Cosmological Consequences of Unconstrained Gravity

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Moriond Electroweak,
2025

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Shadow Matter

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Pre(r)amble

Classical Equations of Motion

$$S = \int dt\, L(q, \dot{q})$$

$$\frac{\delta S}{\delta q} = 0$$

Hamiltonian can be build from: $p \equiv \dfrac{\delta L}{\delta \dot{q}}$

$$H(p, q) = p\dot{q} - L\,|_{\dot{q}(q,p)}$$

Classical: Ehrenfest

$$x \equiv \langle \hat{x} \rangle \qquad \left[\hat{x}, \hat{p}\right] = i$$

From Schrödinger's Eq

for $H = \dfrac{p^2}{2m} + V(x)$

$$\partial_t x = i\,[H, x] = \frac{\partial H}{\partial p} \qquad \partial_t\, x = \frac{p}{m}$$

$$\partial_t p = i\left[H, p\right] = -\frac{\partial H}{\partial x} \qquad \partial_t p = -\,\partial_x V(x)$$

Choose a Hamiltonian to get the classical equations you want

Electromagnetism

EM

classical: $$S = \int d^4x \mathscr{L} = \int d^4x \left[-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - A_\mu J^\mu + \cdots \right]$$

Gauss' Law $$\frac{\delta S}{\delta A_0} = \underbrace{\partial_\mu F^{\mu 0}}_{\nabla \cdot E} - J^0 = 0$$

Ampere's Law $$\frac{\delta S}{\delta A_i} = \underbrace{\partial_\mu F^{\mu i}}_{\dot{E} + \nabla \times B} - J^i = 0$$

EM

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conjugate momenta:

$$\pi^i \equiv \frac{\delta \mathscr{L}}{\delta \dot{A}_i} = \partial^0 A^i - \partial^i A^0 \equiv - E^i$$

$$\pi^0 \equiv \frac{\delta \mathscr{L}}{\delta \dot{A}_0} = 0$$

EM

Hamiltonian: $$H = \int d^3x \mathscr{H}$$

$$\mathscr{H} = \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}(\nabla \times \mathbf{A})^2 - A_0(\nabla \cdot E - J^0) + \mathbf{A} \cdot \mathbf{J} + \cdots$$

EM

Hamiltonian: $$H = \int d^3x \mathscr{H}$$

$$\mathscr{H} = \frac{1}{2}\mathbf{E}^2 + \frac{1}{2}(\nabla \times \mathbf{A})^2 - A_0(\nabla \cdot E - J^0) + \mathbf{A} \cdot \mathbf{J} + \cdots$$

A_0 can be chosen due to gauge freedom.

Simple choice: $A_0 = 0$ (Weyl gauge)

Commutators canonical: $$\left[E^i(\mathbf{x}), A_j(\mathbf{y})\right] = i\,\delta^i_j\,\delta(\mathbf{x} - \mathbf{y})$$

EM

From the Schrödinger Eq:

$$\partial_t \langle \mathbf{A} \rangle = i \langle \left[H, \mathbf{A} \right] \rangle = \langle \mathbf{E} \rangle$$

$$\partial_t \langle \mathbf{E} \rangle = i \langle \left[H, \mathbf{E} \right] \rangle = \langle \nabla \times \mathbf{B} + \mathbf{J} \rangle$$ Ampere's Law

Gauss' Law? $G \equiv \nabla \cdot \mathbf{E} - J^0$

EM

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$$\partial_t \langle \mathbf{A} \rangle = i \langle [H, \mathbf{A}] \rangle = \langle \mathbf{E} \rangle$$

$$\partial_t \langle \mathbf{E} \rangle = i \langle [H, \mathbf{E}] \rangle = \langle \nabla \times \mathbf{B} + \mathbf{J} \rangle$$

Ampere's Law

Gauss' Law? $G \equiv \nabla \cdot \mathbf{E} - J^0$

Note: $[H, G] = 0$

Can require: $G \,|\, \psi \rangle_{\text{phys}} = 0$

EM

Gauss' Law — Can require:

$$G \equiv \nabla \cdot \mathbf{E} - J^0$$

$$G\,|\,\psi\rangle_{\rm phys} = 0$$

EM

Gauss' Law — Can require:

$$G \equiv \nabla \cdot \mathbf{E} - J^0$$

$$G\,|\,\psi\rangle_{\text{phys}} = 0$$

Could instead require: $G\,|\,\psi\rangle_{\text{phys}} = \rho_{\text{sh}}(\mathbf{x})\,|\,\psi\rangle_{\text{phys}}$

Or even: $\langle\psi\,|\,G\,|\,\psi\rangle = 0$ or $\rho_{\text{sh}}(\mathbf{x})$

EM

Gauss' Law — Can require:

$$G \equiv \nabla \cdot \mathbf{E} - J^0$$

$$G\,|\,\psi\rangle_{\rm phys} = 0$$

Could instead require: $G\,|\,\psi\rangle_{\rm phys} = \rho_{\rm sh}(\mathbf{x})\,|\,\psi\rangle_{\rm phys}$

Or even: $\langle\psi\,|\,G\,|\,\psi\rangle = 0$ or $\rho_{\rm sh}(\mathbf{x})$

like a static charge density b.g.

$$\langle\, \nabla \cdot \mathbf{E} - J^0 - \rho_{\rm sh}(\mathbf{x})\rangle = 0$$

equivalent physics to infinite mass charge distribution

General Relativity

GR

$$S = \int d^4x \, \underbrace{(\sqrt{-g}\; M_{\rm pl}^2 R \;+\; \sqrt{-g}\; \mathscr{L}_{\rm matter}(\phi))}_{\mathscr{L}_g} \;+\; S_{GHY}$$

classical:

$$\frac{\delta S}{\delta g_{\mu\nu}} = \sqrt{-g}\; \left(M_{\rm pl}^2 G^{\mu\nu} - T^{\mu\nu}\right) = 0$$

conjugates:

$$\frac{\delta \mathscr{L}_g}{\delta \dot{g}_{ij}} \equiv \pi^{ij} \qquad\qquad \frac{\delta \mathscr{L}_g}{\delta \dot{g}_{0\mu}} = 0$$

GR

$$S = \int d^4x\,\underbrace{(\sqrt{-g}\,M_{\rm pl}^2 R \;+\; \sqrt{-g}\,\mathscr{L}_{\rm matter}(\phi))}_{\mathscr{L}_g} \;+\; S_{GHY}$$

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conjugates:

$$\frac{\delta \mathscr{L}_g}{\delta \dot{g}_{ij}} \equiv \pi^{ij} \qquad\qquad \frac{\delta \mathscr{L}_g}{\delta \dot{g}_{0\mu}} = 0$$

construct Hamiltonian:

$$g_{0\mu} = -\,\delta_{0\mu} \qquad \gamma_{ij} \equiv g_{ij}$$

(synchronous gauge)

$$\left[\gamma_{ij}(\mathbf{x}), \pi^{kl}(\mathbf{y})\right] = i\,\delta^k_{(i}\delta^l_{j)}\,\delta(\mathbf{x}-\mathbf{y})$$

$$\mathscr{H} = \frac{1}{\sqrt{\gamma}}(\pi^{ij}\pi_{ij} - \frac{1}{2}\pi^2) - \sqrt{\gamma}\;{}^{(3)}R + \cdots$$

GR

EOM: $$\partial_t \langle \pi^{ij} \rangle \sim - \left\langle \frac{\delta H}{\delta \gamma_{ij}} \right\rangle \longrightarrow G^{ij} = 8\pi G T^{ij} \qquad \left(\frac{\delta S}{\delta g_{ij}} = 0 \right)$$

(under rug: coherent states, operator ordering)

Remaining equations constraints — impose on the initial states

$\langle \mathscr{H} \rangle = 0$ Hamiltonian $\longrightarrow \sqrt{-g}\left(M_{\rm pl}^2\, G^{00} - T^{00}\right) = 0$

$\langle \chi^i \rangle = 0$ Momentum $\longrightarrow \sqrt{-g}\left(M_{\rm pl}^2\, G^{0i} - T^{0i}\right) = 0$

GR

EOM: $$\partial_t \langle \pi^{ij} \rangle \sim -\left\langle \frac{\delta H}{\delta \gamma_{ij}} \right\rangle \longrightarrow G^{ij} = 8\pi G T^{ij} \qquad \left(\frac{\delta S}{\delta g_{ij}} = 0 \right)$$

(under rug: coherent states, operator ordering)

Remaining equations constraints — impose on the initial states

$$\langle \mathscr{H} \rangle = 0 \quad \text{Hamiltonian} \longrightarrow \sqrt{-g}\left(M_{\rm pl}^2\, G^{00} - T^{00}\right) = 0$$

$$\langle \chi^i \rangle = 0 \quad \text{Momentum} \longrightarrow \sqrt{-g}\left(M_{\rm pl}^2\, G^{0i} - T^{0i}\right) = 0$$

$$\langle \mathscr{H} \rangle = \mathbb{H} \longrightarrow \sqrt{-g}\left(M_{\rm pl}^2\, G^{00} - T^{00}\right) = \mathbb{H}$$

$$\langle \chi^i \rangle = \mathbb{P}^i \longrightarrow \sqrt{-g}\left(M_{\rm pl}^2\, G^{0i} - T^{0i}\right) = \mathbb{P}^i$$

GR

$$G^{00} = 8\pi G_N \left(T^{00} + \frac{\mathbb{H}}{\sqrt{-g}} \right)$$

$$G^{0i} = 8\pi G_N \left(T^{0i} + \frac{\mathbb{P}^i}{\sqrt{-g}} \right) \longrightarrow T^{\mu\nu}_{\rm sh} \equiv \frac{1}{\sqrt{-g}} \begin{pmatrix} \mathbb{H} & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3 \\ \mathbb{P}^1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 \end{pmatrix}$$

$$G^{ij} = 8\pi G_N T^{ij}$$

$\mathbb{H}, \mathbb{P}^i$ are constrained functions

(identity) $\nabla_\mu G^{\mu\nu} = 0$

(EOM) $\nabla_\mu T^{\mu\nu} = 0$

$$\longrightarrow \nabla_\mu T^{\mu\nu}_{\rm sh} = 0 \longrightarrow \begin{aligned} \partial_0 \mathbb{H} + \partial_i \mathbb{P}^i &= 0 \\ \partial_0 (g_{ij} \mathbb{P}^j) &= 0 \end{aligned}$$

GR

$$G^{\mu\nu} = 8\pi G_N (T^{\mu\nu} + T^{\mu\nu}_{\rm sh})$$

$$T^{\mu\nu}_{\rm sh} \equiv \frac{1}{\sqrt{-g}} \begin{pmatrix} \mathbb{H} & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3 \\ \mathbb{P}^1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 \end{pmatrix}$$

(synchronous gauge)

$$\partial_0 \mathbb{H} + \partial_i \mathbb{P}^i = 0$$

$$\partial_0 (g_{ij} \mathbb{P}^j) = 0$$

What do these source terms do?

look at limits

GR

$$\partial_0 \mathbb{H} + \partial_i \mathbb{P}^i = 0$$

$$\partial_0 (g_{ij} \mathbb{P}^j) = 0$$

limit: $\mathbb{P}^i = 0 \quad \rightarrow \partial_0 \mathbb{H} = 0$

$$G^{00} = 8\pi G \, (T^{00} + \frac{\mathbb{H}(\mathbf{x})}{\sqrt{-g}})$$

write as:

$$u^\mu = \{1, 0, 0, 0\} \qquad \rho_{\rm sh} = \mathbb{H}(\mathbf{x})/\sqrt{-g}$$

$$T^{\mu\nu}_{\rm sh} = \rho_{\rm sh} u^\mu u^\nu$$

Looks like pressure-less dust!

GR

if $\mathbb{H}(\mathbf{x}) > 0$ everywhere, then coordinate redefinition:

$$\frac{\mathbb{H}(\mathbf{x})}{\sqrt{-g}} \longrightarrow \frac{\overline{\mathbb{H}}}{\sqrt{-g}}$$

GR

if $\mathbb{H}(\mathbf{x}) > 0$ everywhere, then coordinate redefinition:

$$\frac{\mathbb{H}(\mathbf{x})}{\sqrt{-g}} \longrightarrow \frac{\overline{\mathbb{H}}}{\sqrt{-g}}$$

$$ds^2 = -dt^2 + a(t)^2(\delta_{ij} + h_{ij})dx^i dx^j \qquad h = h_{ij}\delta^{ij}$$

linear order: $$T^{00}_{\rm sh} = (\bar{\rho}_{\rm sh} + \delta\rho_{\rm sh}) = \frac{\overline{\mathbb{H}}}{\sqrt{-g}} \simeq \frac{\overline{\mathbb{H}}}{a^3}(1 - h/2)$$

or $\dot{\delta} = -\dot{h}/2$

Looks like pressure-less dust!

GR

limit: $\mathbb{H}(\mathbf{x}) = \overline{\mathbb{H}} + \delta\mathbb{H}(\mathbf{x})$ $\mathbb{P}^i \equiv \delta\mathbb{P}^i$

Perturbative expansion around homogeneity

GR

limit: $\mathbb{H}(\mathbf{x}) = \overline{\mathbb{H}} + \delta\mathbb{H}(\mathbf{x})$ $\mathbb{P}^i \equiv \delta\mathbb{P}^i$

Perturbative expansion around homogeneity

constraints:

$$\partial_0\mathbb{H} + \partial_i\mathbb{P}^i = 0$$

$$\partial_0(g_{ij}\mathbb{P}^j) = 0$$

at linear order: $\partial_0(a^2\,\delta\mathbb{P}^j) = 0$

$\delta\mathbb{P}^i \sim a^{-2}$ and $\delta\mathbb{P}^i/\sqrt{-g} \sim a^{-5}$

Redshift quickly away outside horizon

GR

general: $\mathbb{H}(x),\ \mathbb{P}^i(x)$

$$T^{\mu\nu}_{\rm sh} \equiv \frac{1}{\sqrt{-g}} \begin{pmatrix} \mathbb{H} & \mathbb{P}^1 & \mathbb{P}^2 & \mathbb{P}^3 \\ \mathbb{P}^1 & 0 & 0 & 0 \\ \mathbb{P}^2 & 0 & 0 & 0 \\ \mathbb{P}^3 & 0 & 0 & 0 \end{pmatrix}$$

$$q^\mu = (0, \mathbb{P}^1, \mathbb{P}^2, \mathbb{P}^3)/\sqrt{-g}$$

$$T^{\mu\nu}_{\rm sh} = \rho_{\rm sh}\, u^\mu u^\nu + q^\mu u^\nu + u^\mu q^\nu$$

q^μ produces a ‘heat flux’ that differs from normal particle dynamics
could appear in smaller structures

GR

Crazy things — $\mathbb{H}(x)$ could be negative in some places mimicking negative mass particles!

In the early universe, if $2\,|\,\mathbb{P}\,|\, > \mathbb{H}$, shadow matter still violates the NEC

Possibilities of wormholes or bouncing cosmology

GR + EM

$$D_\mu F^{\mu\nu} = (J^\nu + J^\nu_{\rm sh}) \qquad\qquad D_\mu J^\mu_{\rm sh} = 0$$

$$\sqrt{-g}\,J^\mu_{\rm sh} \equiv \{\mathbb{J}(\mathbf{x}), 0, 0, 0\}$$

↑

time independent

$$J^\mu_{\rm sh} = \rho^{ch}_{\rm sh} v^\mu \qquad\qquad v^\mu = \{1, 0, 0, 0\}$$

synchronous gauge

shadow charge density follows geodesics and does not respond to electromagnetic fields directly

Shadow Matter

Loosening constraints of GR allows for source terms that could explain why we think there is dark matter

New source terms for EM produce a charged component of the fake dark matter.
Could effect the CMB, BBN, galactic dynamics, and direct detection.
Challenging pheno (plasma dynamics)

New source terms could violate NEC with no microscopic instabilities.
New phenomena possible

If Shadow Matter is most or all of dark matter, it is in conflict with inflation.
Worth exploring new ideas for initial conditions.

Thank you!

Extras?

Non-dynamical d.o.f.

Simple example: $$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\hat{q}_2 - \hat{q}_1)$$

Non-dynamical d.o.f.

Simple example: $$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\hat{q}_2 - \hat{q}_1)$$

$$Q = \frac{q_1 + q_2}{2} \qquad q = q_2 - q_1$$

$$P = p_1 + p_2 \qquad p = \frac{p_2 - p_1}{2}$$

$$\rightarrow H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(q)$$

Non-dynamical d.o.f.

Simple example: $$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\hat{q}_2 - \hat{q}_1)$$

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$$\to H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(q)$$

$$[P, H] = 0 \qquad \hat{P}\,|\,P\rangle = P\,|\,P\rangle$$

Think of eigenstates of P as super-selection sectors

Non-dynamical d.o.f.

Simple example: $$\hat{H} = \frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\hat{q}_2 - \hat{q}_1)$$

$$Q = \frac{q_1 + q_2}{2} \qquad q = q_2 - q_1$$

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$$\rightarrow H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(q)$$

$$[P, H] = 0 \qquad \hat{P}\,|\,P\rangle = P\,|\,P\rangle$$

Think of eigenstates of P as super-selection sectors

Can choose

$$\hat{P}\,|\,\psi\rangle_{\text{phys}} = 0$$

$$\hat{P}\,|\,\psi\rangle_{\text{phys}} = P\,|\,\psi\rangle_{\text{phys}}$$

$$\langle\psi\,|\,\hat{P}\,|\,\psi\rangle_{\text{phys}} = P$$

Non-dynamical d.o.f.

Next example: $$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(q) + \frac{P}{M}\overline{V}(q)$$

Still, $[P, H] = 0$

Non-dynamical d.o.f.

Next example: $$H = \frac{P^2}{2M} + \frac{p^2}{2\mu} + V(q) + \frac{P}{M}\overline{V}(q)$$

Still, $[P, H] = 0$

$$\hat{P}\,|\,\psi\rangle_{\text{phys}} = 0$$

Again, can choose $$\hat{P}\,|\,\psi\rangle_{\text{phys}} = P\,|\,\psi\rangle_{\text{phys}}$$

$$\langle\psi\,|\,\hat{P}\,|\,\psi\rangle_{\text{phys}} = P$$

Measurement allows us to tell what super-selection sector we are in

Gravity (toy)

Gravity: minisuperspace

zero-mode only (FRW): $$ds^2 = -N(t)^2 dt^2 + a(t)^2 d\mathbf{x}^2$$

classical: $$S = \int d^4x \sqrt{-g}\,\Big(\, M_{\rm pl}^2 R \;+ \underbrace{\mathscr{L}_{\rm matter}(\phi)}_{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\cdots}\Big) \;+\; S_{GHY}$$

Gravity: minisuperspace

zero-mode only (FRW): $ds^2 = -N(t)^2 dt^2 + a(t)^2 d\mathbf{x}^2$

classical: $S = \int d^4x \sqrt{-g}\,\left(M_{\rm pl}^2 R + \underbrace{\mathscr{L}_{\rm matter}(\phi)}_{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\cdots} \right) + S_{GHY}$

$$\frac{\delta S}{\delta N} = a^3 \left(6M_{\rm pl}^2 \frac{\dot{a}^2}{N^2 a^2} - \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) = 0$$

1st Friedmann Eq

$$\frac{\delta S}{\delta a} = 3Na^2 \left(4M_{\rm pl}^2 \left(\frac{\ddot{a}}{N^2 a} + \frac{\dot{a}^2}{2N^2 a^2} - \frac{\dot{a}\dot{N}}{N^3 a} \right) + \frac{\dot{\phi}^2}{2N^2} - V(\phi) \right) = 0$$

2nd Friedmann Eq

$$\frac{\delta S}{\delta \phi} = -\frac{a^3}{N} \left(\ddot{\phi} + 3\frac{\dot{a}}{a}\dot{\phi} - \frac{\dot{N}\dot{\phi}}{N} + N^2 \frac{\partial V(\phi)}{\partial \phi} \right)$$

matter EOM

Gravity: minisuperspace

zero-mode only (FRW): $$ds^2 = -N(t)^2 dt^2 + a(t)^2 d\mathbf{x}^2$$

classical: $$S = \int d^4x \sqrt{-g}\,\Big(\, M_{\rm pl}^2 R \;+ \underbrace{\mathscr{L}_{\rm matter}(\phi)}_{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \cdots}\Big) \;+\; S_{GHY}$$

$$\left.\frac{\delta S}{\delta N}\right|_{N=1} = a^3\left(6M_{\rm pl}^2\frac{\dot a^2}{a^2} - \frac{1}{2}\dot\phi^2 - V(\phi)\right) = 0$$ 1st Friedmann Eq

$$\left.\frac{\delta S}{\delta a}\right|_{N=1} = 3a^2\left[4M_{\rm pl}^2\left(\frac{\ddot a}{a} + \frac{\dot a^2}{2a^2}\right) + \frac{1}{2}\dot\phi^2 - V(\phi)\right] = 0$$ 2nd Friedmann Eq

$$\left.\frac{\delta S}{\delta \phi}\right|_{N=1} = -\,a^3\left(\ddot\phi + 3\frac{\dot a}{a}\dot\phi + \frac{\partial V(\phi)}{\partial\phi}\right)$$ matter EOM

Gravity: minisuperspace

classical:
$$S = \int d^4x\sqrt{-g}\,(\;M_{\rm pl}^2 R \;+\; \underbrace{\mathscr{L}_{\rm m}(\phi)}_{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\cdots}\;)\;+\;S_{GHY}$$

conjugate momenta:

$$\pi_a \equiv \frac{\delta\mathscr{L}}{\delta\dot a} = -\,6M_{\rm pl}^2\frac{a\dot a}{N} \qquad \pi_N \equiv \frac{\delta\mathscr{L}}{\delta\dot N} = 0 \qquad \pi_\phi = \frac{\delta\mathscr{L}}{\delta\dot\phi} = \frac{a^3}{N}\dot\phi$$

Gravity: minisuperspace

classical: $$S = \int d^4x\sqrt{-g}\,(\;M_{\rm pl}^2 R \;+\; \underbrace{\mathscr{L}_{\rm m}(\phi)}_{g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi+\cdots}\;) + S_{GHY}$$

conjugate momenta:

$$\pi_a \equiv \frac{\delta\mathscr{L}}{\delta\dot{a}} = -6M_{\rm pl}^2\frac{a\dot{a}}{N} \qquad \pi_N \equiv \frac{\delta\mathscr{L}}{\delta\dot{N}} = 0 \qquad \pi_\phi = \frac{\delta\mathscr{L}}{\delta\dot{\phi}} = \frac{a^3}{N}\dot{\phi}$$

Hamiltonian:

$$H = \left[\pi\dot{a} + \pi_\phi\dot{\phi} - \mathscr{L}\right]_{\dot{a}=\cdots,\dot{\phi}=\cdots} = -\frac{N}{24M_{pl}^2 a}\pi^2 + \frac{N}{2a^3}\pi_\phi^2 + Na^3V(\phi)$$

$$= N\tilde{H}(\pi_a, a, \pi_\phi, \phi)$$

Gravity: minisuperspace

Schrödinger eq:

$$i\frac{d}{dt}|\psi\rangle = N(t)\,\tilde{H}\,|\psi\rangle \qquad \to i\frac{d}{N(t)dt}|\psi\rangle = \tilde{H}\,|\psi\rangle$$

N maintains time reparameterization invariance.

N is a parameter — Can pick it. Simple choice: $N = 1$

Gravity: minisuperspace

Schrödinger eq:

$$i\frac{d}{dt}|\psi\rangle = N(t)\,\tilde{H}\,|\psi\rangle \quad \rightarrow i\frac{d}{N(t)dt}|\psi\rangle = \tilde{H}\,|\psi\rangle$$

N maintains time reparameterization invariance.

N is a parameter — Can pick it. Simple choice: $N = 1$

$$\partial_t\langle a\rangle = i\langle[H,a]\rangle \rightarrow \langle\pi_a\rangle = \langle -6M_{\rm pl}a\dot{a}\rangle$$

(choosing operator ordering wisely)

$$\partial_t\langle\pi_a\rangle = i\langle\left[H,\pi_a\right]\rangle \rightarrow$$

$$\longrightarrow \frac{\ddot{a}}{a} + \frac{1}{2}\left(\frac{\dot{a}}{a}\right)^2 = \frac{1}{M_{\rm pl}^2}p$$

2nd Friedmann Eq

(replacing π_a's with $\dot{a}$'s, assuming classical states)

$$p = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

Gravity: minisuperspace

1st Friedmann Eq? From classical:

$$0 = \frac{\delta S}{\delta N} = \frac{\delta \int \pi_a \dot{a} + \pi_\phi \dot{\phi} - N\tilde{H}}{\delta N} \longrightarrow \tilde{H} = 0$$

The ‘Hamiltonian Constraint’ *is* the 1st Friedmann Eq

Standard treatment (like Gauss): $\tilde{H} \,|\, \psi\rangle_{\text{phys}} = 0$ (Wheeler-deWitt)

Gravity: minisuperspace

1st Friedmann Eq? From classical:

$$0 = \frac{\delta S}{\delta N} = \frac{\delta \int \pi_a \dot{a} + \pi_\phi \dot{\phi} - N\tilde{H}}{\delta N} \longrightarrow \tilde{H} = 0$$

The ‘Hamiltonian Constraint’ *is* the 1st Friedmann Eq

Standard treatment (like Gauss): $\tilde{H}\,|\psi\rangle_{\text{phys}} = 0$ (Wheeler-deWitt)

??? No Schrödinger equation ???

The “problem of time” in quantum gravity

Simple fix: $\langle\psi\,|\,H\,|\,\psi\rangle = 0$

Gravity: minisuperspace

1st Friedmann Eq:

$$\langle \psi | \tilde{H} | \psi \rangle = 0$$

For classical states:

$$\tilde{H} = a^3 \left(3M_{\rm pl}^2 \left(\frac{\dot{a}}{a} \right)^2 - \rho \right) = 0$$

$$\rho = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

Gravity: minisuperspace

Could choose:

$$\langle \psi | \tilde{H} | \psi \rangle = \mathbb{C}$$

$\mathbb{C}$ is constant as $[\tilde{H}, \tilde{H}] = 0$

For classical states:

$$\tilde{H} = a^3 \left(3M_{\rm pl}^2 \left(\frac{\dot{a}}{a} \right)^2 - \rho \right) = \mathbb{C}$$

$$\longrightarrow \left(\frac{\dot{a}}{a} \right)^2 = \frac{1}{3M_{\rm pl}^2} \left(\rho + \frac{\mathbb{C}}{a^3} \right)$$

behaves as another matter component

GR + EM

additional modification to Einstein's equations

$$T^{\mu\nu} = \mathscr{E}^{\mu\nu} + T^{\mu\nu}_{matter} \equiv F^{\mu\lambda}F^{\nu}_{\lambda} - \frac{1}{4}g^{\mu\nu}F^{\lambda\sigma}F_{\lambda\sigma} + T^{\mu\nu}_{matter}$$

$$\begin{aligned} \nabla_\mu T^{\mu\nu} &= \nabla_\mu \mathscr{E}^{\mu\nu} + \nabla_\mu T^{\mu\nu}_{matter} \\ &= F^{\nu}{}_{\lambda}(J^{\lambda} + J^{\lambda}_{\mathrm{sh}}) - F^{\nu}{}_{\lambda}J^{\lambda} \end{aligned}$$

modified constraints

$$\partial_0 \mathbb{H} + \partial_i \mathbb{P}^i = 0$$

$$\partial_0 (g_{ij}\mathbb{P}^j) = -F_{i0}\,\mathbb{J}$$