Simplicity and Universality in (Binary) BH dynamics: an integrability f-Airy tale

(An "asymptotic-reasoning" hierarchical research program)

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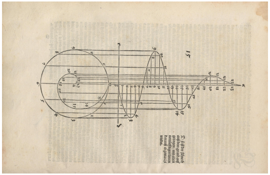


Black Holes and their Symmetries

Institut Denis Poisson, Tours, 2-4 July 2025

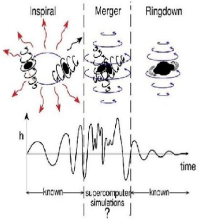
Scheme

1. The problem: simplicity and universality of BBH waveforms
   - Simplicity, universality & 'linearity': hints for an Airy/Painlevé-II BBH model
   - A BBH waveform hierarchical program
2. Hyperboloidal approach to scattering: a zeroth-order "Wave-Mean Flow"
   - Cauchy formulation: isospectrality and Lax pairs
   - Hyperboloidal framework: "bulk" and "boundary" sectors
   - "Fast DoFs": non-normal hyperboloidal dynamics
   - "Slow DoFs": the hyperboloidal "Lax-pair" problem
3. Heuristics of BBH dynamics: an "asymptotic" fAiry tale approach
   - Universal diffraction patterns on caustics: from rainbows to Airy
   - Universality from integrability: the Painlevé-II thread to BBH dynamics
   - A "Wave-Mean Flow" approach: scattering on integrable backgrounds
4. Perspectives

Scheme

The fundamental Problem

The waveform of a binary black hole system...



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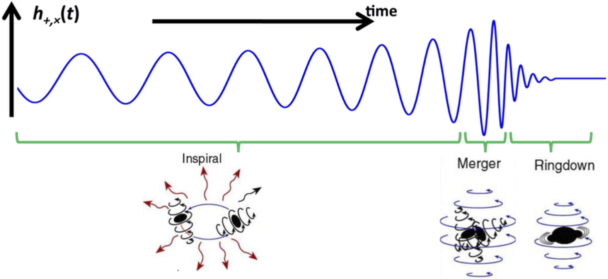

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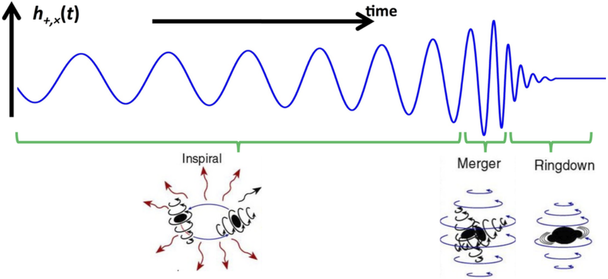

...is elegant or, rather, boring?

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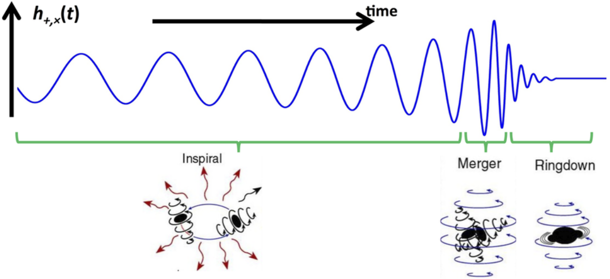

...is elegant or, rather, boring?

Elegant and "intriguing": a possible "key/probe" into the structure of the theory?

- The system behaves more "linearly" than expected [Price & Pullin 94; Baker et al. 02; Buonanno et al. 07, Berti et al. 07; ...; Choquet-Bruhat 69, 08, Touati 23 ("transparency" phenomenon in "WKB" treatmenet of gravity/geometric optics); Frauendiener et al. 25... for me Sopuerta!].
- **Simplicity** and **Universality** in the BBH merger waveform.

Quest for properties with some kind of **"structural stability"** for **"universality"**.

Methodology: "Asymptotic Reasoning"

Asymptotic reasoning: remove details, unveil patterns...[Batterman 01]

A methodological approach to a given theory (or a regime/layer of the theory):

- Typically in certain limits of a small/large 'asymptotic' parameter.
- Explicitly **sacrificing precission and exactness by eliminating details**.
- Aiming at making underlying **patterns explicitly apparent** in the appropriate range of the asymptotic parameter.
- Often entailing a gain in the mathematical tractability.

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Remarkable examples

- Renormalization group.
- Semi-classical expansions in ODEs/PDEs (WKB(J), micro-local analysis...)
- **"Catastrophe optics": diffraction on caustics**.

Methodology: “Asymptotic Reasoning”

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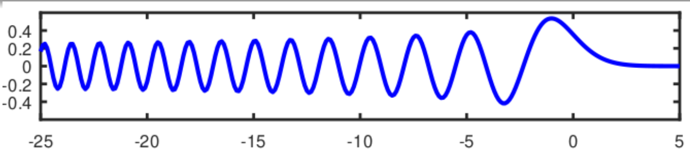


A (significant) example: the Airy function (solution to $\ddot{u} - tu = 0$)

Airy's series (convergent):

$$\mathrm{Ai}(x) = A\Big(1 + \frac{9x+3}{2\cdot 3} + \frac{9^2 x^6}{2\cdot 3\cdot 5\cdot 6} + \frac{9^3 x^9}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} + \ldots\Big) + B\Big(x + \ldots\Big)$$

Stokes' series (asymptotic!):

$$\begin{aligned}
\mathrm{Ai}(x) &\sim \frac{1}{\sqrt{\pi}}\frac{1}{(-x)^{1/4}}\cos\left(\frac{2}{3}(-x)^{3/2} - \frac{\pi}{4}\right)(1+\ldots) \quad (x \to -\infty) \\
\mathrm{Ai}(x) &\sim \frac{1}{2\sqrt{\pi}}\frac{1}{x^{1/4}}e^{-\frac{2}{3}x^{3/2}}(1+\ldots) \qquad (x \to +\infty)
\end{aligned}$$

Plan

A model: universal wave patterns in dispersive hydrodyn.

PDE "phase transitions" and universality

- *Universal wave patterns* [e.g. Miller 16]:
  a PDE analogue of universality features in (critical) phase transitions.

Example of the Burgers' equation: **"All viscous shocks look the same"**

$$\partial_t u + u\partial_x u = 0$$

Generates shocks, with discontinuous jumps $\Delta(t)$ in u along $x = x(t)$.
Regularization with viscosity term:

$$\partial_t u^\epsilon + u^\epsilon \partial_x u^\epsilon = \epsilon \partial^2_{xx} u^\epsilon$$

Then, the following limit exists

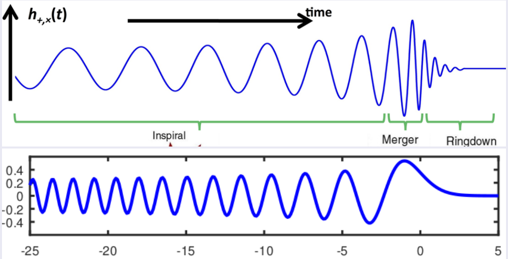
$$U(y,t) = \lim_{\epsilon\to 0} u^\epsilon(x(t) + \epsilon y, t) = \bar{u} - \frac{\Delta(t)}{2} \tanh\left(\frac{y - y_o(t)}{4\Delta(t)^{-1}}\right)$$

All viscous shocks "look the same" and are described by the "universal" special function $\tanh(x)$*, and this special function is independent of the initial data.*

Example of universal wave pattern.

Special function at the (critical) transition: Airy function

BBH merger waveform and Airy direct comparison: ignore ringdown and phase



Problem: assess the BBH merger waveform as a modulated Airy function.

Dynamical transition: from oscillatory to decaying behaviour

$$\boxed{\ddot{u} - tu = 0} \qquad \text{(Airy equation)}$$

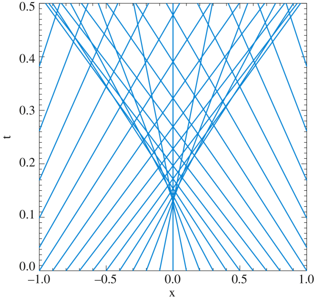
interpolating between oscillation and damped regimes

$$\begin{aligned} \ddot{u} + \omega^2 u &= 0 \qquad \text{(oscillatory)} \\ \ddot{u} - \omega^2 u &= 0 \qquad \text{(exponentially “decaying”)} \end{aligned}$$

Special function at the (critical) transition: Airy function

Universality in (linear) dispersive wave equations: **"All caustics look the same"**

Universality for arbitrary **linear dispersive wave PDEs**, for arbitrary initial data, near "caustics" for (appropriate) characteristic lines: **described by Airy**.



Semiclassical ("WKB") treatment, with small parameter ϵ [e.g. Miller 16]

$$\psi(t,x) \sim \frac{e^{-i\pi/4}}{\sqrt{2\pi\epsilon t}} \int_{-\infty}^{\infty} e^{iI(y;x,t)/\epsilon} \sqrt{\rho_0(y)}\, dy$$

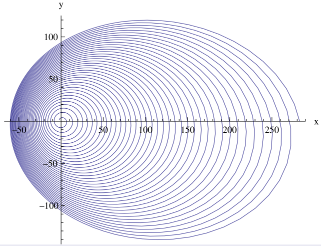
Stationary points $I'(y;x,t) = 0$ lead to characteristics (and "caustics") above.

Airy and radiation damped orbital: Painlevé-II

Damped orbital motion of charged particle in Coulomb potential [Rajeev 08; O. Lisovyy!]

Landau-Lifshitz equations (in the non-relativistic limit) for a charged particle in a Coulomb potential **solved exactly in terms of the Painlevé-II equation**:

$$\frac{d^2L}{dz^2} = -\frac{1}{2k^2}L^3 - \frac{\tau}{k}zL$$



$$\boxed{\frac{d^2w}{dt^2} - tw - 2w^3 = 0}$$

(Painlevé-II, "non-linear Airy")

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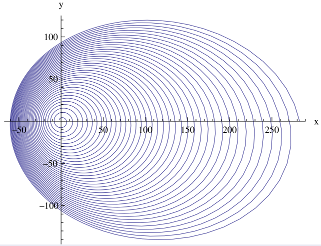

$$\boxed{\frac{d^2w}{dt^2} - tw - 2w^3 = 0}$$ (Painlevé-II, "non-linear Airy")

EMRI transition to plunge: Painlevé-I dynamics [Ori & Thorne 00; Compère and Küchler 21]

Dynamics $\frac{d^2X}{dT^2} = -X^2 - T$ solved by $\frac{d^2w}{dt^2} = 6w^2 + t$ (Painlevé-I)

Plan

A bottom-up approach to BBH merger waveforms

"Asymptotic reasoning" and "structural stability"

- **Simplicity: Asymptotic reasoning**. Address qualitive mechanisms underlying simplicity, by filtering overload of details encumbering structural features in "full and exact approaches" [Batterman 01]
- **Universality: Structural stability**. Aim at accounting for universality features, as stable properties in generic configurations.

A bottom-up approach to BBH merger waveform: Airy as a diffraction caustic

[JLJ, Krishnan 22]

	Asymptotic reasoning simplication	Structural stability realisation
Fold-caustic model	Geometric optics (ray physics)	Arnol'd-Thom's theorem classification of stable caustics
Airy function model (high frequency)	Fresnel's diffraction approximation to wave theory (diffraction on caustics)	Universal difraction patterns in caustics (scaling laws)
Airy function model (general case)	turning points ODE models (linear case)	asymptotic theory of ordinary differential equations
(effective) PDE model	?	?

A bottom-up approach to BBH merger waveform

A bottom-up approach to the BBH merger waveform: beyond Airy to PDEs

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics Catastrophe (singularity) Theory	Arnol'd-Thom's Theorem Classification of Stable Caustics
Airy function model	Fresnel's Diffraction Semiclassical Theory asymptotic ODE theory	Universal Diffraction Patterns in Caustics linear ODE turning points
Painlevé-II model	Painlevé Transcendents and Integrability Self-force calculations and EMRBs	Painlevé property Non-linear Turning Points
KdV-like model	Inverse Scattering Transform and Integrability Dispersive Non-linear PDEs Critical Phenomena in Dispersive PDEs	Painlevé test, Lax pairs Darboux transformations Soliton Scattering Universal Wave Patterns Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability Twistorial techniques	(anti-)Self-Dual DoF Instantons, Tunneling Penrose Transform, 'Twistor' BBH data

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The hierarchical BBH program: a "Wittgenstein's ladder" [JLJ, Krishnan & Sopuerta 23]

Resulting proposal:
"Wave-Mean Flow" approach with "fast" degrees of freedom (DoF) **"linearly"** propagating/interacting on a "slow" degrees of freedom **integrable** background.

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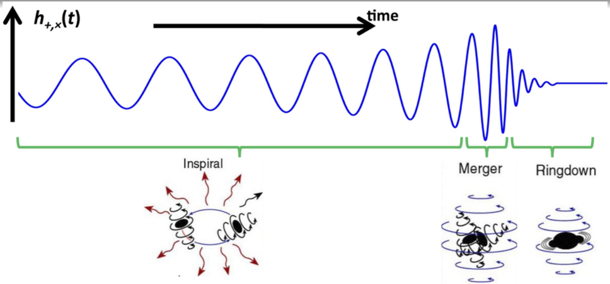
"Integrability": Lax-pairs and late scattering



"Hidden symmetries" in Schwarzschild: Darboux covariance and KdV

- **"Vertical" QNM Isospectrality**, between axial/polar modes: **Darboux transformations**, role of **algebraically special QNMs** [Chandrasekhar & Detweiler 75, Chandrasekhar 83; ...; Glampedakis, Johnson & Kennefick 17, ; Lenzi & Sopuerta 21]
- **"Horizontal" QNM Isospectrality**, along each axial/polar branch: Schrödinger equation and KdV conserved charges [Chandrasekhar; IST theory...; Lenzi & Sopuerta 21]
- **Lax pair**, Virasoro underlying **bulk** symmetry, infinite PDE hierarchy... [...; Semenov-Tian-Shansky, JLJ, Lenzi & Sopuerta 24]

"Integrability": Lax-pairs and late scattering

Quasi-Normal Mode (QNM) ("formal") spectral problem

From equations on master functions Ψ, QNMs satisfy the spectral problem (under outgoing boundary condition) for the "Schrödinger" operator $\mathcal{L} = (\partial_x^2 - V)$:

$$\left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0 \implies \mathcal{L}\Psi = (\partial_x^2 - V)\Psi = -\omega^2\Psi$$

Korteweg-de Vries (KdV) and QNM isospectrality

Deform V and Ψ according to:

$$\begin{aligned}\partial_\sigma V - 6V\partial_x V + \partial_x^3 V &= KdV[V] = 0 \\ \partial_\sigma \Psi &= \left(-4\partial_x^3 + 6V\partial_x + 3V_{,x}\right)\Psi = P\Psi\ .\end{aligned}$$

$\mathcal{L}$ and P form a "Lax pair": isospectrality (and integrability)

$$\partial_\sigma \mathcal{L} - [P, \mathcal{L}] = -\mathrm{KdV}[V]\cdot \mathrm{Id} = 0$$

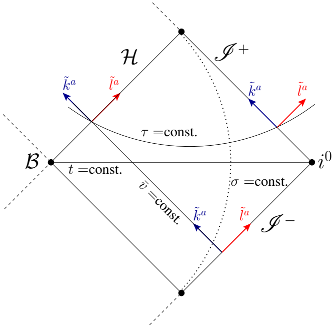
$$\boxed{\partial_\sigma \mathcal{L} = [P, \mathcal{L}]} \qquad \text{(Lax pair equation)}$$

Plan

Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

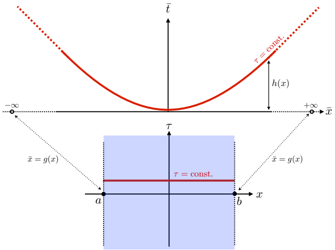
- **Spectral problem**: homogeneous wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at $\mathscr{I}^+$.
- Outgoing BCs actually "incorporated" at $\mathscr{I}^+$:
    - Geometrically: null cones outgoing.
    - Analytically: BCs encoded into a singular operator, **"BCs as regularity conditions"**.
- **Eigenfunctions** do not diverge when $x \to \infty$: actually **integrable**. Key to Hilbert space.



Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to QNMs

- B. Schmidt [Schmidt 93; cf. also Friedman & Schutz 75]
- Analysis in the conformally compactified picture [Friedrich; Frauendiener,...]. Micro. Analysis [Vasy 13]
- Framework for BH perturbations [Zenginoglu 11].
- QNMs of asymp. AdS spacetimes [Warnick 15].
- QNM definition as operator eigenvalues [Bizoń...; Bizoń, Chmaj & Mach 20].
- Schwarzschild QNMs [Ansorg & Macedo 16]. (cf. also Reissner-Nordström [Macedo, JLJ, Ansorg 18]).
- "Gevrey" [Gajic & Warnick 20; Galkowski & Zworski 21].



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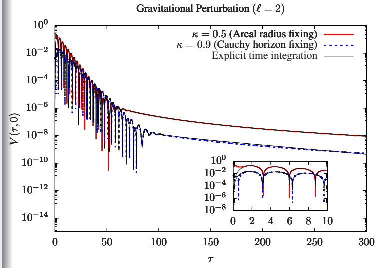

Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

BH QNM instability: Spacetime asymptotics

- Asymptotically flat [L. Al Sheikh Ph.D thesis 22] [JLJ, R. P. Macedo, L. Al Sheikh 21; E. Gasperin, JLJ 22; ...; K. Destounis et al 21; V. Boyanov et al 22; JLJ 22].
- Asymptotically de Sitter [S. Sarkar, M. Rahman, S. Chakraborty 23; JLJ, R. P. Macedo, L. Al Sheikh 21]
- Asymptotically Anti-de Sitter
  - Hyperboloidal slicing [D. Areán, D. García-Fariña, K. Landsteiner 23]
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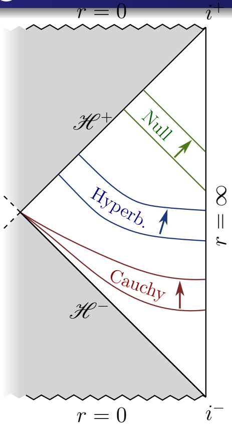

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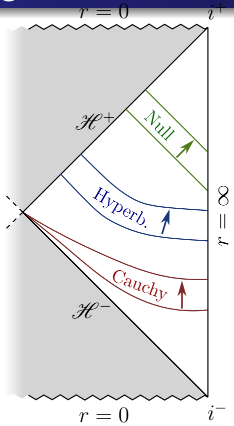

1. Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\; , \; r^* \in]-\infty, \infty[$$

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

Starting point: (scalar) wave equation in "tortoise" coordinates

On a stationary spatime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \ ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Hyporboloidal approach

$$\left\{ \begin{array}{lcl} \bar{t} & = & \tau - h(x) \\ \bar{x} & = & f(x) \end{array} \right. \quad .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \mathrm{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future $\mathscr{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda\partial_t = \partial_{\bar{t}} = \partial_\tau$.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\boxed{\partial_\tau u_{\ell m} = i L u_{\ell m}} \quad , \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$\begin{aligned} L_1 &= \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)} \\ L_2 &= \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right) = \frac{1}{w(x)}\left(\gamma(x)\partial_x + \partial_x(\gamma(x)\cdot)\right) \end{aligned}$$

with $w(x) = \dfrac{f'^2 - h'^2}{|f'|} > 0 \;\; , \;\; p(x) = \dfrac{1}{|f'|} \;\; , \;\; q(x) = |f'|\, V_\ell \;\; , \;\; \gamma(x) = \dfrac{h'}{|f'|}.$

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$\boxed{L\, v_n = \omega_n\, v_n}\ .$$

with v_n the "right-eigenvectors", where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

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Hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is "singular": **BCs "in-built" in** L.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

A "natural" scalar product: initial value problem and physical content

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1, u_2 \rangle_{_E} = \frac{1}{2}\int_a^b \Big(w(x)\bar{\psi}_1\psi_2 + p(x)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}_\ell\bar{\phi}_1\phi_2 \Big) dx \ ,$$

associated with the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||^2_{_E} = \langle u, u \rangle_{_E} = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau\phi) t^a n^b d\Sigma_\tau \ ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L: not selfadjoint.
- L_2: dissipative term encoding the energy leaking at $\mathscr{I}^+$.
- Energy flux at $\mathscr{I}^+$: $\boxed{F(\tau) = \int_{\mathcal{S}^{\mathcal{H}}} \gamma|\partial_\tau\phi|^2 dS + \int_{\mathcal{S}^\infty} \gamma|\partial_\tau\phi|^2 dS}$
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

QNM: (spherically symm.) "general case" [JLJ, Macedo, Al Sheikh 21]

Adjoint operator

$$L^\dagger = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 + L_2^\partial \end{array}\right)$$

where

$$L_2^\partial = 2\frac{\gamma}{w}\Big(\delta(x-a) - \delta(x-b)\Big)$$

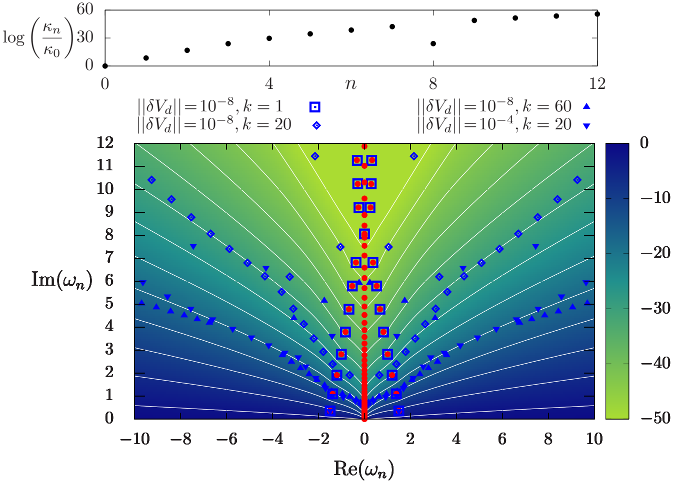
Loss of "self-adjointness" happens at the boundaries (as expected)

Loss of "self-adjointness": an "infrared" phenomenon

- "Ultraviolet" QNM overtone instability connected with a essentially asymptotic ("infrared") structure.
- "Missing degrees of freedom and asymptotic symmetries".

Black Hole QNM spectral instability

Schwarzschild ($s = 2$, $\ell = 2$):



Spectral QNM overtone instability

Low regularity phenomenon, need to control high-derivatives: Sobolev H^p spaces

Towards a geometric description of QNMs

Compactified hyperboloidal slicing: geometric data

i) Choose a slicing $\{\mathcal{S}^\infty_\tau\}$ of $\mathscr{I}^+$. Given the generator k^a along $\mathscr{I}^+$ consider, at each slice $\mathcal{S}^\infty_\tau$, the only null normal ℓ^a to $\mathcal{S}^\infty_\tau$ satisfying $k^a\ell_a = -1$. Repeat on the BH horizon.

ii) Choose a function γ on null infinity sections $\mathcal{S}^\infty_\tau$ (respectively, also a function γ on $\mathcal{S}^{\mathcal{H}}_\tau$ in BH spacetimes) and consider the null vector $\gamma^a = \gamma\ell^a$, outgoing from $\mathcal{M}$.

Repeat on the BH horizon.

iii) Extend the vector γ^a arbitrarily to the bulk (under appropriate conditions).

iv) Slices Σ_τ in $\tilde{\mathcal{M}}$ are compact manifolds with boundary, with boundaries given by spheres.

v) Resulting data:

$$(\{\Sigma_\tau\}, \gamma_{ab}, K_{ab}, w; \{\mathcal{S}^\infty_\tau\}, \{\mathcal{S}^{\mathcal{H}}_\tau\}, \gamma)$$

Bulk and Boundary algebraic-structures separation

Evolution equations

$$\partial_\tau u = iLu \quad , \quad u = \begin{pmatrix} \phi \\ \psi = \partial_\tau \phi \end{pmatrix} \quad , \quad L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)$$

$$L_1 = \frac{1}{w}\left(\tilde{\Delta} - \tilde{V}(x)\right), \qquad L_2 = \frac{1}{w}\left(2\gamma \cdot \tilde{D} + \tilde{D} \cdot \gamma\right)$$

Bulk + Boundary separation

- L_1 controls de "bulk" degree of freedom V. QNM-isospectral (adiabatic?) deformations of V [JLJ, Lenzi & Sopuerta 24]:

  KdV - Virasoro - Schwarzian derivative

- L_2 controls the boundary, where the energy leaking happens.

The adjoint operator and adjoint spectral problem

Adjoint infinitesimal evolution operator

$$L^\dagger = L + L^\partial \quad , \quad L^\partial = \frac{1}{i}\left(\begin{array}{c|c} 0 & 0 \\ \hline 0 & L_2^\partial \end{array}\right)$$

$$L_2^\partial = -2\frac{\gamma^a k_a}{w}\left(\delta_{S^{\mathcal{H}}} - \delta_{S^\infty}\right) = -2\frac{(\gamma k^a)\ell_a}{w}\left(\delta_{S^{\mathcal{H}}} - \delta_{S^\infty}\right)$$

Adjoint eigenvalue problem ("left-eigenvectors")

Note that the spectral problem can be (and must be) written also in terms of the adjoint $L^\dagger$

$$\boxed{L^\dagger \alpha_n = \bar{\omega}_n \alpha_n}$$

Missing degrees of freedom and BMS symmetry

Non-selfadjointness and missing degrees of freedom

- **Non-selfadjointness in dynamical problems generically reflects that some degrees of freedom are missing or are being lost**: the system is not isolated or is part of a larger system, leading to non-conservative dynamics.
- To cast such systems in terms of a selfadjoint problem, one must "complete" the system by adding an appropriate set of degrees of freedom.

How could we render our BH perturbation problem selfadjoint?

- In our scattering case, degrees of freedom flow to infinity and through the BH horizon.
- Consider adding formal degrees of freedom at future null infinity $\mathscr{I}^+$ and at the BH horizon $\mathcal{H}$, in such a way that they account for the degrees of freedom and energy leaving the system.
- Placing **"Geiger counters" at the infinite and horizon boundaries**, such that the total number of degrees of freedom is conserved.

Coupling boundary and bulk degrees of freedom

Being anchored to the boundary, such boundary degrees of freedom would be non-propagating ones (in the bulk)

- Flux at boundaries:

$$F(\tau) = \int_{\mathcal{S}^{\mathcal{H}}} \gamma |\partial_\tau \phi|^2 dS + \int_{\mathcal{S}^{\infty}} \gamma |\partial_\tau \phi|^2 dS$$

  Bondi-Sachs flux of energy-momentum at null infinity if: i) γ is built from the $\ell = 0, 1$ spherical harmonics and ii) we identify $\partial_\tau \phi$ as a news function $\mathcal{N}$.

- Here, γ arbitrary function on $\mathcal{S}^{\infty}_{\tau}$: supertranslation in the asymptotic BMS group.

$$\xi^a = \gamma k^a \quad , \quad \mathcal{L}_k \gamma = 0$$

  corresponds to a supertranslation in the asymptotic BMS group.

- Flux term (arbitrary $\gamma(x)$) ["Coupling" action: BCs as action boundary terms (Roberto!!!]:

$$H_\xi = \int_{\mathscr{I}^+} \left(\gamma \partial_\tau \phi \partial_\tau \phi + \gamma \partial_\tau \phi + \partial_\tau \phi \Delta_{\mathcal{S}^\infty} \gamma \right) d\tau dS$$

  corresponding to the **BMS supertranslation** ξ^a **and generating symplectomorphisms in the (appropriate) phase space of degrees of freedom that "live on the boundary"**.

Missing degrees of freedom and BMS symmetries

A "heuristic" proposal

Missing degrees of freedom can be encoded in BMS supertranslation degrees of freedom:

i) **BMS supertranslations** ξ^a **would stand as a dynamical symmetry**, acting on (and generating) the **phase space of degrees of freedom on the boundary** (the "clicks" in the Geiger counter analogy), with γ **a degree of freedom living on the null boundaries**.

ii) The whole geometric construction starts from the **choice of slicing of null boundaries**. This is **arbitrary**, but all choices are **related by an appropriate BMS supertranslation**.

iii) **The set of degrees of freedom** (ϕ, γ) **would be complete.** By this we mean that, as ϕ flows away, degrees of freedom γ are "activated" through a coupling controlled by the Hamiltonian (1), so **a total energy of the type** $E_o = E(\phi, \psi) + H_\xi$ **would be conserved**.

Missing degrees of freedom and BMS symmetries

"Hopes":

- From the perspective of the extended space of degrees of freedom (ϕ, γ), one could gain **insight into the observed instability of BH QNMs**.

  Paraphrasing this in terms of inverse scattering theory: **additional BMS degrees of freedom could provide necessary additional data at the boundaries** —complementary to (transmission and) reflection coefficients and (possible) bound energy levels— needed to determine the scattering potential [cf. Ashtekar's "Gravitational Wave Tomography"].

- Conservative formulation of the dynamics can be constructed in the extended space of degrees of freedom (ϕ, γ), then a notion of **global spacetime normal modes**, as eigenfunctions of the time evolution generator, could be envisaged. This could be of **interest both from a phenomenological and from a fundamental (quantization) perspective.**

- Role of "metriplectic (Leibniz) structures" in geometry of dissipative systems? [Kabel & Wieland 22, Wieland 24...].

Plan

Non-normal dynamics

Hyperboloidal (linear) evolution problem

$$\begin{cases} \partial_\tau u = iLu \ , \\ u(\tau = 0, x) = u_0(x) \ \ , \ \ ||u_0|| < \infty \ , \end{cases}$$

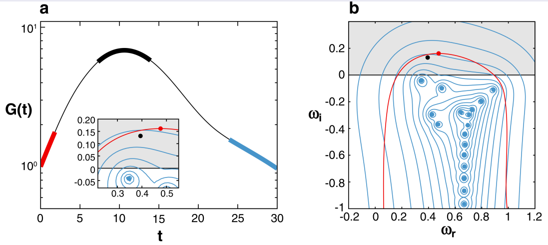
Non-normal dynamics: non-modal analysis

$$\boxed{[L, L^\dagger] \neq 0}$$

- No spectral theorem: no "normal modes" (no Hilbert basis).
- Eigenfunctions of L non-orthornormal.
- Non-modal (linear) **transient growths**.
- Tools such as the ϵ-Pseudospectrum, essentially the norm of resolvent $||R_L(\omega)||$, and the Growth function $G(\tau)$.

Pseudospectrum and non-modal analysis

Growth factor $G(t) = ||e^{itL}||$ and pseudospectrum (eg. Poiseuille flow [Schmid 07])



Growth function $G(\tau)$: maximum possible amplification

$$G(\tau) = \sup_{u_0 \neq 0} \frac{||u(\tau)||}{||u_0||} = \sup_{u_0 \neq 0} \frac{||e^{i\tau L} u_0||}{||u_0||} = ||e^{i\tau L}||$$

Optimal excitation u_0: eigenfunction of the maximum (generalised) eigenvalue in the Singular Value Decomposition (eigenfunction of the $\max \sigma[(e^{i\tau L})^\dagger e^{i\tau L})]$.

Pseudospectrum and non-modal analysis

Beyond spectral analysis: some elements

- **QNM spectrum** $\sigma(L)$: possibility of spectral instabilities.
- **Numerical range** $W(L)$ **and numerical abscisa** $\omega(L)$-

$$
\begin{aligned}
W(L) &= \{\langle u, Lu\rangle, \text{with } ||u|| = 1, u \in H\} \\
\omega(L) &= \sup \mathrm{Im}\big(W(L)\big) \\
\omega(L) &= \frac{d}{dt}\,||e^{tL}||\Big|_{t=0}
\end{aligned}
$$

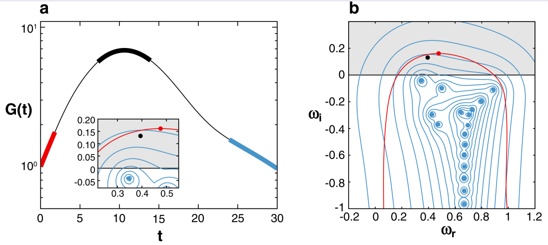
- **Intermediate/maximum transient, Kreiss constant** $\mathcal{K}(L)$:

$$
\begin{aligned}
\sup_{t\geq 0} G(t) = \sup_{t\geq 0} ||e^{tL}|| &\geq \mathcal{K}(L) \\
\mathcal{K}(L) &= \sup_{\mathrm{Im}(z)>0} \{|\mathrm{Im}(z)| \cdot ||(L - zI)^{-1}||\}
\end{aligned}
$$

- **Pseudo-resonances**: $R_{\max}(\omega) = \sup_{s\neq 0} \frac{||u^{(2)}||}{||s||} = e^{-\mathrm{Im}(\omega)\tau}||R_L(\omega)||$

Pseudospectrum and non-modal analysis

Growth factor $G(t) = ||e^{itL}||$ and pseudospectrum (eg. Poiseuille flow [Schmid 07])



Spectral instability, Transients and Pseudoresonances [Trefethen et al. 93, Tref. & Embrée 05, ...]

- **Late times**: QNM spectrum $\sigma(L)$.
- **Transients** (no source, $u^{(1)}$): consequence of non-orthogonality of QNMs.
  - **Initial times**: numerical range $W(L)$ and spectral abscisa $\omega(L)$.
  - **Intermediate/maximum**: pseudospectrum $\sigma_\epsilon(L)$ and Kreiss constant $\mathcal{K}(L)$.

  [JLJ 22, Boyanov, Destounis et al. 23, J. Carballo & B. Withers: Transient dynamics of QNM sums, 24, J.-N. Chen,L.-B. Wu & Z.-K. Guo 24, Besson et al (in prep.) 24; cf. J. Redondo-Yuste's talk]
- **Pseudo-resonances** (source present, $u^{(2)}$): $R_{\max}(\omega)$ (with $\omega \in \mathrm{Re}(\omega)$).

Keldysh QNM decomposition [Besson & JLJ 25]

Dual spectral problems: vectors and covectors

$$Lv_n = \omega_n v_n \quad , \quad L^t \alpha_n = \omega_n \alpha_n \quad , \quad v_n \in \mathcal{H}, \alpha_n \in \mathcal{H}^*$$

If a scalar product available: spectral and adjoint spectral problem

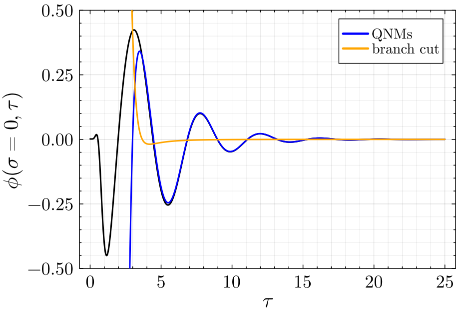
$$L\hat{v}_n = \omega_n \hat{v}_n \quad , \quad L^\dagger \hat{w}_n = \overline{\omega}_n \hat{w}_n \quad , \quad \hat{v}_n, \hat{w}_n \in \mathcal{H}$$

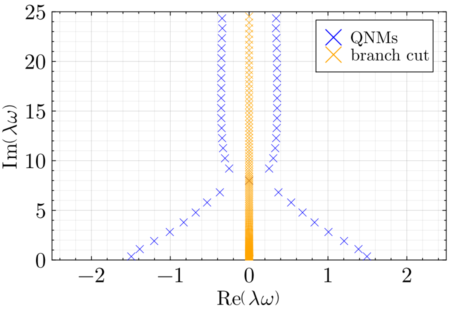
Keldysh expansion [Besson & JLJ 25]

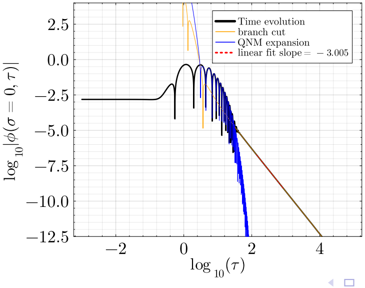
$$\begin{aligned} u(\tau, x) &= \sum_{n=0}^{N_{\mathrm{QNM}}} e^{i\omega_n \tau} \langle \alpha_n, u_0 \rangle v_n(x) + E_{N_{\mathrm{QNM}}}(\tau; u_0) \\ &= \sum_{n=0}^{N_{\mathrm{QNM}}} e^{i\omega_n \tau} \kappa_n \langle \hat{w}_n, u_0 \rangle_G \hat{v}_n(x) + E_{N_{\mathrm{QNM}}}(\tau; u_0) \end{aligned}$$

with $\quad ||E_{N_{\mathrm{QNM}}}(\tau; u_0)|| \leq C(N_{\mathrm{QNM}}, L) e^{-a_{N_{\mathrm{QNM}}} \tau} ||u_0||$,

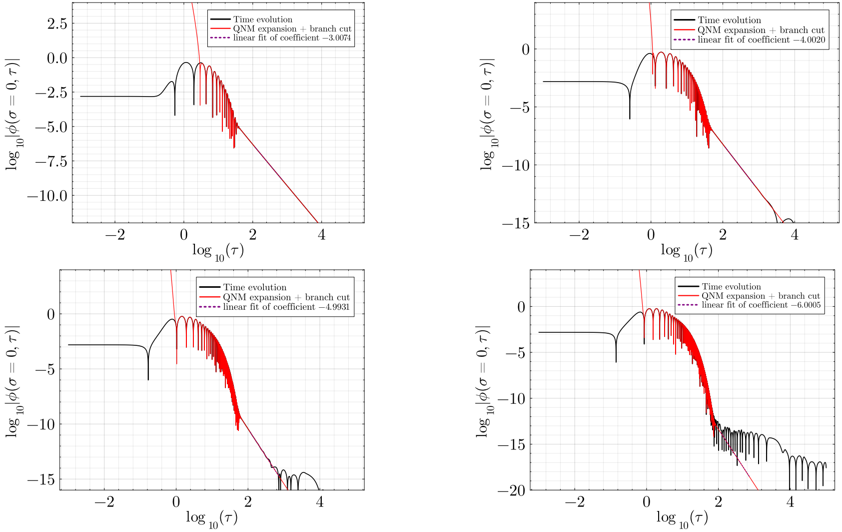
Keldysh QNM decomposition [Besson & JLJ 25]





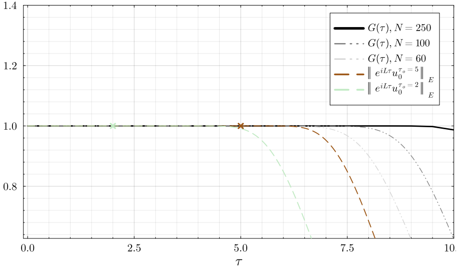


Keldysh QNM decomposition [Besson & JLJ 25]

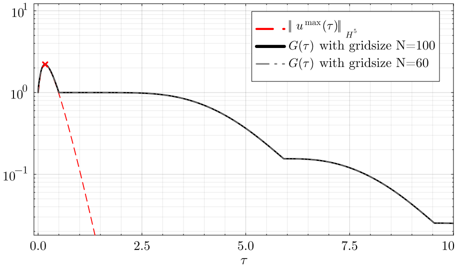


Non-normal transient growths and distributions [Besson & JLJ 25]

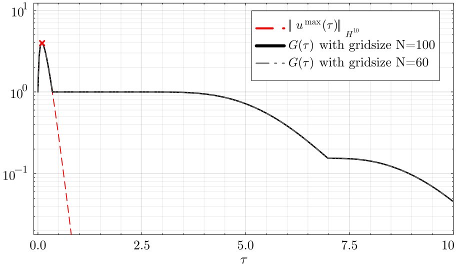
Energy norm



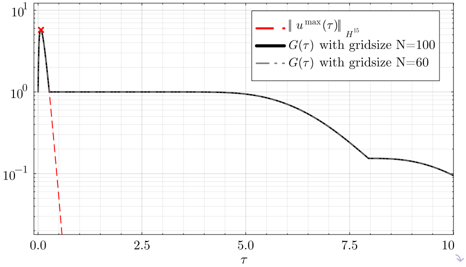
Sobolev H^5-norm



Sobolev H^{10}-norm

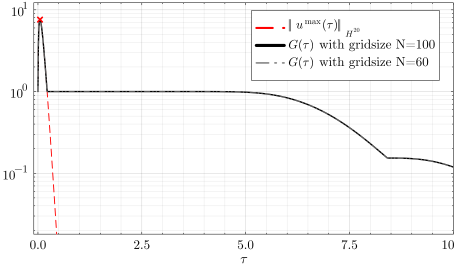


Sobolev H^{15}-norm

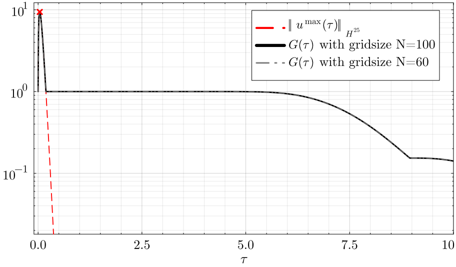


Non-normal transient growths and distributions [Besson & JLJ 25]

Sobolev H^{20}-norm



Sobolev H^{25}-norm



H^p growth transients: distributions at large p

$$\tau_{\max} \sim \frac{1}{p} \quad , \quad G_{\max} \sim p$$

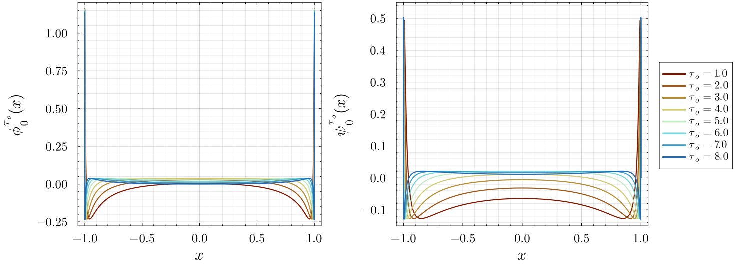
In the limit $p \to \infty$:

$$\boxed{\lim_{p\to\infty} G(\tau) \sim \delta(\tau)}$$

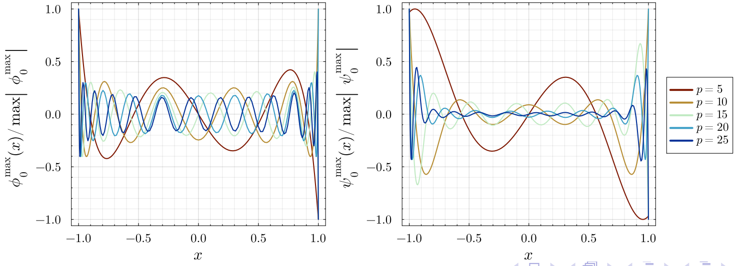
Distributional (in time) 'impulsive disturbance': key in "response function" in linear response theory.

Optimal initial data

$u_0^{\tau_o}(x)$ as a function of x (energy norm)

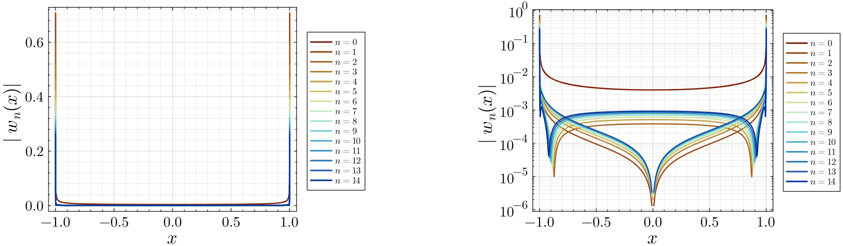


$u_0^{\max}(x)$ as a function of x for different H^p norms

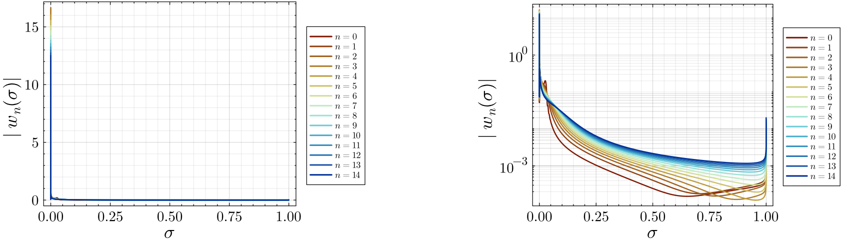


Co-modes $\hat{w}_n$ of $L^\dagger$: disctributions peaked at the boundary

eigenfunctions $\hat{w}_n$ of $L^\dagger$: Pöschl-Teller case



eigenfunctions $\hat{w}_n$ of $L^\dagger$: Schwarzschild case



Main message to retain: role of "co-modes" and "dual spectral problem" for $L^\dagger$

Relevance of distributions, dual space, in the hyperboloidal scattering problem

- For the study of purely dynamical features of the "fast" DoF it is crucial to take into account "distributional functions" in the dual space.
- Relevant objects with $\delta(t - t_o)$ and $\delta(x - x_o)$ structure: distributional structure peaked at the integration boundary.
- This will impact the guideline in the Lax pair problem.

Algebraically special QNM

More to be said (next time) concerning:

- Relation to Darboux generating function between axial/polar modes.
- Evolution by m-KdV.
- Keldysh decomposition and role in premerger dynamics.

Plan

The hyperboloidal "Lax-pair" problem [ongoing work C. Vitel 25]

The problem

Find a Lax pair P for the non-selfadoint generator L:

$$\boxed{\partial_\sigma L = [P, L]}$$

The hyperboloidal "Lax-pair" problem [ongoing work C. Vitel 25]

The problem

Find a Lax pair P for the non-selfadoint generator L:

$$\boxed{\partial_\sigma L = [P, L]}$$

Aiming at integrable dynamics:

- KdV-like dynamics for bulk V.
- Dynamics for boundary γ.
- Dynamics on extra fields...?
- Algebraic (symmetry) structures in the bulk and boundary, semidirect action of the bulk of the boundary:

$$\boxed{Sym = Sym_{Bulk} \otimes_S Sym_{Boundary}}$$

The hyperboloidal "Lax-pair" problem [ongoing work C. Vitel 25]

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The (alternative) equivalent adjoint Lax-pair problem

Find a Lax pair $\tilde{P}$ for the non-selfadoint generator $L^\dagger$:

$$\boxed{\partial_\sigma L^\dagger = [\tilde{P}, L^\dagger]}$$

The hyperboloidal "Lax-pair" problem [ongoing work C. Vitel 25]

The problem

Find a Lax pair P for the non-selfadoint generator L:

$$\boxed{\partial_\sigma L = [P, L]}$$

Note "resonances" with other appaoches [Marios Petropoulos, Wolfgang Wieland]:

- Search for (radiative) dynamics on the boundary: fields & conjugate momenta (phase space).
- (Finite) radiative sector, (Infinite) flat sector: Wave-Mean Flow.
- Integrability sector (link to electric-magnetic duality).

The (alternative) equivalent adjoint Lax-pair problem

Find a Lax pair $\tilde{P}$ for the non-selfadoint generator $L^\dagger$:

$$\boxed{\partial_\sigma L^\dagger = [\tilde{P}, L^\dagger]}$$

Lax pair and hyperboloidal function flows [C. Vitel]

"Weak" adjoint Lax-pair problem: hierarchy of deformations [Vitel 25]

Defining a (integrated) weak spectral problem from $\partial_\sigma L^\dagger = [\tilde{P}, L^\dagger]$, and introducing

$$\boxed{u_0 = -\tilde{V},\, u_1 = -\gamma_{,\xi},\, u_2 = \gamma^2 - 1}$$

the resulting Lax-pair equation is equivalent to the hierarchy [Antonowicz & Fordy 89] of equations

$$\begin{cases} u_{2,\sigma_m} = J_0 P_{m-2} + J_1 P_{m-1} + J_2 P_m \\ u_{1,\sigma_m} = J_0 P_{m-1} + J_1 P_m \\ u_{0,\sigma_m} = J_0 P_m \end{cases}$$

where $J_0 P_{k-2} + J_1 P_{k-1} + J_2 P_k = 0$ and the J_k are known.

Integrable hierarchy on V, γ and $\partial_x \gamma$ [Vitel 25]

- For $m = 1$: KdV for V and flow for γ and $\partial_x \gamma$.
- Zero-curvature formulation.

Scheme

1. The problem: simplicity and universality of BBH waveforms
   - Simplicity, universality & 'linearity': hints for an Airy/Painlevé-II BBH model
   - A BBH waveform hierarchical program
2. Hyperboloidal approach to scattering: a zeroth-order "Wave-Mean Flow"
   - Cauchy formulation: isospectrality and Lax pairs
   - Hyperboloidal framework: "bulk" and "boundary" sectors
   - "Fast DoFs": non-normal hyperboloidal dynamics
   - "Slow DoFs": the hyperboloidal "Lax-pair" problem
3. Heuristics of BBH dynamics: an "asymptotic" fAiry tale approach
   - Universal diffraction patterns on caustics: from rainbows to Airy
   - Universality from integrability: the Painlevé-II thread to BBH dynamics
   - A "Wave-Mean Flow" approach: scattering on integrable backgrounds
4. Perspectives

An "asymptotic-reasoning" hierarchical research program

A bottom-up approach to the BBH merger waveform [JLJ, Krishnan & Sopuerta 22, 23]

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics Catastrophe (singularity) Theory	Arnol'd-Thom's Theorem Classification of Stable Caustics
Airy function model	Fresnel's Diffraction Semiclassical Theory asymptotic ODE theory	Universal Diffraction Patterns in Caustics linear ODE turning points
Painlevé-II model	Painlevé Transcendents and Integrability Self-force calculations and EMRBs	Painlevé property Non-linear Turning Points
KdV-like model	Inverse Scattering Transform and Integrability Dispersive Non-linear PDEs Critical Phenomena in Dispersive PDEs	Painlevé test, Lax pairs Darboux transformations Soliton Scattering, "Soliton Resolution" Universal Wave Patterns Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability Twistorial techniques	(anti-)Self-Dual DoF Instantons, Tunneling Penrose Transform, 'Twistor' BBH data

Plan

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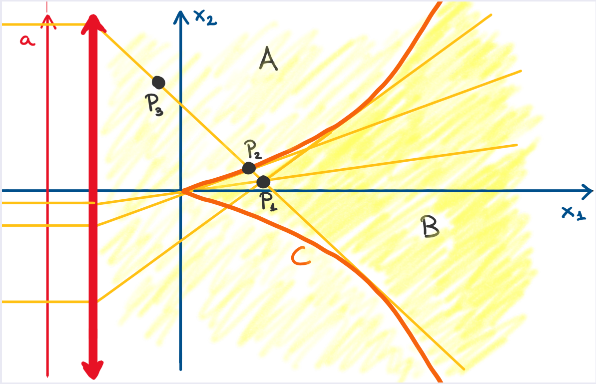
4. Perspectives

1. "Geometric optics": examples of caustics

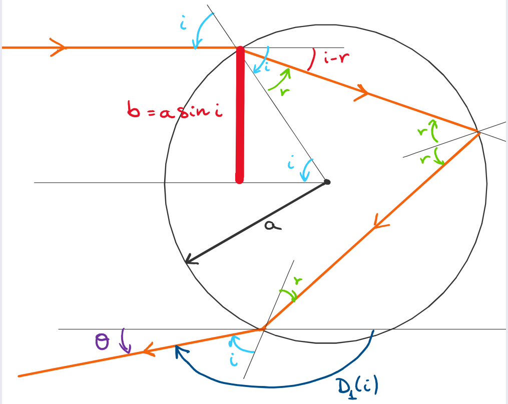
(Cusp) caustic in an Axisymmetric lens



- **"State" variable**: a.
- **"Control" variable**: x_1, x_2.

1. "Geometric optics": examples of caustics

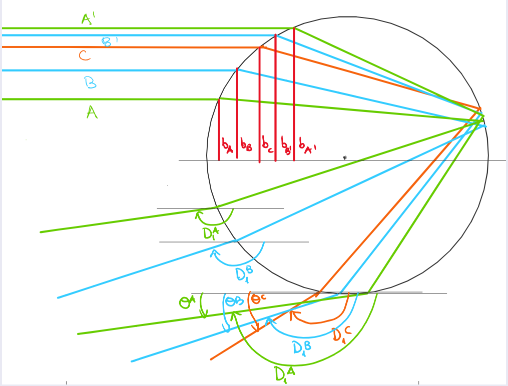
(Fold) caustic in the rainbow



- **"State" variable**: b.
- **"Control" variable**: θ (or D).

1. "Geometric optics": examples of caustics

(Fold) caustic in the rainbow



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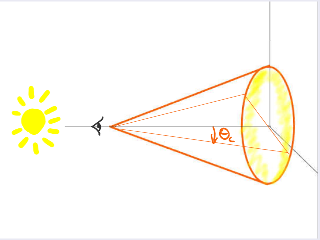

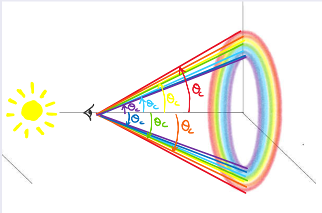

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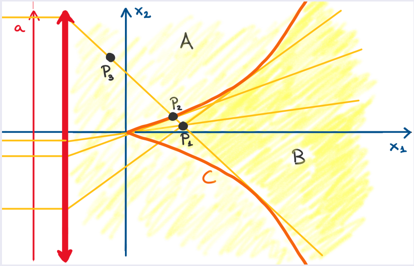
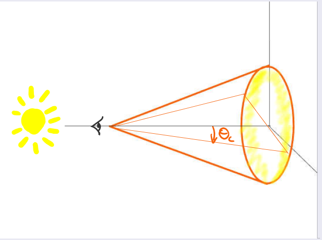




Key points here

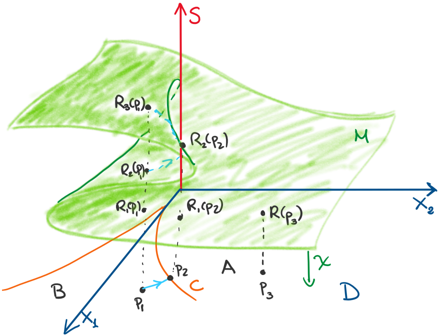
- Only observable: **intensity** of light.
- Caustic **"interior" higher intensity** of light.
- Intensity grows towards the caustic.
- Intensity diverges at the caustic.

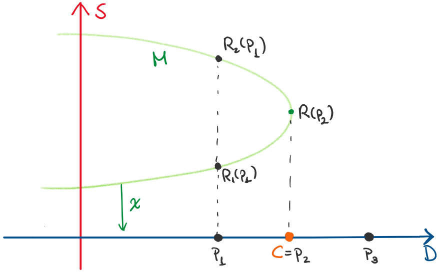
1. "Geometric optics": caustics in singularity theory

Caustics in a nutshell (catastrophe theory approach)

- **State space** S: $\boldsymbol{R} = (R_1, \ldots, R_m)$. **Control space** D: $\boldsymbol{X} = (X_1, \ldots, X_n)$.
- **Generating function**: $\Phi : S \times D \to \mathbb{R}$.
$$\Phi(\boldsymbol{R}, \boldsymbol{X}) = \sum_{\alpha=1}^{m} R_\alpha X_\alpha + f(\boldsymbol{R}; X_{m+1}, \ldots, X_n)$$
- **Stationary, solution or catastrophe manifold** M (n-dimensional):
$$\nabla_{R_\alpha}\Phi = 0 \quad , \quad \alpha \in \{1, \ldots, m\}, \qquad X_\alpha = -\nabla_{R_\alpha} f \quad , \quad \alpha \in \{1, \ldots, m\}$$
- **Magnification matrix** $\mathcal{M}_{\alpha\beta}(\boldsymbol{X})$:
$$\mathcal{M}_{\alpha\beta}(\boldsymbol{X}) = \left(\frac{\partial X_\alpha}{\partial R_\beta}\right)^{-1}_{(X_{m+1}, \ldots, X_n)} (\boldsymbol{R}(\boldsymbol{X})) = -\left(\frac{\partial^2 \Phi}{\partial R_\alpha \partial R_\beta}\right)^{-1}_{(X_1, \ldots, X_n)} (\boldsymbol{R}(\boldsymbol{X}))$$
- **Intensity** I:
$$I(\boldsymbol{X}) = \sum_i |\mathrm{det}\mathcal{M}_{\alpha\beta}|(\boldsymbol{R}_i(\boldsymbol{X}))$$
- **Caustic** C:
set in D where $|\mathrm{det}\mathcal{M}_{\alpha\beta}|$ diverges or where $\left(\frac{\partial^2\Phi}{\partial R_\alpha \partial R_\beta}\right)^{-1} (\boldsymbol{R}(\boldsymbol{X}))$ has a kernel.
More precisely:
Given the projection $\pi : M \to D$, C is the image (by π) of the critical values of π (i.e. where $d\pi$ has a kernel).

1. "Geometric optics": caustics in singularity theory





$$\Phi_{\text{lens}}(a; x_1, x_2) = \frac{(a-x_2)^2\omega}{2fx_1c}\left(f - x_1 + \frac{a^2x_1}{2s^2}\right)$$

$$\Phi_{\text{rainbow}}(\delta;\Delta)_{\text{lens}} = -\frac{1}{3}\alpha\beta\delta^3 + \beta\delta\Delta$$

where $\delta = b - b_c$ and $\Delta = D - D_c$

1. "Geometric optics": universality of caustics

Elementary (stable) caustics (Arnol'd-Thom theorem)

Co-dimension	Caustic name	Generating function $\phi(\boldsymbol{r}, \boldsymbol{x})$
1	Fold	$r_1 x_1 + \frac{1}{3} r_1^3$
2	Cusp	$r_1 x_1 + \frac{1}{2} x_2 r_1^2 + \frac{1}{4} r_1^4$
3	Swallow tail	$r_1 x_1 + \frac{1}{2} x_2 r_1^2 + \frac{1}{3} x_3 r_1^3 + \frac{1}{5} r_1^5$
3	Elliptic umbilic	$-r_1 x_1 - r_2 x_2 - x_3 (r_1^2 + r_2^2) - 3 r_1 r_2^2 + r_1^3$
3	Hyperbolic umbilic	$-r_1 x_1 - r_2 x_2 + x_3 r_1 r_2 + r_1^3 + r_2^3$

"Elementary (stable) catastrophes" classifying, in particular, structurally stable caustics with co-dimension $n \leq 3$.

More state variables can be introduced in each case, but they "irrelevant".

The fold caustic

The only stable caustic with a **only one control variable** x_1 is **the fold**, and has only one ("irreducible") state variable r_1.

2. "Diffraction on caustics": stationary point approximation (WKB)

Between geometric optics and full wave theory: Fresnel diffraction

Fresnel representation of wave field, superposition of ray contributions [Duistermaat 74, Berry 86, Kravtsov & Orlov, Guillemin & Sternberg,...]

$$\psi(\boldsymbol{X}) = \left(\frac{k}{2\pi}\right)^{\frac{m}{2}} \int d^m\boldsymbol{R}\, e^{ik\Phi(\boldsymbol{R},\boldsymbol{X})} a(\boldsymbol{R},\boldsymbol{X})$$

with $k = 2\pi/\lambda$, $\Phi(\boldsymbol{R},\boldsymbol{X})$ generating function accounting for the phase delay, $a(\boldsymbol{R},\boldsymbol{X})$ slowly varying function in $(\boldsymbol{R},\boldsymbol{X})$.

Diffraction on non-degenerate stationary points

For $k \gg 1$, stationary phase point evaluation

$$\psi(\boldsymbol{X}) \sim \sum_i \frac{e^{i(kS_i(\boldsymbol{X})+\alpha_i\pi/4)}}{|\det(\mathcal{M}^{-1}_{\alpha\beta}(\boldsymbol{X}))|^{\frac{1}{2}}} a(\boldsymbol{R}_i(\boldsymbol{X}),\boldsymbol{X})$$

2. "Diffraction on caustics": universal diffraction patterns on caustics

Uniform approximation: "elementary diffraction caustics" [Berry 76, Berry & Upstill, Kravtsov & Orlov... ; Torres et al.22]

- The stationary point approximation fails at the caustics, where stationary points merge.
- A "uniform asymptotic approximation", depending on the topological features of stationary point, reduces the evaluation to canonical $\phi(\boldsymbol{r}, \boldsymbol{x})$

$$\boxed{j(\boldsymbol{x}) = \left(\frac{k}{2\pi}\right)^{\frac{m}{2}} \int d^m\boldsymbol{r}\; e^{ik\phi(\boldsymbol{r},\boldsymbol{x})}}$$

Universal diffraction patterns on caustics: Arnol'd-Thom theorem

Topological equivalence of generating func. Φ and canonical ϕ [e.g. Poston & Stewart]

$$\Phi(\boldsymbol{R}, \boldsymbol{X}) = C(\boldsymbol{X}) + \phi(\boldsymbol{r}(\boldsymbol{R}), \boldsymbol{x}(\boldsymbol{X}))$$

Wave field $\psi(\boldsymbol{X})$ in terms of the elementary $j(\boldsymbol{x})$ and its derivatives

$$\psi(\boldsymbol{X}) = e^{ikC(\boldsymbol{X})} \left(g_0(\boldsymbol{x}(\boldsymbol{X}))j(\boldsymbol{x}(\boldsymbol{X})) + \frac{g_1(\boldsymbol{x}(\boldsymbol{X}))}{ik} \cdot \nabla_{\boldsymbol{x}} j(\boldsymbol{x}(\boldsymbol{X})) \right)$$

Interference regularizes the caustic divergence through universal diffraction patterns that "clothe the skeleton provided by stable caustics". [Berry 76]

2. "Diffraction on caustics": the fold caustic and Airy

Diffraction catastrophe scaling laws (Arnol'd index)

Introducing the k-independent function $J(\boldsymbol{x})$

$$J(\boldsymbol{x}) = \left(\frac{1}{2\pi}\right)^{\frac{m}{2}} \int d^m\boldsymbol{r}\ e^{i\phi(\boldsymbol{r},\boldsymbol{x})}$$

elementary $j(\boldsymbol{x})$ presents the following (self-similar) law: $\boxed{j(x) = k^\beta J(k^{\sigma_i} x_i)}$

The elementary fold integral caustic: the Airy function

For the fold caustic, $\phi = \frac{1}{3}r^3 + xr$, and from this

$$J(x) = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{-\infty}^{\infty} dr\ e^{ik(\frac{1}{3}r^3 + xr)} \sim \mathrm{Ai}(x) \quad \text{and} \quad j(x) = k^{\frac{1}{6}}(2\pi)^{\frac{1}{2}}\mathrm{Ai}(k^{\frac{2}{3}}x)$$

Then, for a Φ topologically equivalent to a fold (and making $x \to t$ and $k \to \omega$)

$$\boxed{h(t) := \psi(t) = e^{i\omega C(t)}\left(\omega^{\frac{1}{6}}\tilde{g}_0(t)\mathrm{Ai}(\omega^{\frac{2}{3}}\tilde{t}(t)) - i\omega^{-\frac{1}{6}}\tilde{g}_1(t)\mathrm{Ai}'(\omega^{\frac{2}{3}}\tilde{t}(t))\right)}$$

The wave field diffracted on a fold-caustic can be written as a modulation of the Airy function Ai and its derivative Ai'.

3. "Turning point equations": linear case

Beyond "geometric optics" limit $k \gg 1$: "asymptotic reasoning"

The tenet of asymptotic reasoning is that the asymptotic limits unveil the structural patterns. Here, the "geometric optics" limit identify the Airy function.
Claim: the Airy function pattern extends to lower frequencies.

Oscillation-to-damped transition: linear "turning points"

Airy as archetype of linear "turning point"

$$\frac{d^2u}{dt^2} - tu = 0$$

A more general linear ODEs describing, in an effective way, a turning point problem

$$\epsilon^2 \frac{d^2u}{d\tau^2} - \left(\tau\phi(\tau) - \epsilon\psi(\tau,\epsilon)\right) u = 0 \quad , \quad \phi(0) \neq 0$$

This equation can always be solved in terms of Ai and Ai'

$$\boxed{h(t) := u(t) = M(t)\left(P_1 \mathrm{Ai}(\phi(t)) + P_2(t)\mathrm{Ai}'(\phi(t))\right)}$$

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   - Simplicity, universality & 'linearity': hints for an Airy/Painlevé-II BBH model
   - A BBH waveform hierarchical program
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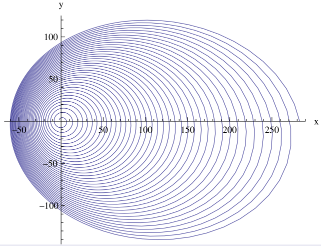
4. Perspectives

Painlevé-II BBH inspiral (EMRI) dynamics

Damped orbital motion of a charged particle in a Coulomb potential [Rajeev 08]

Landau-Lifshitz equations (in the non-relativistic limit) for a charged particle in a Coulomb potential **solved exactly in terms of the Painlevé-II equation**:

$$\frac{d^2L}{dz^2} = -\frac{1}{2k^2}L^3 - \frac{\tau}{k}zL$$



$$\boxed{\frac{d^2w}{dt^2} - tw - 2w^3 = 0}$$

(Painlevé-II, "non-linear Airy")

Painlevé-II BBH inspiral (EMRI) dynamics

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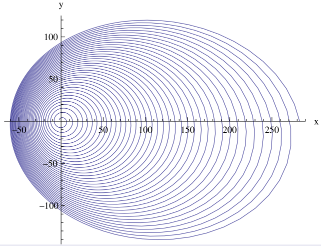


$$\boxed{\frac{d^2w}{dt^2} - tw - 2w^3 = 0}$$ (Painlevé-II, "non-linear Airy")

EMRI transition to plunge: Painlevé-I dynamics [Ori & Thorne 00; Compère and Küchler 21]

Dynamics $\frac{d^2X}{dT^2} = -X^2 - T$ solved by $\frac{d^2w}{dt^2} = 6w^2 + t$ (Painlevé-I)

Painlevé-II and BBH waveform

BBH dynamics and Painlevé-II

In the EMRI limit, part of the dynamics captured by so-called Painlevé transcendents.

Oscillation-to-damped transition: Painlevé-II non linear "turning points"

Non linear "turning points" equation, non-linear version of Airy:

$$\boxed{\frac{d^2w}{dt^2} - tw - 2w^3 = 0} \qquad \text{(Painlevé-II)}$$

Tantalizing proposal for the BBH waveform, $P_{II}(t) = w(t)$:

$$\boxed{h(t) = M(t)\left(P_1 \mathrm{P_{II}}(\phi(t)) + P_2(t)\mathrm{P}'_{\mathrm{II}}(\phi(t))\right)}$$

It does offer an avenue for the phase issue... **[BUT!]**

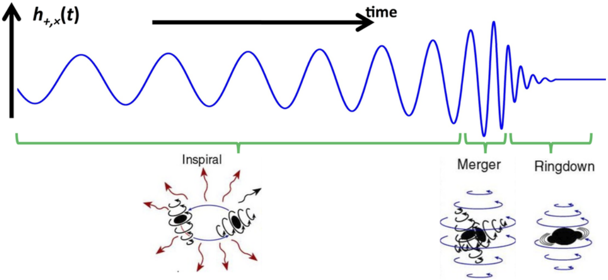
Painlevé transcendents and Integrability Theory

A smoking gun of an underlying **integrable structure**... and a path to PDEs.

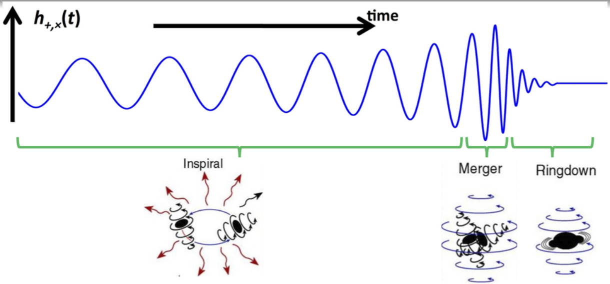
Universality from integrability

Airy-like model: restricted to inspiral-merger

- Caustic-diffraction mechanism: only valid for the **inspiral-merger** transition, at the "catastrophe/coalescence" point.
- For **ringdown**, need of another mechanism: perturbation of a finite object.



Universality from integrability



Hidden symmetries: Darboux covariance and KdV [Lenzi & Sopuerta]

- **“Vertical” QNM Isospectrality**, between axial/polar modes: **Darboux transformations**, role of **algebraically special QNMs**. ‘
- **“Horizontal” QNM Isospectrality**, along each axial/polar branch: Schrödinger equation and KdV conserved charges [Chandrasekhar, ...; Lenzi & Sopuerta]
- **Lax pair**, Virasoro, infinite PDE hierarchy...

Universality from integrability

Quasi-Normal Mode (QNM) ("formal") spectral problem

From equations on master functions Ψ, QNMs satisfy the spectral problem (under outgoing boundary condition) for the "Schrödinger" operator $\mathcal{L} = (\partial_x^2 - V)$:

$$\left(-\partial_t^2 + \partial_x^2 - V\right)\Psi = 0 \implies \mathcal{L}\Psi = (\partial_x^2 - V)\Psi = -\omega^2\Psi$$

Korteweg-de Vries (KdV) and QNM isospectrality

Deform V and Ψ according to:

$$\begin{aligned} \partial_\sigma V - 6V\partial_x V + \partial_x^3 V &= KdV[V] = 0 \\ \partial_\sigma \Psi = \left(-4\partial_x^3 + 6V\partial_x + 3V_{,x}\right)\Psi &= P\Psi \ . \end{aligned}$$

$\mathcal{L}$ and P form a "Lax pair": isospectrality (and integrability)

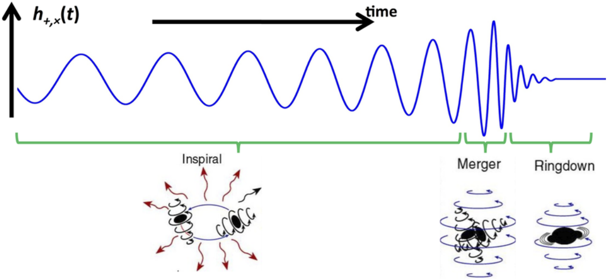
$$\partial_\sigma\mathcal{L} - [P, \mathcal{L}] = -\mathrm{KdV}[V]\cdot\mathrm{Id} = 0$$

$$\boxed{\partial_\sigma\mathcal{L} = [P, \mathcal{L}]} \qquad \text{(Lax pair equation)}$$

Universality from integrability

Guideline for passage from ODEs to PDEs: Painlevé test

"Painlevé test" for PDE integrability by inverse scattering: integrable PDE reducible by (self-similar) Ansatz to a "Painlevé transcendent", solution to one of the six Painlevé equations.



Idea to explore for BBH merger

BBH merger described by effective integrable PDE, admitting self-similar solutions satisfying Painlevé-II.

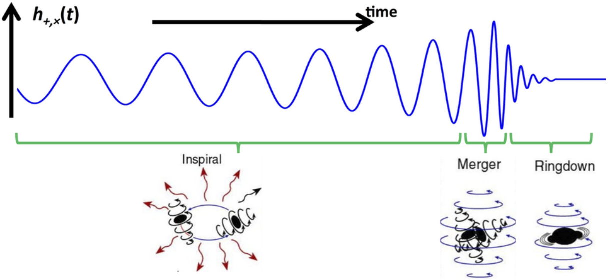
Universality from integrability

Painlevé-II: structural thread between pre-merger and post-merger BBH dynamics, passing through the merger

i) The orbital (in particular, EMRB-type) dynamics in the inspiral phase.

ii) The self-similar (part of the) solutions to KdV-like equations through the merger.

iii) The rationale to understand the KdV-related *hidden symmetries*.



Main ingredientes to retain

Main ingredients in our BBH problem: a brainstorm

- **Effective Dispersive character**: feature of Airy (Fourier transf.: $e^{\frac{i}{3}(2\pi k)^3}$). Consequence of (asymptotic) "filtering" degrees of freedom.
- **Effective linear dynamics**: linearization around an (effective) soliton-like metastable (instanton?) non-linear configuration?
  - *Effective dynamics are simple*: "inverse scattering transform" to reconstruct effective potential [Gelfand & Levitan 51, Marchenko 55].
    GLM equation, role in soliton theory [Gardner et al. 67, "KdV"].
- **Transition between two dynamical regimes (phases)**: possibility of approaching the "universality" features in terms of "critical phase transitions"?
  - *"Phase transitions" in PDE problems*: **universality** features (e.g. dispersive hydrodynamics).
- **Integrable systems**: universality classes contain a representative that is integrable. "Conserved quantities" explaining the simplicity of the waveform?
  - *Asymptotic reasoning*: filtering of "irrelevant" features for approximation: effective equations quasi-integrable?

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The Painlevé-II integrability thread to dispersive PDEs: KdV-like BBH models

"Wave-mean flow" approach

i) Fast degrees of freedom (Ψ): linear (wave) dynamics on a background.

ii) Slow degrees of freedom (u): non-linear integrable systems, constituting the background for the propagation of slow degree of freedom.

Integrability: non-linear background dynamics

Ingredients:

- Weakly dispersive.
- Weakly non-linear PDE dynamics.

Leading-order asymptotic equations "universally given" [Ablowitz] by Korteweg-De Vries equation:

$$\partial_t u - 6u\partial_x u + \partial^3_{xxx} u = 0$$

"Indeed, we now know that both the KdV and NLS equations are "universal" models. It can be shown that KdV-type equations arise whenever we have weakly dispersive and weakly nonlinear systems as the governing system." [Ablowitz 11]

Painlevé-II: from critical phenomena to Inverse Scattering

Universal wave patterns in dispersive (non-linear) PDEs: PDE critical phenomena

PDE-like critical phenomenon: profile of the PDE solution 'at the critical point' is independent of the initial data and given by a **universal special function** (cf. Dubrovin's(-like) conjectures).

BBH context: Painlevé-II as universal special functions, is the main motivation for an effective non-linear dispersive PDE BBH model

Painlevé test and the Inverse Scattering Transform (IST): Self-similarity and modified KdV

Modified-KdV (m-KdV), from KdV through Miura transformation: $u = \partial_x \tilde{u} + \tilde{u}^2$:

$$\tilde{u}_t - 6\tilde{u}^2 \partial_x \tilde{u} + \partial^3_{xxx} \tilde{u} = 0$$

"Self-similar" Ansatz $\tilde{u}(t,x) \sim \frac{1}{(3t^p)} u\left(\frac{x}{(3t)^q}\right)$ fixes $p = q = 1/3$ and reduces m-KdV to the Painlevé-II equation.

Painlevé-II: from critical phenomena to Inverse Scattering

Scattering problem

Consider the one-dimensional scattering problem:

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\phi(x)=k^2\phi(x)$$

with asymptotic conditions

$$\frac{\phi(x,k)}{a(k)}\sim\begin{cases}\frac{e^{-ikx}}{a(k)}\ , & (x\to-\infty)\\ e^{-ikx}+\frac{b(k)}{a(k)}e^{ikx} & (x\to\infty)\end{cases}$$

Transmission and reflection coefficients are given by

$$t(k)=\frac{1}{a(k)}\quad,\quad r(k)=\frac{b(k)}{a(k)}$$

Consider the possible presence of (finite number of) bound states:

$$\left(-\frac{d^2}{dx^2}+V(x)\right)\phi_n(x)=-\chi_n^2\phi_n(x)$$

with ϕ_n square-integrable.

Painlevé-II: from critical phenomena to Inverse Scattering

Inverse scattering: Gelfand-Levitan-Marchenko (GLM) algorithm

Scattering data: reflection coefficients and energy levels:

$$r(k), \{\chi_1, \ldots, \chi_N\}$$

GLM algorithm:

1. Set:

$$F(x) = \sum_{n=1}^{N} \frac{b_n e^{-\chi_n x}}{ia'(i\chi_n)} + \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{ikx} dk$$

2. GLM integral equation for $K(x, y)$:

$$K(x, y) + F(x + y) + \int_{x}^{\infty} K(x, z) F(z + y) dz = 0$$

3. Then:

$$V(x) = -2 \frac{d}{dx} K(x, x)$$

Application to the test-screen approach to the BBH problem?

Can an effective potential $V(x)$ (in some "appropriate coordinates") be retrieved from data h_{inn} an h_{out} on "inner" and "outer" screens? In which sense is $V(x)$ meaningful?

Effective Wave-Mean Flow: Scattering on solitons

Non-linear dispersive hydrodynamics effective picture: scattering on solitons

Effective separation of slow degrees of freedom $u(t,x)$:

$$\begin{cases} \left(-\Box + V_{\text{even,odd}}(t,x;u)\right)\Psi_{\text{even,odd}} = S_{\text{even,odd}}(t,x;u) \\ \partial_t u = F(t,x;u,u_x,u_{xx},\ldots) \end{cases}$$

- Integrability is only on slow DoF, u.
- Fast DoF subject to "catastrophe-caustic" PDE critical phenomena.
- Relying ultimate in "soliton resolution" picture [T. Tao, P. Bizoń...].

Effective Wave-Mean Flow: Scattering on solitons

Non-linear dispersive hydrodynamics effective picture: scattering on solitons

Separation of slow degrees of freedom $u(t,x)$:

$$u(t,x) \sim u_{\text{conservative}}(t,x) + u_{\text{dispersive}}(t,x) \sim u^{\text{potential}}_{\text{solitonic}}(t,x) + u^{P_{II}}_{\text{self}-\text{similar}}(t,x)$$

Then, the methodological approach is to address the problem as:

$$\begin{cases} \left(-\Box + V_{\text{even,odd}}(t,x;u^{\text{potential}}_{\text{solitonic}})\right)\Psi_{\text{even,odd}} = S_{\text{even,odd}}(t,x;u^{P_{II}}_{\text{self}-\text{similar}}) \\ \partial_t u = F(t,x;u,u_x,u_{xx},\ldots) \end{cases}$$

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Why this? Lesson from turbulence in geophysical fluids

- Turbulence appears in full simulations Navier-Stokes/Boussinesq.
- Understanding of turbulence out of reach.
- Asymptotic models by filtering DoF: **Quasi-Geostrophic equations**.
- Turbulence (inverse) cascades understood by conservation of "enstrophy" in the asymptotic effective theory: asymptotic reasoning.

Moral: focus on the phenomenon and filter the non-relevant degrees of freedom.

Slow dynamics in the BBH setting: the stationary pool

Dual frame: corrotating and non-rotating frame

Need of choosing coordinates: not a covariant formulation.

- Fast degrees of freedom Ψ: described in a non-rotating (with respect to infinity) coordinate system.
- Slow degrees of freedom u: described in a corrotation system

"Near-zone" stationary pool: stationarity independent of boundary conditions [Whelan, Price...] (ellucidate the role of light cylinder).

Bottom-up and Top-down stationary dynamics

- Bottom-up: Disperssive non-linear (hydro-like) PDEs (KdV et al...)
- Top-Down: Hellical Killing vector and "adiabatics":
  - i) Two commuting Killings: Ernst equation, sigma model.
  - ii) One hellical Killing: Gravity with matter given by $SL(2,\mathbb{R})/SO(1,1)$ sigma model [Klein 04].

Full (Conformal) Einstein equations: semi-linear system with "Wave-Mean Flow" structure [Friedrich...; Frauendiener...; Valiente-Kroon; here: Frauendiener, Stevens & Thwala 25]

Subsystem 1, "Slow" degrees of freedom: transport equations

$$e_a(c_b^\mu) - e_b(c_a^\mu) = \hat{\Gamma}_{ab}{}^c c_c^\mu - \hat{\Gamma}_{ba}{}^c c_c^\mu,$$

$$e_a(\hat{\Gamma}_{bc}{}^d) - e_b(\hat{\Gamma}_{ac}{}^d) = (\hat{\Gamma}_{ab}{}^e - \hat{\Gamma}_{ba}{}^e)\hat{\Gamma}_{ec}{}^d - \hat{\Gamma}_{bc}{}^e\hat{\Gamma}_{ae}{}^d + \hat{\Gamma}_{ac}{}^e\hat{\Gamma}_{be}{}^d$$

$$+\Theta K_{abc}{}^d - 2\eta_{c[a}\hat{P}_{b]}{}^d + 2\delta_{[a}{}^d\hat{P}_{b]c} - 2\hat{P}_{[ab]}\delta_c{}^d,$$

$$\hat{\nabla}_a\hat{P}_{bc} - \hat{\nabla}_b\hat{P}_{ac} = b_e K_{abc}{}^e,$$

Subsystem 2, "Fast" degrees of freedom: symmetric hyperbolic system

$$\hat{\nabla}_e K_{abc}{}^e = b_e K_{abc}{}^e,$$

Wittgenstein's ladder in action

Forget the rest of the talk and just solve "Subsystem 1" + "Subsystem 2", with "Subsystem 1" enforced to be (quasi-)integrable.

Integrability and Ward's conjecture: BBH scattering over self-dual backgrounds (instantons and twistors)

"First-principles" separation of (slow) background and (fast) dynamics?

Ward's conjecture: integrability and self-duality

Conjecture: fundamental relation between integrable (or solvable) differential equation systems and reductions of (anti-)self-dual Yang-Mills equations.
In GR setting:

- Connection between self-dual Einstein vacuum equations and the self-dual Yang-Mills equations [Mason& Newman; Woodhouse; Dunajski...].
- Ansatz: **'Backgrounds' in the (anti-)self-dual sector of GR.** [Compare approach by Mason et al.]

Scattering on instantons: self-dual sector controlled by "twistor tools"

Gravitational (anti-)self-dual solutions, either complex or real in Euclidean GR:

- The notion of 'background' needs to be relaxed/generalised.
- Idea: *BBH merger waveforms addressed in terms of an appropriate notion of classical scattering on a (anti-)self-dual gravitational instanton.*

Scheme

1. The problem: simplicity and universality of BBH waveforms
   - Simplicity, universality & 'linearity': hints for an Airy/Painlevé-II BBH model
   - A BBH waveform hierarchical program
2. Hyperboloidal approach to scattering: a zeroth-order "Wave-Mean Flow"
   - Cauchy formulation: isospectrality and Lax pairs
   - Hyperboloidal framework: "bulk" and "boundary" sectors
   - "Fast DoFs": non-normal hyperboloidal dynamics
   - "Slow DoFs": the hyperboloidal "Lax-pair" problem
3. Heuristics of BBH dynamics: an "asymptotic" fAiry tale approach
   - Universal diffraction patterns on caustics: from rainbows to Airy
   - Universality from integrability: the Painlevé-II thread to BBH dynamics
   - A "Wave-Mean Flow" approach: scattering on integrable backgrounds
4. Perspectives

Asymptotic reasoning: a hierarchical approach to BBH

"Boring or Elegant?": Actually, perhaps both...

Universality because:

- **"Nothing happens" (boring)**: Airy as universal behaviour through a "caustic" ("fast DoF").
- **Something very special (and elegant) happens**: integrability extending through the whole BBH dynamics (Painlevé-II...), but critically at the transition merger-ringdown ("slow DoF").

Asymptotic reasoning: universal patterns and degrees of freedom

- **Phenomenological application**: Airy BBH waveforms.
- **Fundamental degrees of freedom in Gravity:**
  - **Bottom-up**: hydrodynamic flavour, multiscale expansions, self-dual sector.
  - **Top-down**: start from the full theory and "filter" (multiscale asymptotic expansion"), using fundamental structures to "zoom".
    Reduce full Einstein equations to effective (quasi-)integrable PDE system (KdV-like?) by appropriate (asymptotic) approximations in the BBH problem.

Asymptotic reasoning: a hierarchical approach to BBH

"Asymptotic reasoning circularity"

Bottom-up:

Asymptotic reasoning $\longrightarrow$ Universal patterns $\longrightarrow$ Fundamental structures,

Top-down:

Universal patterns $\longleftarrow$ Asymptotic reasoning $\longleftarrow$ Fundamental structures.

Asymptotic reasoning: a hierarchical approach to BBH

"Asymptotic reasoning circularity"

Bottom-up:

Asymptotic reasoning $\longrightarrow$ Universal patterns $\longrightarrow$ Fundamental structures,

Top-down:

Universal patterns $\longleftarrow$ Asymptotic reasoning $\longleftarrow$ Fundamental structures.

Asymptotic reasoning and fundamental structures

"To understand these things
is to have a pattern and a model,
and to understand the pattern and the model
is mysterious power."

Lao Tzu, Tao Te Ching
("translation"/adaptation by Ursula K. Le Guin)

Towards a top-down approach to BBH merger waveforms

Integrability and Fundamental Gravity: a brainstorm (**revisiting elegance...**)

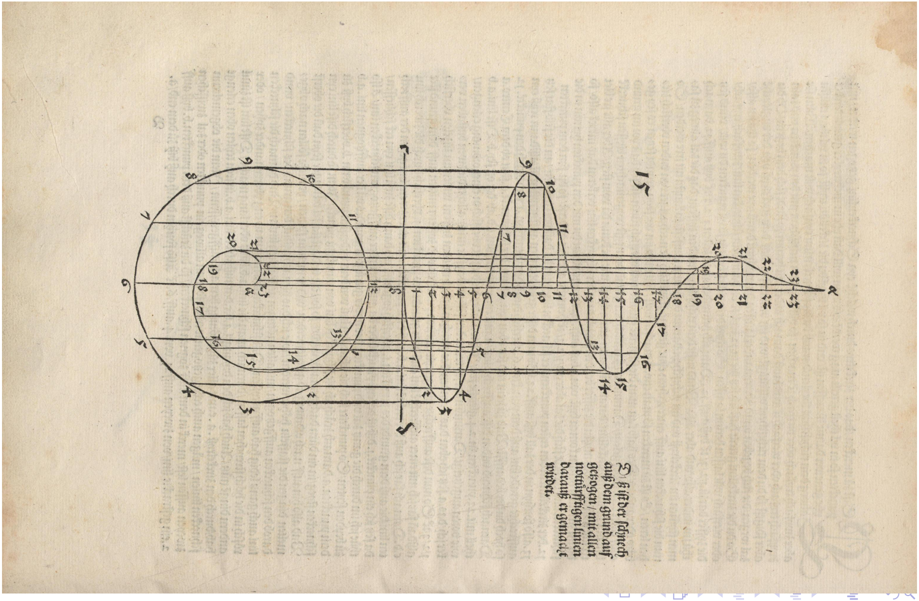
- Gravitational wave detection as a caustic in time.
- Caustic diffraction patterns and Airy function BBH waveform model.
- Painlevé-II turning point model and EMRIs. Painlevé-II-thread though all BBH dynamics ("hidden symmetry").
- "Wave-Mean Flow" methodology:
  - **Non-linear dispersive PDE integrability, universal waveforms and scattering on solitons ('soliton resolution conjecture').**
  - **Conformal Einstein equations as semi-linear PDE system, enforcing integrable "mean flow" sector.**
  - **Scattering in non-linear (integrable) self-dual backgrounds (twistors).**
- Current Problem: Lax pair in the hyperboloidal linear scattering picture.

Ablowitz and Segur (1981):

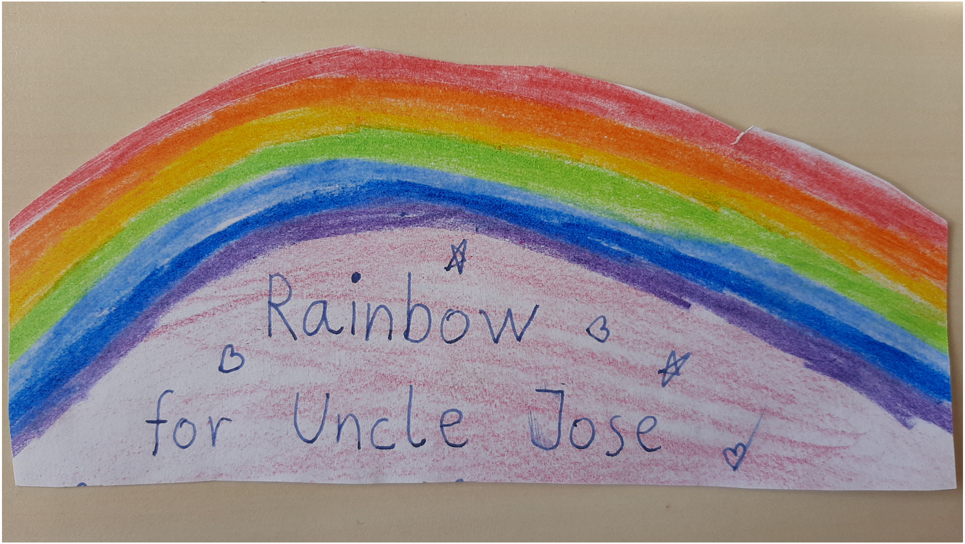
"Certain nonlinear problems have a surprisingly simple underlying structure, and can be solved by essentially linear methods".

Application to simplicity and universality in BBH dynamics?

“Golden” (self-similar) spirals, Fairies and Rainbows



"Golden" (self-similar) spirals, Fairies and Rainbows



“Golden” (self-similar) spirals, Fairies and Rainbows

"Golden" (self-similar) spirals, Fairies and Rainbows

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