Third law of black hole mechanics for supersymmetric black holes

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A. McSharry & HSR (to appear)

Black hole mechanics

In 1973, Bardeen, Carter and Hawking formulated the laws of black hole mechanics.

Zeroth law. The surface gravity κ of a stationary (time-independent) black hole is constant across the event horizon.

First law. Linear perturbations of a stationary black hole obey $dM = \frac{\kappa}{8\pi} dA + \Omega_H dJ$

Second law. The horizon area A is a non-decreasing function of time.

The remarkable similarity to the corresponding laws of thermodynamics is explained by Hawking's discovery that black holes emit thermal radiation at temperature $T_H = \frac{\hbar\kappa}{2\pi}$ and so have entropy $S_{BH} = A/4\hbar$.

Third law

Nernst's "unattainability" statement of the **third law of thermodynamics**:

It is impossible for any procedure, no matter how idealized, to reduce the temperature of a system to absolute zero in a finite number of operations.

Bardeen-Carter-Hawking's (unproved) statement of the **third law of black hole mechanics**:

It is impossible by any procedure, no matter how idealized, to reduce κ *to zero by a finite sequence of operations.*

Israel's proof

It is impossible by any procedure, no matter how idealized, to reduce κ to zero by a finite sequence of operations.

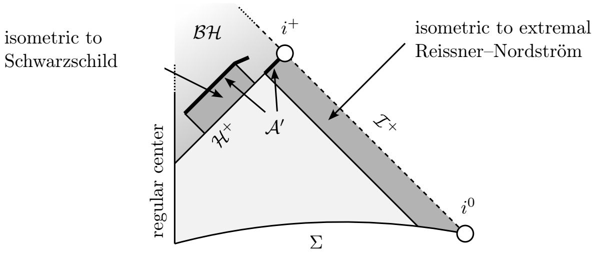
Israel (1986): "finite sequence of operators" should mean "in finite advanced time", i.e., finite time for an observer near the horizon.

1970's: spherical gravitational collapse of a thin shell of charged matter onto a non-extremal black hole can produce an exactly extremal Reissner-Nordström black hole in finite time, in violation of third law.

In such examples, the apparent horizon jumps outwards discontinuously when crossed by the shell. However, Israel argued that the third law holds if the matter is sufficiently smooth (and obeys weak energy condition), and presented a proof.

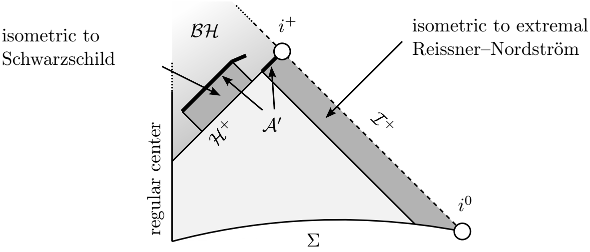
Kehle-Unger third-law violating solutions (2022)

Einstein-Maxwell theory coupled to a massless charged scalar field. Spherically symmetric gravitational collapse of the scalar field can result in formation of an exactly extremal Reissner-Nordström black hole in finite advanced time, with an intermediate phase in which the spacetime is exactly Schwarzschild at the horizon:



(image credit: Kehle and Unger)

Kehle-Unger third-law violating solutions (2022)

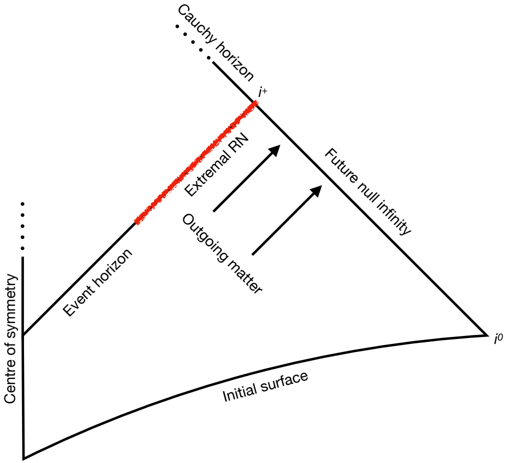


(image credit: Kehle and Unger)

The apparent horizon is discontinuous. This is why Israel's proof fails. The third law does not hold for the Einstein-Maxwell massless charged scalar theory.

Solutions non-extremal at spatial infinity

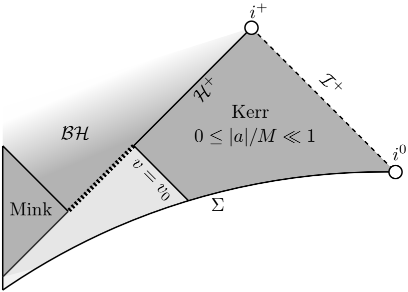
The solutions of Kehle and Unger are extremal RN all the way out to spatial infinity. Very likely that there exist other solutions that are extremal RN on the horizon but non-extremal everywhere outside the horizon:



Rotating black holes

Kehle and Unger conjecture that it should be possible to form a black hole that is exactly *extremal Kerr* after a finite advanced time, starting from regular *vacuum* initial data (gravitational collapse of gravitational waves).

Kehle and Unger (2023) prove that regular vacuum initial data can give a spacetime that is exactly a slowly-rotating ($|a| \ll M$) non-extremal Kerr black hole after a finite advanced time:



(image credit: Kehle and Unger)

Bounded charge to mass ratio

The third-law violating solutions of Kehle and Unger involve a massless charged scalar field: matter with large charge to mass ratio.

What happens if the charge to mass ratio of matter is bounded?

Local mass-charge inequality

We'll consider Einstein-Maxwell theory coupled to matter satisfying the local mass-charge inequality of Gibbons & Hull (1981):

$$T_{00}^{(m)} \geq \sqrt{T_{0i}^{(m)} T_{0i}^{(m)} + J_0^2 + \tilde{J}_0^2}$$

where indices $(0, i)$ refer to an arbitrary orthonormal frame, $T_{ab}^{(m)}$ is the energy-momentum tensor of matter (excluding the Maxwell field) and J_a, $\tilde{J}_a$ are the electric and magnetic currents of matter.

This is a strengthened version of the dominant energy condition.

For a scalar field of mass m and charge q it is equivalent to $m \geq |q|$.

Global mass-charge inequality

If matter satisfies the local mass-charge inequality then a strengthened version of the positive mass theorem applies.

If Σ is a complete asymptotically flat hypersurface then the ADM mass M and electric and magnetic charges Q, P measured at spatial infinity satisfy the "BPS bound" (Gibbons & Hull 1981)

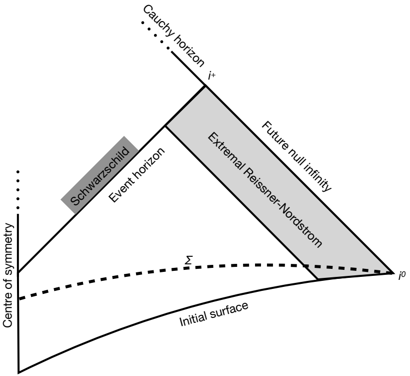
$$M \geq \sqrt{Q^2 + P^2}$$

The proof involves spinors. If the inequality is saturated then the spacetime is said to be supersymmetric and there exists a "supercovariantly constant" spinor ϵ:

$$\hat{\nabla}_a \epsilon \equiv \nabla_a \epsilon + \frac{1}{4} F_{bc} \gamma^b \gamma^c \gamma_a \epsilon = 0$$

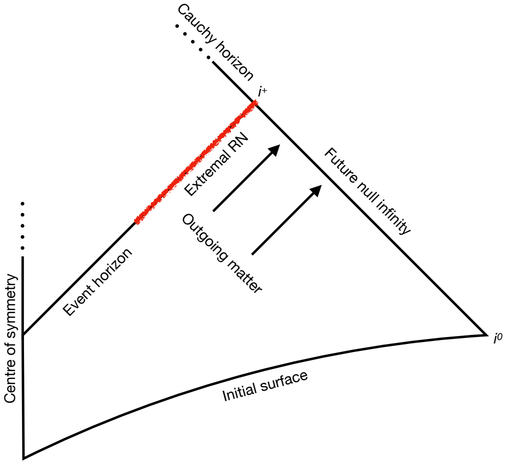
(The terminology comes from $N = 2$ supergravity but we don't need to assume that the theory is supersymmetric.)

This immediately excludes spacetimes of the form constructed by Kehle and Unger:



Spacetime has $M = |Q|$ (and $P = 0$) so saturates BPS bound. Hence there exists a supercovariantly constant spinor on Σ. But from ϵ we can construct a causal vector $X^a \equiv \bar{\epsilon}\gamma^a\epsilon$ which is *Killing*. So spacetime is time-independent on (arbitrary) Σ: contradiction!

This argument excludes third-law violating spacetimes that are exactly extremal RN near spatial infinity. But what about third-law violating spacetimes that have $M > \sqrt{Q^2 + P^2}$ at spatial infinity but, after finite advanced time, are exactly extremal RN at the horizon?



Idea: can we generalize the BPS bound to a situation where M, Q, P are defined *at the horizon*, and somehow argue as above to exclude extremal RN at the horizon?

For an arbitrary closed 2-surface S we can define the charges enclosed by S in the usual way:

$$Q = \frac{1}{4\pi}\int_S \star F \qquad\qquad P = \frac{1}{4\pi}\int_S F$$

but defining M is the problem of defining a *quasi-local mass* for S.

Dougan-Mason mass

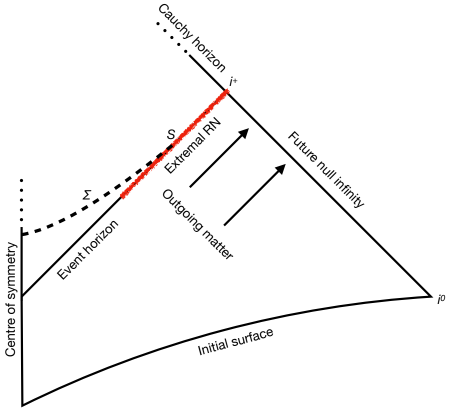
We're looking for a definition of quasi-local mass for which we can use spinors to prove a quasi-local BPS bound.

In 1991, Dougan & Mason presented a spinorial definition of quasi-local mass M for a closed 2-surface S. Assuming (i) $S = \partial\Sigma$ where Σ is a compact spacelike surface and (ii) the dominant energy condition is satisfied on Σ, they proved $M \geq 0$. Later work (Szabados 1993) showed that if $M = 0$ then the spacetime in $D(\Sigma)$ (the domain of dependence of Σ) must admit a covariantly constant spinor, and therefore must be flat or a pp-wave.

Our approach is essentially to take the work of Dougan & Mason and modify it in the same way that Gibbons & Hull modified the earlier positive energy theorem of Witten (1981), i.e., "replace covariant derivatives with supercovariant derivatives."

Compact interior

We are considering the possibility of forming an extremal RN black hole in gravitational collapse. In such a situation, the black hole would have *compact interior*, i.e., a horizon cross-section S would have $S = \partial\Sigma$ for a compact spacelike surface Σ. (The maximal analytic extension of extremal RN does *not* have compact interior because of the singularity at $r = 0$.)



The main results: third law

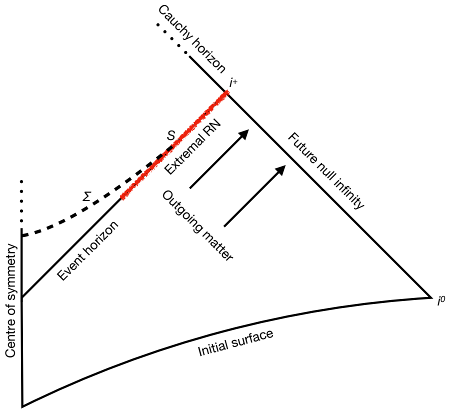
A *supersymmetric surface* is a 2-surface S such that on S we have a non-trivial solution of $t^a \hat{\nabla}_a \epsilon = 0$ where t^a is any vector tangent to S and $\hat{\nabla}$ is the supercovariant derivative.

Examples: (a) Any 2-surface in the extremal RN spacetime. (b) A 2-surface S in a non-supersymmetric spacetime for which there exists a local diffeo mapping the metric, extrinsic curvature, and Maxwell field on S to the corresponding quantities on a 2-surface in extremal RN.

Theorem. Let S be a supersymmetric surface with $S = \partial\Sigma$ where Σ is a compact spacelike surface. Assume that matter satisfied the local mass-charge inequality on Σ, and that S is *marginally trapped*. Then the electric and magnetic charges on Σ must vanish, i.e., S has $Q = P = 0$.

Proof: combine Gibbons-Hull with Dougan-Mason and Tod...

Corollary (third law). If matter satisfies the local mass-charge inequality and S has the same metric, Maxwell field and extrinsic curvature as a horizon cross-section of extremal RN then one cannot write $S = \partial\Sigma$ with Σ a compact spacelike surface, i.e., S does not have compact interior.



The main results: quasi-local BPS bound

For a symmetry 2-sphere in the RN spacetime (not necessarily extremal), the Dougan-Mason quasi-local mass differs from the ADM mass. I show how to define a modification ϖ of the Dougan-Mason quasi-local mass such that (i) ϖ agrees with the ADM mass in RN; (ii) a quasi-local BPS inequality holds:

Theorem. Let $S = \partial\Sigma$ where Σ is a compact spacelike surface. Assume that matter satisfied the local mass-charge inequality on Σ and that the ingoing null geodesics normal to S are converging. Then

$$\varpi \geq \sqrt{Q^2 + P^2}$$

with equality only if there exists a supercovariantly constant spinor on Σ and the local mass-charge inequality is saturated on Σ.

Generalisation (with McSharry)

I've shown that if matter satisfies the local mass-charge inequality then an extremal RN black hole cannot form in gravitational collapse.

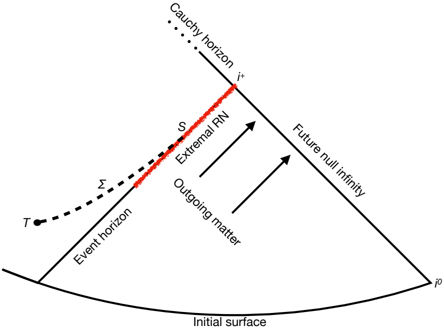
Hence a non-extremal black hole cannot become extremal if the initial black hole was formed from collapse.

What if the initial black hole was not formed in collapse e.g. if we start with a a *two-sided* non-extremal black hole? Could we make it extremal by throwing in charged matter?

Can adapt work of Gibbons, Hawking, Horowitz & Perry (1982) to cover this situation.

Israel: "non-extremal" means "there exists a trapped surface"

Theorem. Let Σ be a compact spacelike surface with $\partial\Sigma = S \cup T$ where T is a trapped surface. Assume that matter satisfied the local mass-charge inequality on Σ. Then S cannot be a marginally trapped supersymmetric surface. In other words, this is impossible:



Anti-de Sitter black holes (with McSharry)

$d = 4$ Einstein-Maxwell theory with negative cosmological constant and charged matter satisfying local mass-charge inequality.

Supercovariant derivative is now the one used in *gauged* $N = 2$ supergravity.

Extremal RN is not supersymmetric but there exists a 1-parameter family of supersymmetric (and extremal) Kerr-Newman-AdS black holes $M = M(Q)$, $J = J(Q)$, $P = 0$ (Kostalecky-Perry 1995) and also static supersymmetric black holes with magnetic charge and horizons of genus > 1 (Caldarelli-Klemm 1998).

Our results can be adapted to prove that the third law holds for these classes of black holes.

Discussion

The third law can be violated for Einstein-Maxwell theory coupled to matter with a large charge to mass ratio.

I've proved that the third law holds for *supersymmetric* black holes if matter satisfies the local mass-charge inequality. This covers extremal Reissner-Norstrom (if $\Lambda = 0$) or supersymmetric Kerr-Newman-AdS (if $\Lambda < 0$).

I think this should generalize to supersymmetric black holes for various other theories in various dimensions.

It is possible that a black hole could approach a supersymmetric black hole *asymptotically*, i.e., at infinite advanced time. (Such solutions exist with an uncharged scalar field (Murata-Tanahashi-HSR 2012).)

Is the result sharp? If the local mass-charge inequality is violated (in a sensible matter model) then do there exist third law violating solutions? (Kehle student: yes, for massive charged dust)

Kehle and Unger (2024) prove that (when third law violated) extremal RN is a "critical solution", i.e., in the moduli space of solutions it lies on the boundary between solutions that collapse to black holes and solutions that disperse. The boundary also contains more familiar Christodoulou/Choptuik naked singularities.

What about supersymmetric black holes in theories satisfying the local mass charge inequality? Maybe there are critical solutions that are asymptotically supersymmetric.

Kehle and Unger conjecture that extremal Kerr can be formed in finite advanced time in collapse of vacuum initial data. If correct then this implies that extremal Kerr will violate third law even in theories where extremal RN does not, i.e., third law violated by non-susy black holes but not by susy holes.

A different version of the third law asserts that entropy should vanish at zero temperature. Violated classically but recent work (Iliesu, Turiaci, ...) suggests that quantum effects might enforce this version of the third law for non-susy holes but not for susy ones, i.e., the opposite of the above situation!