\Delta_x - \Delta_r}{\hat{\omega}\Omega^2\mathcal{R}^2} , \qquad \gamma(r,x) := \frac{1}{2}\log\left(\frac{\Delta_r - \hat{\omega}^2\Delta_r}{\Omega^4}\right) ,$$

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$$\mathcal{R}(r,x) := \sqrt{r^2 + \hat{\omega}^2 x^2} \qquad \Omega(r,x) := 1 - \alpha r x ,$$

$$\Delta_r(r) := -\hat{\omega}(e^2 + p^2 + k\hat{\omega}^2) + 2m\hat{\omega}r - \epsilon\hat{\omega}r^2 + 2\hat{n}\alpha r^3 + k\alpha^2\hat{\omega}r^4 ,$$

$$\Delta_x(x) := -k\hat{\omega} - 2\hat{n}x + \epsilon\hat{\omega}x^2 - 2m\alpha\hat{\omega}x^3 + \alpha^2\hat{\omega}(e^2 + p^2 + k\hat{\omega}^2)x^4 ,$$

$$A_\mu(r,x) = \left(-\frac{er + \hat{\omega}px}{\mathcal{R}^2} , 0 , 0 , \frac{e\hat{\omega}rx^2 - pxr^2}{\mathcal{R}^2}\right) .$$

P-D seed in terms of Ernst complex gravitational and electromagnetic potentials

$$\mathcal{E}(r,x) = \frac{r\hat{\omega}\Delta_x + i\{\Omega^2\hat{\omega}[-ikr\hat{\omega} + x(e^2 + p^2 + k\hat{\omega}^2)] + x\Delta_y\}}{rx\Omega^2\hat{\omega}(-ir + \hat{\omega}x)} ,$$

$$\mathbf{\Phi}(r,x) = -\frac{e + ip}{r + i\hat{\omega}x} .$$

Plebanski-Demianski with a double NUT parameter

Ehlers[Plebansky-Demianski]:

$$
\begin{aligned}
\bar{\mathcal{E}}(r,x) &= -\frac{i\big[\Omega^2\hat{\omega}(-ikr\hat{\omega}+x(e^2+p^2+k\hat{\omega}^2)\big]-ir\hat{\omega}\Delta_x+x\Delta_y}{\Omega^2\hat{\omega}\big[-ickr\hat{\omega}+xr(ir-x\hat{\omega})+cx(e^2+p^2+k\hat{\omega}^2)\big]-icr\hat{\omega}\Delta_x+cx\Delta_r} \ , \\
\bar{\mathbf{\Phi}}(r,x) &= \frac{(p-ie)xr\Omega^2\hat{\omega}}{\Omega^2\hat{\omega}\big[-ickr\hat{\omega}+xr(ir-x\hat{\omega})+cx(e^2+p^2+k\hat{\omega}^2)\big]-icr\hat{\omega}\Delta_x+cx\Delta_r} \ .
\end{aligned}
$$

$$
ds^2 = -\frac{f(r,x)}{|1+ic\mathcal{E}|^2}\left[dt-\bar{\omega}(r,x)d\varphi\right]^2+\frac{|1+ic\mathcal{E}|^2}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)}+\frac{dx^2}{\Delta_x(x)}\right)+\rho^2(r,x)d\varphi^2\right] .
$$

$\gamma(r,x)$ function remains invariant under the Ehlers transformation, while

$$
\bar{\omega} = \omega(r,x)+c^2\,\frac{r^2\Delta_r(\Delta_x+k\Omega^2\hat{\omega})^2+x^2\Delta_x[\Delta_r+\Omega^2\hat{\omega}(q^2+k\hat{\omega}^2)]^2}{x^2r^2\hat{\omega}\Omega^4(\Delta_r-\hat{\omega}^2\Delta_x)}+\frac{2cr^3\alpha\Delta_r(\Delta_x+k\Omega^2\hat{\omega})}{xr^2\alpha\hat{\omega}\Omega^2(\Delta_r-\hat{\omega}^2\Delta_x)}
$$

$$
+2c\frac{x\Omega(\Delta_r-\Delta_x\hat{\omega}^2)[\Delta_r+\Omega\hat{\omega}(q^2+k\hat{\omega}^2-2mr)]+xr\hat{\omega}\{x\alpha\Delta_r[\Delta_x\hat{\omega}+\Omega^2(q^2+k\hat{\omega}^2)]\}}{xr^2\alpha\hat{\omega}\Omega^2(\Delta_r-\hat{\omega}^2\Delta_x)}+\omega_0 \ .
$$

The non-null component of the electromagnetic vector results

$$
\begin{aligned}
\bar{A}_t &= \frac{xr\hat{\omega}\Omega^2\{cer\hat{\omega}\Delta_x+cpx\Delta_r+\hat{\omega}\big[-xr(er+px\hat{\omega})+c(pqx+ker\hat{\omega}+kpx\hat{\omega}^2)\big]\Omega^2\}}{|1+ic\mathcal{E}|^2} \\
\bar{A}_\varphi &= +\frac{er\hat{\omega}\Delta_x+px\Delta_r}{\hat{\omega}^2\Delta_x-\Delta_r}+c\,\frac{(er+px\hat{\omega})\Delta_x\Delta_r+\hat{\omega}[px\hat{\omega}(q^2+k\hat{\omega}^2)\Delta_x+ker\Delta_r]\Omega^2}{xr\hat{\omega}\Omega^2(\Delta_r-\hat{\omega}^2\Delta_x)}-\bar{\omega}\bar{A}_t \ .
\end{aligned}
$$

Plebanski-Demianski with a double NUT parameter

Ehlers[Plebansky-Demianski]:

$$\bar{\mathcal{E}}(r,x) = -\frac{i\big[\Omega^2\hat\omega(-ikr\hat\omega + x(e^2+p^2+k\hat\omega^2)\big] - ir\hat\omega\Delta_x + x\Delta_y}{\Omega^2\hat\omega\big[-ickr\hat\omega + xr(ir - x\hat\omega) + cx(e^2+p^2+k\hat\omega^2)\big] - icr\hat\omega\Delta_x + cx\Delta_r} ,$$

$$\bar{\boldsymbol{\Phi}}(r,x) = \frac{(p-ie)xr\Omega^2\hat\omega}{\Omega^2\hat\omega\big[-ickr\hat\omega + xr(ir - x\hat\omega) + cx(e^2+p^2+k\hat\omega^2)\big] - icr\hat\omega\Delta_x + cx\Delta_r} .$$

$$ds^2 = -\frac{f(r,x)}{|1+ic\mathcal{E}|^2}\left[dt - \bar\omega(r,x)d\varphi\right]^2 + \frac{|1+ic\mathcal{E}|^2}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] .$$

$\gamma(r,x)$ function remains invariant under the Ehlers transformation, while

$$\bar\omega = \omega(r,x) + c^2\,\frac{r^2\Delta_r(\Delta_x + k\Omega^2\hat\omega)^2 + x^2\Delta_x[\Delta_r + \Omega^2\hat\omega(q^2+k\hat\omega^2)]^2}{x^2r^2\hat\omega\Omega^4(\Delta_r - \hat{\omega^2}\Delta_x)} + \frac{2cr^3\alpha\Delta_r(\Delta_x + k\Omega^2\hat\omega)}{xr^2\alpha\hat\omega\Omega^2(\Delta_r - \hat{\omega^2}\Delta_x)}$$

$$+2c\,\frac{x\Omega(\Delta_r - \Delta_x\hat\omega^2)[\Delta_r + \Omega\hat\omega(q^2+k\hat\omega^2 - 2mr)] + xr\hat\omega\{x\alpha\Delta_r[\Delta_x\hat\omega + \Omega^2(q^2+k\hat\omega^2)]\}}{xr^2\alpha\hat\omega\Omega^2(\Delta_r - \hat{\omega^2}\Delta_x)} + \omega_0 .$$

The non-null component of the electromagnetic vector results

$$\bar A_t = \frac{xr\hat\omega\Omega^2\{cer\hat\omega\Delta_x + cpx\Delta_r + \hat\omega\big[-xr(er+px\hat\omega) + c(pqx + ker\hat\omega + kpx\hat\omega^2)\big]\Omega^2\}}{|1+ic\mathcal{E}|^2}$$

$$\bar A_\varphi = +\frac{er\hat\omega\Delta_x + px\Delta_r}{\hat\omega^2\Delta_x - \Delta_r} + c\,\frac{(er+px\hat\omega)\Delta_x\Delta_r + \hat\omega[px\hat\omega(q^2+k\hat\omega^2)\Delta_x + ker\Delta_r]\Omega^2}{xr\hat\omega\Omega^2(\Delta_r - \hat{\omega^2}\Delta_x)} - \bar\omega\bar A_t .$$

Removal of the Minsner string

Convenient parametrization

$$\begin{aligned}
\hat{\omega} &\rightarrow \frac{a^2+\ell^2}{a} \\
\hat{n} &\rightarrow \frac{\ell(a^2+\ell^2)^2 + am(\ell^4-a^4)\alpha}{(a^2+\ell^2)^2 - 2am\ell(a^2+\ell^2)\alpha + a^2\ell^2(a^2+e^2+p^2-\ell^2)\alpha^2} \\
\epsilon &\rightarrow \frac{(a^2+\ell^2)^2 + 4am\ell(a^2+\ell^2)\alpha - a^2(a^2-\ell^2)(a^2+e^2+p^2-\ell^2)\alpha^2}{(a^2+\ell^2)^2 - 2am\ell(a^2+\ell^2)\alpha + a^2\ell^2(a^2+e^2+p^2-\ell^2)\alpha^2} \\
k &\rightarrow \frac{a^2(a^2-\ell^2)}{(a^2+\ell^2)^2 - 2am\ell(a^2+\ell^2)\alpha + a^2\ell^2(a^2+e^2+p^2-\ell^2)\alpha^2} \\
c &\rightarrow \frac{-m+\sqrt{m^2+n^2-2n\ell}}{n} \quad .
\end{aligned}$$

To remove the discontinuity on the axis of symmetry

$$\Delta\omega = \lim_{x\to 1}\omega(r,x) - \lim_{x\to -1}\omega(r,x) = 4(n-\ell) = 0 \ .$$

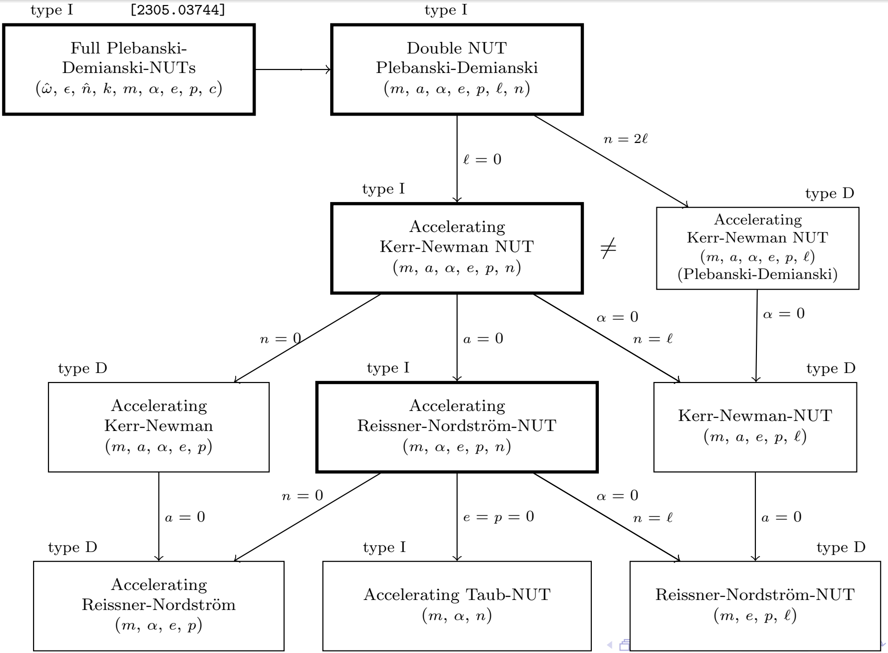
horizons from

$$Q(r) = (r-\bar{r}_+)(r-\bar{r}_-)\left[1 - \frac{a(a-\ell)(r-\hat{r}_-)\alpha}{a^2+\ell^2}\right]\left[1 + \frac{a(a+\ell)(r-\hat{r}_-)\alpha}{a^2+\ell^2}\right] \ ,$$

where

$$\bar{r}_\pm := m \pm \sqrt{m^2+(n-\ell)^2-a^2-e^2-p^2} \qquad \text{and} \qquad \hat{r}_\pm := m \pm \sqrt{m^2+n^2-2n\ell} \quad ;$$

Plebanski-Demianski NUTs family



Harrison and Ehlers transformations commute

Ehlers transformation

$$(III): \qquad \mathcal{E} \longrightarrow \bar{\mathcal{E}} = \frac{\mathcal{E}}{1+ic\mathcal{E}} \quad , \qquad \mathbf{\Phi} \longrightarrow \bar{\mathbf{\Phi}} = \frac{\mathbf{\Phi}}{1+ic\mathcal{E}} \ .$$

Harrison transformation

$$(V): \qquad \mathcal{E} \longrightarrow \bar{\mathcal{E}} = \frac{\mathcal{E}}{1-2\alpha^*\mathbf{\Phi}-\alpha\alpha^*\mathcal{E}} \quad , \qquad \mathbf{\Phi} \longrightarrow \bar{\mathbf{\Phi}} = \frac{\mathbf{\Phi}+\alpha\mathcal{E}}{1-2\alpha^*\mathbf{\Phi}-\alpha\alpha^*\mathcal{E}} \ .$$

Ehlers and Harrison transformations commute $[(III)\ ,\ (V)] = 0$

$$(III)\circ(V) = \begin{cases} \mathcal{E} \quad \longrightarrow \quad \bar{\mathcal{E}} = \dfrac{\mathcal{E}}{1+ic\mathcal{E}-\alpha\alpha^*\mathcal{E}-2\alpha^*\mathbf{\Phi}} \\ \\ \mathbf{\Phi} \quad \longrightarrow \quad \bar{\mathbf{\Phi}} = \dfrac{\mathbf{\Phi}+\alpha\mathcal{E}}{1+ic\mathcal{E}-\alpha\alpha^*\mathcal{E}-2\alpha^*\mathbf{\Phi}} \end{cases} .$$

PD can thus be further generalised to the presence of extra NUT and electromagnetic field $(\bar{\mathcal{E}}, \bar{\mathbf{\Phi}})$

$$\frac{-ir\hat{\omega}\Delta_x + x\Delta_r + \hat{\omega}[-ikr\hat{\omega}+x)(q^2+k\hat{\omega}^2)]\Omega^2}{(c+is^2)(r\hat{\omega}\Delta_x + ix\Delta_r) - \hat{\omega}[r^2x + 2(ip+e)rsx + ir(x^2+ikc-ks^2)\hat{\omega} + (s^2-icx)(q^2+k\hat{\omega}^2)]\Omega^2},$$

$$\frac{-irs\hat{\omega}\Delta_x + sx\Delta_r + \hat{\omega}[(ip+e)rx + (q^2+k\hat{\omega}^2)sx - ikrs\hat{\omega}]\Omega^2}{(c+is^2)(r\hat{\omega}\Delta_x + ix\Delta_r) - \hat{\omega}[r^2x + 2(ip+e)rsx + ir(x^2+ikc-ks^2)\hat{\omega} + (s^2-icx)(q^2+k\hat{\omega}^2)]\Omega^2}.$$

Plebanski-Demianski-NUTs in electromagnetic Rindler background

In metric form

$$d\bar{s}^2 = -\bar{f}(r,x)\left[dt - \bar{\omega}(r,x)d\varphi\right]^2 + \frac{1}{\bar{f}(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] .$$

$$\bar{f} = \frac{f}{1 + (ic - s^2)\mathcal{E} - 2s\mathbf{\Phi}} \quad ,$$

where $\bar{\omega}(r,x) =$

$$\frac{1}{r^2x^2\hat{\omega}(\hat{\omega}^2\Delta_x - \Delta_r)(1-\Omega)\Omega^4}\Big\{c^2(\Omega-1)\Big[r^2\Delta_r(\Delta_x + k\hat{\omega}\Omega^2)^2 + x^2\Delta_x(\Delta_r + \hat{\omega}(q^2+k\hat{\omega}^2)\Omega^2)^2\Big]$$
$$-2crx\Omega^2\Big[r^2\Delta_r(\Omega-1)(\Delta_x + k\hat{\omega}\Omega^2) + 2r\Big(mx^2\hat{\omega}(\Delta_r - \hat{\omega}^2\Delta_x)\Omega^2 + es\Delta_r(\Omega-1)(\Delta_x + k\hat{\omega}\Omega^2)\Big)$$
$$+x\Big(-x\Delta_r^2\Omega + \hat{\omega}^2(q^2+k\hat{\omega}^2)\Delta_x\Omega^2(x\hat{\omega}^2 - 2ps + 2ps\Omega) + \hat{\omega}\Delta_r\Big(x(q^2+k\hat{\omega}^2)(\Omega-2)\Omega^2$$
$$+\Delta_x\left(2(ps + x\hat{\omega})\Omega - 2ps - x\hat{\omega}\right)\Big)\Big)\Big] + (\Omega-1)\Big[x^2\hat{\omega}^2\Big(r^4 + 4er^3s + 6q^2r^2s^2 + 4ers^3(q^2+k\hat{\omega}^2)\Big)\Delta_x\Omega^4$$
$$+s^4x^2\Delta_r^2\Delta_x + \Delta_r\Big(r^4s^4\Delta_x^2 + 2s^3\Big(kr^2s\hat{\omega} - 2psr^2x + (2er + q^2s)x^2\hat{\omega} + ksx^2\hat{\omega}^3\Big)\Delta_x\Omega^2$$
$$+r^2\omega(4psx^3 - 4kps^3x + k^2s^4\hat{\omega} + x^4\hat{\omega})\Omega^4\Big)\Big]\Big\} .$$

Plebanski-Demianski-NUTs in electromagnetic Rindler background

In metric form

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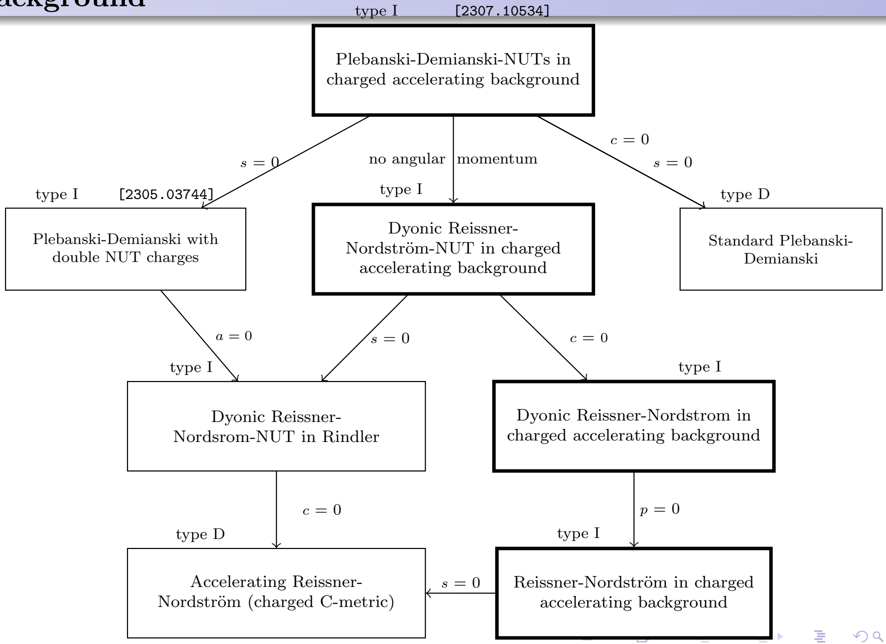
$$\bar{f} = \frac{f}{1 + (ic - s^2)\mathcal{E} - 2s\mathbf{\Phi}} \quad ,$$

where $\bar{\omega}(r,x) =$

$$\frac{1}{r^2x^2\hat{\omega}(\hat{\omega}^2\Delta_x - \Delta_r)(1-\Omega)\Omega^4}\Big\{c^2(\Omega-1)\Big[r^2\Delta_r\left(\Delta_x + k\hat{\omega}\Omega^2\right)^2 + x^2\Delta_x\left(\Delta_r + \hat{\omega}(q^2 + k\hat{\omega}^2)\Omega^2\right)^2\Big]$$
$$-2crx\Omega^2\Big[r^2\Delta_r(\Omega-1)(\Delta_x + k\hat{\omega}\Omega^2) + 2r\Big(mx^2\hat{\omega}(\Delta_r - \hat{\omega}^2\Delta_x)\Omega^2 + es\Delta_r(\Omega-1)(\Delta_x + k\hat{\omega}\Omega^2)\Big)$$
$$+x\Big(-x\Delta_r^2\Omega + \hat{\omega}^2(q^2 + k\hat{\omega}^2)\Delta_x\Omega^2(x\hat{\omega}^2 - 2ps + 2ps\Omega) + \hat{\omega}\Delta_r\Big(x(q^2 + k\hat{\omega}^2)(\Omega-2)\Omega^2$$
$$+\Delta_x\left(2(ps + x\hat{\omega})\Omega - 2ps - x\hat{\omega}\right)\Big)\Big)\Big] + (\Omega-1)\Big[x^2\hat{\omega}^2\Big(r^4 + 4er^3s + 6q^2r^2s^2 + 4ers^3(q^2 + k\hat{\omega}^2)\Big)\Delta_x\Omega^4$$
$$+s^4x^2\Delta_r^2\Delta_x + \Delta_r\Big(r^4s^4\Delta_x^2 + 2s^3\Big(kr^2s\hat{\omega} - 2psr^2x + (2er + q^2s)x^2\hat{\omega} + ksx^2\hat{\omega}^3\Big)\Delta_x\Omega^2$$
$$+r^2\omega(4psx^3 - 4kps^3x + k^2s^4\hat{\omega} + x^4\hat{\omega})\Omega^4\Big)\Big]\Big\} .$$

Plebanski-Demianski-NUTs in electromagnetic Rindler background



Generalised accelerating black holes from binary systems

For n even, the general metric that describes $n/2$ axially aligned, stationary rotating and axisymmetric black holes

$$ds^2 = g_{ab}(\rho, z) dx^a dx^b + f(\rho, z)(d\rho^2 + dz^2) \tag{5}$$

with $a, b, c \in \{0, 1\}$ and $k, l \in \{1, ..., n\}$, so $x^a = \{t, \varphi\}$, where

$$g_{ab}(\rho, z) = \frac{1}{\rho^n} \left(\prod_{k=1}^{n} \mu_k \right) \left[\overset{\circ}{g}_{ab} - \sum_{k,l=1}^{n} \frac{(\Gamma^{-1})_{kl} L_a^{(k)} L_b^{(l)}}{\mu_k \mu_l} \right], \tag{6a}$$

$$f(\rho, z) = \frac{16\, C_f \overset{\circ}{f}_0}{\rho^{n^2/2}} \left(\prod_{k=1}^{n} \mu_k \right)^{n+1} \left[\prod_{k>l=1}^{n} (\mu_k - \mu_l)^{-2} \right] \det \Gamma\,, \tag{6b}$$

where $L_a^{(k)} = m_c^{(k)} \overset{\circ}{g}_{ca}$, the background metric is given by $\overset{\circ}{f} = 1$ and $\overset{\circ}{g}$ as follows

$$\overset{\circ}{g}_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2 \end{pmatrix}, \qquad \Gamma_{kl} = \frac{m_a^{(k)}\, \overset{\circ}{g}_{ab}\, m_b^{(l)}}{\rho^2 + \mu_k \mu_l}, \qquad m_a^{(k)} = \left(C_0^{(k)}, \frac{C_1^{(k)}}{\mu_k} \right),$$

n solitons bring in the metric $2n$ physical integration constants

$$C_1^{(2i-1)} C_0^{(2i)} - C_0^{(2i-1)} C_1^{(2i)} = \sigma_i\,, \qquad C_1^{(2i-1)} C_0^{(2i)} + C_0^{(2i-1)} C_1^{(2i)} = -m_i\,,$$
$$C_0^{(2i-1)} C_0^{(2i)} - C_1^{(2i-1)} C_1^{(2i)} = \ell_i\,, \qquad C_0^{(2i-1)} C_0^{(2i)} + C_1^{(2i-1)} C_1^{(2i)} = a_i\,.$$

with $\sigma_i^2 \equiv m_i^2 - a_i^2 + \ell_i^2$.

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$$g_{ab}(\rho, z) = \frac{1}{\rho^n}\left(\prod_{k=1}^{n} \mu_k\right)\left[\mathring{g}_{ab} - \sum_{k,l=1}^{n} \frac{(\Gamma^{-1})_{kl} L_a^{(k)} L_b^{(l)}}{\mu_k \mu_l}\right], \tag{6a}$$

$$f(\rho, z) = \frac{16\, C_f \mathring{f}_0}{\rho^{n^2/2}}\left(\prod_{k=1}^{n} \mu_k\right)^{n+1}\left[\prod_{k>l=1}^{n} (\mu_k - \mu_l)^{-2}\right] \det\Gamma\,, \tag{6b}$$

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$$\mathring{g}_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & \rho^2 \end{pmatrix}\,, \qquad \Gamma_{kl} = \frac{m_a^{(k)}\, \mathring{g}_{ab}\, m_b^{(l)}}{\rho^2 + \mu_k \mu_l}\,, \qquad m_a^{(k)} = \left(C_0^{(k)}, \frac{C_1^{(k)}}{\mu_k}\right)\,,$$

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with $\sigma_i^2 \equiv m_i^2 - a_i^2 + \ell_i^2$.

Here m_i, a_i, ℓ_i are respectively related to the mass, angular momentum and the NUT parameters, with $i \in 1, n/2$.
The ordered poles w_k, with $w_k < w_{k+1}$, are taken as follows

$$w_1 = z_1 - \sigma_1 \ , \quad w_2 = z_1 + \sigma_1 \ , \quad \dots \qquad w_{2i-1} = z_i - \sigma_i \ , \quad w_{2i} = z_i + \sigma_i \ , \tag{8}$$

New Form of Plebanski Demianski of type D

Thanks to the inverse scattering we can build a Kerr-NUT solution with a pure Rindler horizon, which enhance the Plebanski-Demianski but remaining of Type-D

$$ds^2 = -f(r,x)\,[dt - \omega(r,x)d\varphi]^2 + \frac{1}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] \;,$$

with

$$f = \frac{(1+\alpha r x)^{-2}\{[1+\alpha^2(\ell^2-a^2)x^2]^2\Delta_r - [a+2\alpha\ell r + a\alpha^2 r^2]^2\Delta_x\}}{\{\alpha^2\ell^4x^2 + 2a\ell(\alpha r - x)(1-\alpha r x) + (1+\alpha^2a^2)(r^2+a^2x^2) + \ell^2[1+\alpha x(\alpha x(r^2-2a^2)-4r)]\}}$$

$$\omega = \frac{(a-2\ell x + a x^2)[1-(b^2-a^2)x^2]\Delta_r + (r^2+\ell^2-a^2)(a+2\alpha\ell r + \alpha^2 a r^2)\Delta_x}{[1+\alpha^2(\ell^2-a^2)x^2]^2\Delta_r - [a+2\alpha\ell r + a\alpha^2r^2]^2\Delta_x} + \omega_0 \;,$$

$$\gamma = \frac{1}{2}\log\left\{\frac{[1+\alpha^2x^2(\ell^2-a^2)]^2\Delta_r - (a+2\alpha\ell r + a\alpha^2r^2)^2\Delta_x}{(1+\alpha^2a^2)(1+\alpha r x)^4}\right\} \;,$$

$$\Delta_r = (1-\alpha^2r^2)[(r-m)^2-m^2-\ell^2+a^2] \;, \qquad \Delta_x = (1-x^2)[(1+\alpha m x)^2-\alpha^2x^2(m^2+\ell^2-a^2)] \,.$$

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$$f = \frac{(1+\alpha r x)^{-2}\{[1+\alpha^2(\ell^2-a^2)x^2]^2\Delta_r - [a+2\alpha\ell r + a\alpha^2 r^2]^2\Delta_x\}}{\{\alpha^2\ell^4x^2 + 2a\ell(\alpha r - x)(1-\alpha r x) + (1+\alpha^2a^2)(r^2+a^2x^2) + \ell^2[1+\alpha x(\alpha x(r^2-2a^2)-4r)]\}}$$

$$\omega = \frac{(a-2\ell x + a x^2)[1-(b^2-a^2)x^2]\Delta_r + (r^2+\ell^2-a^2)(a+2\alpha\ell r + \alpha^2 a r^2)\Delta_x}{[1+\alpha^2(\ell^2-a^2)x^2]^2\Delta_r - [a+2\alpha\ell r + a\alpha^2r^2]^2\Delta_x} + \omega_0 \ ,$$

$$\gamma = \frac{1}{2}\log\left\{\frac{[1+\alpha^2x^2(\ell^2-a^2)]^2\Delta_r - (a+2\alpha\ell r + a\alpha^2r^2)^2\Delta_x}{(1+\alpha^2a^2)(1+\alpha r x)^4}\right\} \ ,$$

$$\Delta_r = (1-\alpha^2r^2)[(r-m)^2 - m^2 - \ell^2 + a^2] \ , \qquad \Delta_x = (1-x^2)[(1+\alpha m x)^2 - \alpha^2x^2(m^2+\ell^2-a^2)] \ .$$

Remarkably all the limit to the type-D black hole are well defined including the elusive **accelerating Taub-NUT** spacetime, i.e. for $a \to 0$

New Form of Plebanski Demianski of type D

Thanks to the inverse scattering we can build a Kerr-NUT solution with a pure Rindler horizon, which enhance the Plebanski-Demianski but remaining of Type-D

$$ds^2 = -f(r,x)\,[dt - \omega(r,x)d\varphi]^2 + \frac{1}{f(r,x)}\left[e^{2\gamma(r,x)}\left(\frac{dr^2}{\Delta_r(r)} + \frac{dx^2}{\Delta_x(x)}\right) + \rho^2(r,x)d\varphi^2\right] \ ,$$

with

$$f = \frac{(1+\alpha r x)^{-2}\{[1+\alpha^2(\ell^2-a^2)x^2]^2\Delta_r - [a+2\alpha\ell r + a\alpha^2 r^2]^2\Delta_x\}}{\{\alpha^2\ell^4x^2 + 2a\ell(\alpha r - x)(1-\alpha r x) + (1+\alpha^2a^2)(r^2+a^2x^2) + \ell^2[1+\alpha x(\alpha x(r^2-2a^2)-4r)]\}}$$

$$\omega = \frac{(a-2\ell x + a x^2)[1-(b^2-a^2)x^2]\Delta_r + (r^2+\ell^2-a^2)(a+2\alpha\ell r + \alpha^2 a r^2)\Delta_x}{[1+\alpha^2(\ell^2-a^2)x^2]^2\Delta_r - [a+2\alpha\ell r + a\alpha^2r^2]^2\Delta_x} + \omega_0 \ ,$$

$$\gamma = \frac{1}{2}\log\left\{\frac{[1+\alpha^2x^2(\ell^2-a^2)]^2\Delta_r - (a+2\alpha\ell r + a\alpha^2r^2)^2\Delta_x}{(1+\alpha^2a^2)(1+\alpha r x)^4}\right\} \ ,$$

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$$f = \frac{(1+\alpha^2\ell^2x^2)^2\Delta_r - 4\alpha^2\ell^2r^2\Delta_x}{(1+\alpha r x)^2\{r^2+\ell^2[1+\alpha x(\alpha x(\ell^2+r^2)-4r)]\}} \ ,$$

$$\omega = \frac{2\ell[\alpha r(r^2+\ell^2)\Delta_x - x(1+A^2\ell^2x^2)\Delta_r]}{(1+\alpha^2\ell^2x^2)^2\Delta_r - 4\alpha^2\ell^2r^2\Delta_r} + \omega_0 ,$$

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Summary, Conclusions & Pespectives

- Accelerating black holes describe a limit of a wider system of binary black holes where one of the two sources is much bigger with respect to the other. They can be thought as the near horizon limit, close to the bigger black hole, whose event horizon become the accelerating horizon of the solution.
- The Plebanski-Demianski family of solution can be generalised in order to contain an extra independent electric and magnetic charge or NUT parameter.
- The extra charges (with respect to PD) are related to the accelerating background and are reminiscent of the charges of an infinitely inflated big black hole close to the main small one.
- In case of rotating black holes the Misner string defect can be erased by fine-tuning the two independent NUT parameters (without completely eliminating the NUT parameter, in fact the metric remains of type I)
- This picture opens to further generalisations of accelerating black holes. These can be obtained by the more general Kerr-Newman-NUT binary system (Belinski-Alekseev 2019). For instance when also the big black hole of the binary is endowed with angular momentum.

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That's all

¡Thank You!

Extra – Example: From Schwarzschild to Reissner-Nordstrom

We use the Harrison transformation (V) to charge the Schwarzschild black hole

$$ds^2 = -\left(1 - \frac{2m}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2m}{r}} + r^2 \left[\frac{dx^2}{1 - x^2} + (1 - x^2) d\varphi^2\right]$$

By comparison with the LWP metric we deduce $\omega = 0 \implies h = 0$, $f = 1 - \frac{2m}{r}$, so the seed Ernst potential are

$$\mathbf{\Phi} = A_t + i\tilde{A}_\varphi = 0 \quad , \qquad \mathcal{E} = f - \mathbf{\Phi}\mathbf{\Phi}^* + ih = 1 - \frac{2m}{r} \ .$$

The new solution, generated by (V) with $\alpha = s \in \mathbb{R}$, become

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$$r \to \frac{\bar{r} - 2s^2 m}{1 - s^2} \ , \qquad s \to \frac{\bar{m} - \sqrt{\bar{m}^2 - \bar{e}^2}}{\bar{e}} \ , \qquad m \to \frac{1}{2}\left(\bar{m} - \sqrt{\bar{m}^2 - \bar{e}^2}\right)$$

we recognise the Reissner-Nordstrom black hole

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Disclaimer

There exist another non-equivalent form of the LWP metric

$$ds^2 = -f\,(dt - \omega d\varphi)^2 + f^{-1}\left[\rho^2 d\varphi^2 + e^{2\gamma}\left(d\rho^2 + dz^2\right)\right] \ .$$

that can be used to construct the Ernst equations.

The *magnetic* LWP:

$$d\bar{s}^2 = \bar{f}(d\phi - \bar{\omega} d\tau)^2 + \bar{f}^{-1}\left[-\rho^2 d\tau^2 + e^{2\bar{\gamma}}(d\rho^2 + dz^2)\right], \qquad (9)$$

which is obtained from the standard LWP by a *conjugation* transformation

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The associated Ernst equations are identical and therefore are invariant under the same set of symmetry transformations obtained for the standard LWP metric.

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Disclaimer

There exist another non-equivalent form of the LWP metric

$$ds^2 = -f\,(dt - \omega d\varphi)^2 + f^{-1}\left[\rho^2 d\varphi^2 + e^{2\gamma}\left(d\rho^2 + dz^2\right)\right] \; .$$

that can be used to construct the Ernst equations.

The *magnetic* LWP:

$$d\bar{s}^2 = \bar{f}(d\phi - \bar{\omega} d\tau)^2 + \bar{f}^{-1}\left[-\rho^2 d\tau^2 + e^{2\bar{\gamma}}(d\rho^2 + dz^2)\right] , \qquad (9)$$

which is obtained from the standard LWP by a *conjugation* transformation

$$W := \left\{ f \to \frac{\rho^2}{\bar{f}} - \bar{f}\bar{\omega}^2 \, , \quad \omega \to \frac{\bar{f}^2\bar{\omega}}{\bar{f}^2\bar{\omega}^2 - \rho^2} \, , \quad e^{2\gamma} \to e^{2\bar{\gamma}} \left(\frac{\rho^2}{\bar{f}^2} - \bar{\omega}^2 \right) \right\}$$

The associated Ernst equations are identical and therefore are invariant under the same set of symmetry transformations obtained for the standard LWP metric.

However an Harrison transformation in this context transforms the Schwarzschild black hole, not in RN, but into Schwarzschild-Melvin.