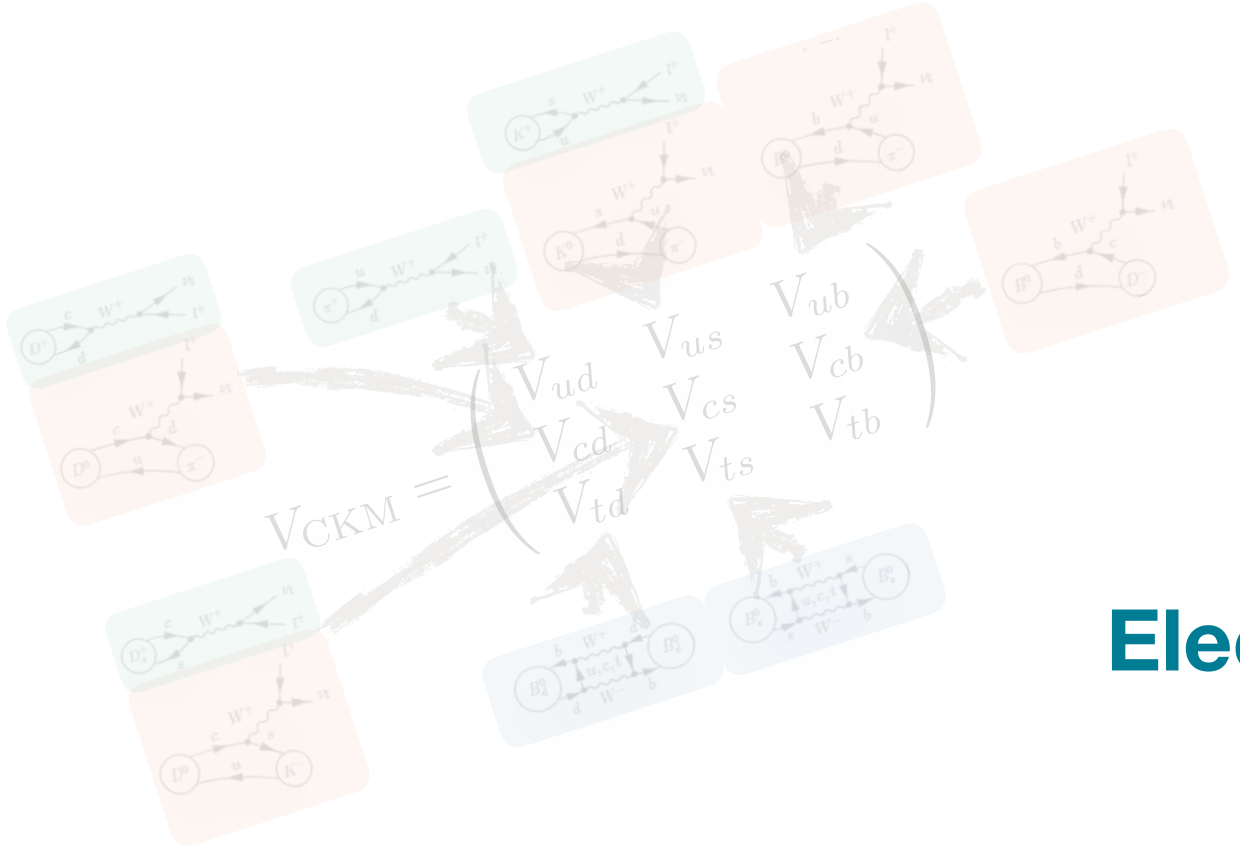
Exclusive or inclusive? The lattice might have the answer...

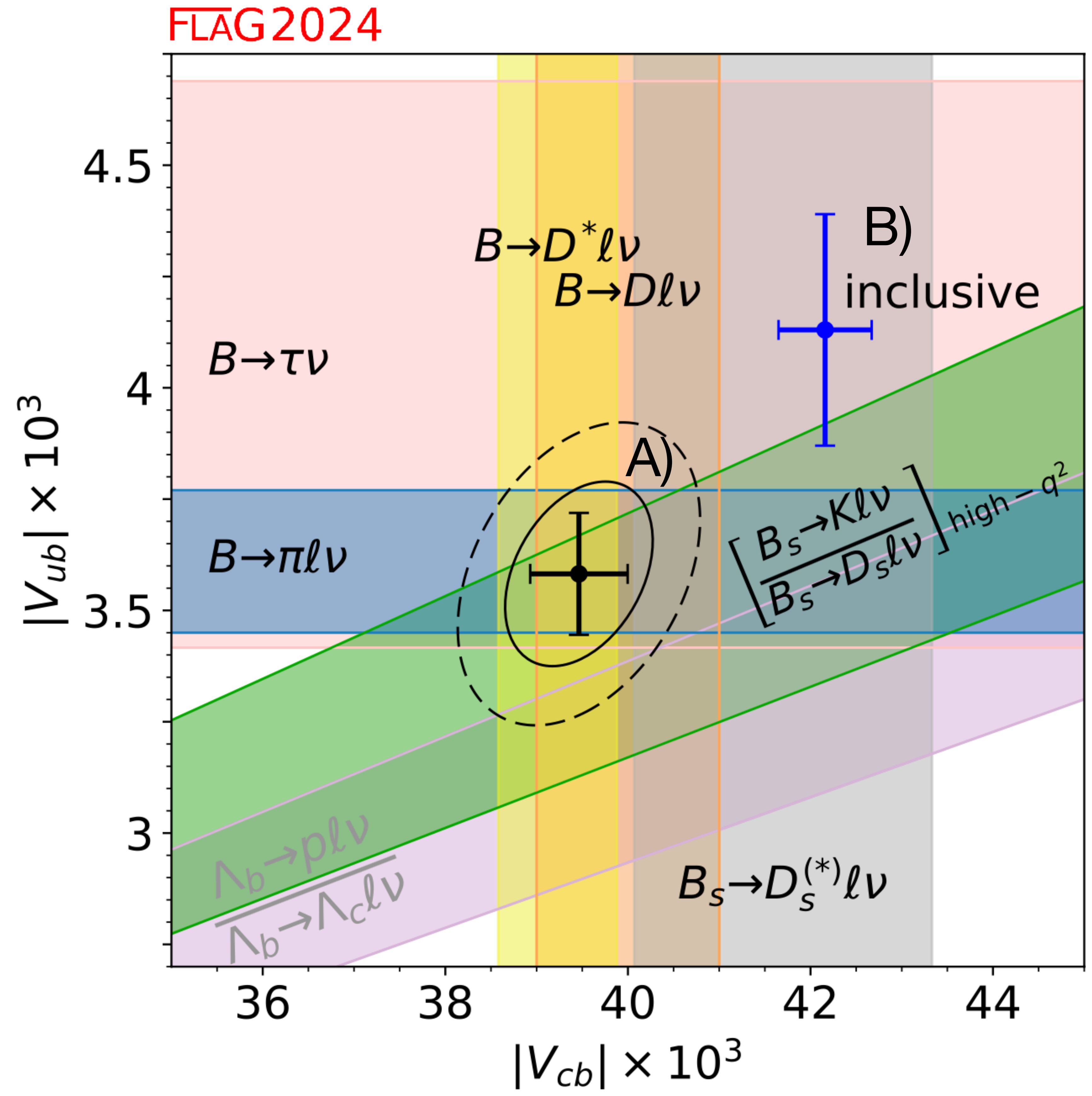


Electroweak Interactions & Unified Theories 2025

Andreas Jüttner

CERN, University of Southampton

CKM puzzle



$|V_{cb}|$ tension — ambiguous SM tests (e.g. ϵ_K, SM fits, …)

A) **lattice + experiment:** exclusive decay:

- new quality of experimental data
- new quality of lattice data
- new analysis techniques†

B) **continuum + experiment:** inclusive decay:

- existing determinations OPE based

†[Bernlochner et al. PRD 100 (2019) 1, 013005], [Bigi PLB 769 (2017) 441-445, JHEP 11 (2017) 061], [Bordone, AJ EPJC 85 (2025)], [Di Carlo et al. PRD 104 (2021) 5, 054502] [Fedele et al. PRD 108 (2023) 5, 5], [Flynn, AJ, Tsang JHEP 12 (2023)], [Gambino PLB 795 (2019) 386-390] [Martinelli et al. EPJC 85 (2025), EPJC 84 (2024) 4, 400, PRD 106 (2022) 9, 093002, EPJC 82 (2022) 12, PRD 105 (2022) 3, 034503, PRD 104 (2021) 9, 094512]

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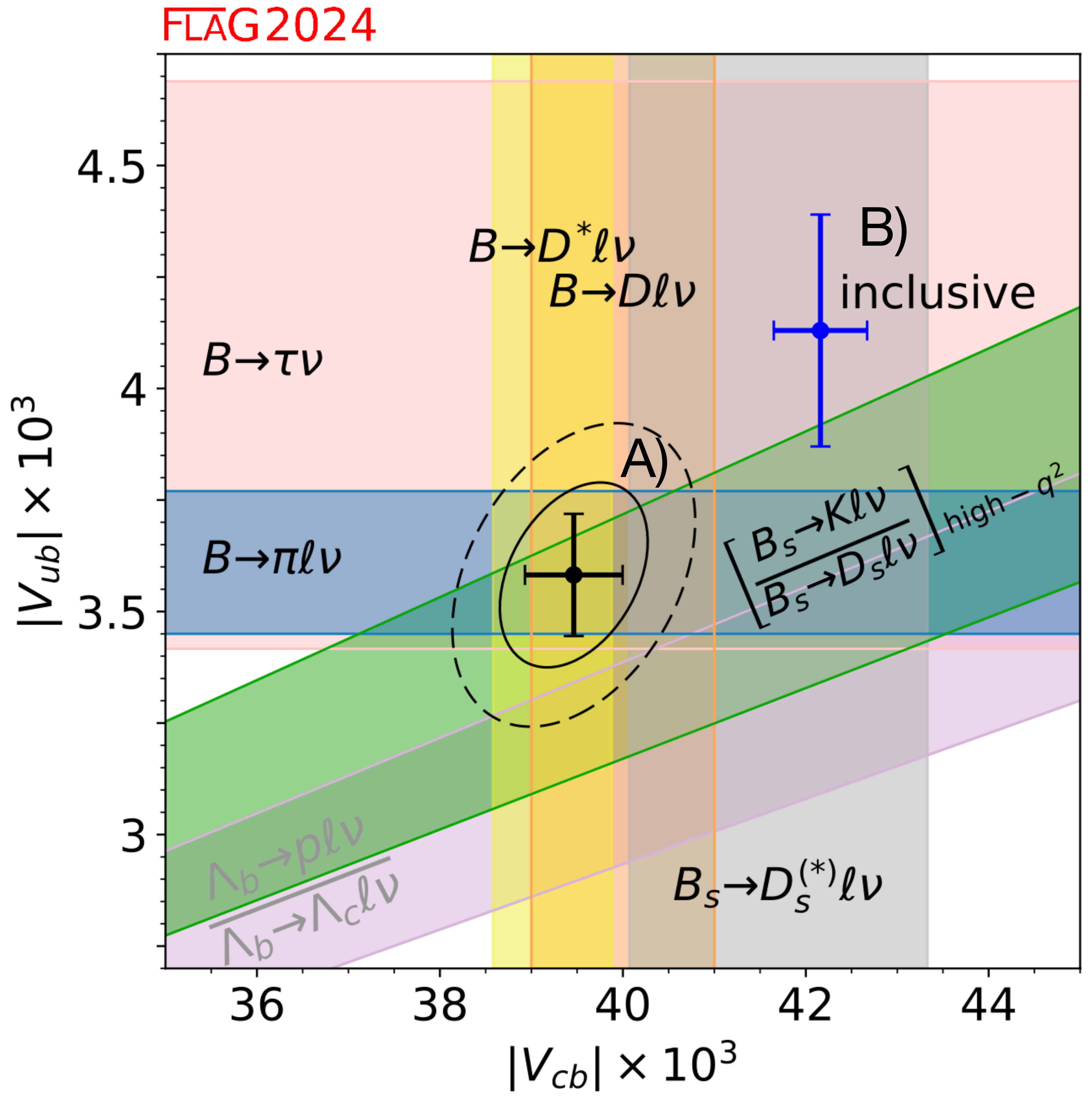

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This talk

C) **lattice + experiment:** inclusive decay $B \to X_c \ell \bar{\nu}_\ell$:

- new ideas making this calculation feasible

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Inclusive SL decays

Lattice is good at doing a few exclusive channels for stable hadrons

Inclusive SL decays

$B^0 \to X_c \ell^+ \nu_l$

Semileptonic and leptonic modes PDGlive

Γ_1	$\ell^+\nu_\ell X$	[1]	$(10.99 \pm 0.28)\%$		
Γ_2	$e^+\nu_e X_c$		$(10.8 \pm 0.4)\%$		
Γ_3	$D\ell^+\nu_\ell X$		$(9.7 \pm 0.7)\%$		
Γ_4	$\overline{D}^0\ell^+\nu_\ell$	[1]	$(2.35 \pm 0.09)\%$		2310
Γ_5	$\overline{D}^0\tau^+\nu_\tau$		$(7.7 \pm 2.5) \times 10^{-3}$		1911
Γ_6	$\overline{D}^*(2007)^0\ell^+\nu_\ell$	[1]	$(5.66 \pm 0.22)\%$		2258
Γ_7	$\overline{D}^*(2007)^0\tau^+\nu_\tau$		$(1.88 \pm 0.20)\%$	S=1.0	1839
Γ_8	$D^-\pi^+\ell^+\nu_\ell$		$(4.4 \pm 0.4) \times 10^{-3}$		2306
Γ_9	$\overline{D}_0^*(2420)^0\ell^+\nu_\ell$, $\overline{D}_0^{*0} \to D^-\pi^+$		$(2.5 \pm 0.5) \times 10^{-3}$		
Γ_{10}	$\overline{D}_2^*(2460)^0\ell^+\nu_\ell$, $\overline{D}_2^{*0} \to D^-\pi^+$		$(1.53 \pm 0.16) \times 10^{-3}$	S=1.0	2065
Γ_{11}	$D^{(*)}$ n $\pi\ell^+\nu_\ell$ (n $\geq$ 1)		$(1.88 \pm 0.25)\%$		
Γ_{12}	$D^{*-}\pi^+\ell^+\nu_\ell$		$(6.0 \pm 0.4) \times 10^{-3}$		2254
Γ_{13}	$\overline{D}_1(2420)^0\ell^+\nu_\ell$, $\overline{D}_1^0 \to D^{*-}\pi^+$		$(3.03 \pm 0.20) \times 10^{-3}$		2084
Γ_{14}	$\overline{D}_1'(2430)^0\ell^+\nu_\ell$, $\overline{D}_1'^0 \to D^{*-}\pi^+$		$(2.7 \pm 0.6) \times 10^{-3}$		
Γ_{15}	$\overline{D}_2^*(2460)^0\ell^+\nu_\ell$, $\overline{D}_2^{*0} \to D^{*-}\pi^+$		$(1.01 \pm 0.24) \times 10^{-3}$	S=2.0	2065
Γ_{16}	$\overline{D}^0\pi^+\pi^-\ell^+\nu_\ell$		$(1.7 \pm 0.4) \times 10^{-3}$		2301
Γ_{17}	$\overline{D}^{*0}\pi^+\pi^-\ell^+\nu_\ell$		$(8 \pm 5) \times 10^{-4}$		2248
Γ_{18}	$D_s^{(*)-}K^+\ell^+\nu_\ell$		$(6.1 \pm 1.0) \times 10^{-4}$		
Γ_{19}	$D_s^-K^+\ell^+\nu_\ell$		$(3.0^{+1.4}_{-1.2}) \times 10^{-4}$		2242
Γ_{20}	$D_s^{*-}K^+\ell^+\nu_\ell$		$(2.9 \pm 1.9) \times 10^{-4}$		2185
Γ_{21}	$\pi^0\ell^+\nu_\ell$		$(7.80 \pm 0.27) \times 10^{-5}$		2638
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Γ_{24}	$\eta'\ell^+\nu_\ell$		$(2.3 \pm 0.8) \times 10^{-5}$		2553
Γ_{25}	$\omega\ell^+\nu_\ell$	[1]	$(1.19 \pm 0.09) \times 10^{-4}$		2582
Γ_{26}	$\omega\mu^+\nu_\mu$				2581
Γ_{27}	$\rho^0\ell^+\nu_\ell$	[1]	$(1.58 \pm 0.11) \times 10^{-4}$		2583
Γ_{28}	$p\bar{p}\ell^+\nu_\ell$		$(5.8^{+2.6}_{-2.3}) \times 10^{-6}$		2467
Γ_{29}	$p\bar{p}\mu^+\nu_\mu$		$< 8.5 \times 10^{-6}$	CL=90%	2446
Γ_{30}	$p\bar{p}e^+\nu_e$		$(8.2^{+4.0}_{-3.3}) \times 10^{-6}$		2467
Γ_{31}	$e^+\nu_e$		$< 9.8 \times 10^{-7}$	CL=90%	2640
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But recently new ideas!

Hansen et al. (2017) PRD 96 094513 (2017)
Hashimoto PTEP 53-56 (2017)
Bailas et al. PTEP 43-50 (2020)
Gambino and Hashimoto PRL 125 32001 (2020)
Barone et al. JHEP 07 (2023) 145

Inclusive SL decay in the SM

hadronic tensor

$$\frac{d\Gamma}{dq^2 dq^0 dE_l} = \frac{G_F^2 |V_{qQ}|^2}{8\pi} L^{\mu\nu} W_{\mu\nu}$$

leptonic tensor

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leptonic tensor

$$\Gamma(B_s \to X_c l\nu) = \frac{G_F^2 |V_{cb}|^2}{24\pi^3} \int_0^{\mathbf{q}^2_{\max}} d\mathbf{q}^2 \sqrt{\mathbf{q}^2} \int_{\omega_{\min}}^{\omega_{\max}} d\omega \, W_{\mu\nu}(\omega, \mathbf{q}) k^{\mu\nu}(\omega, \mathbf{q})$$

$\bar{X}(\mathbf{q}^2)$

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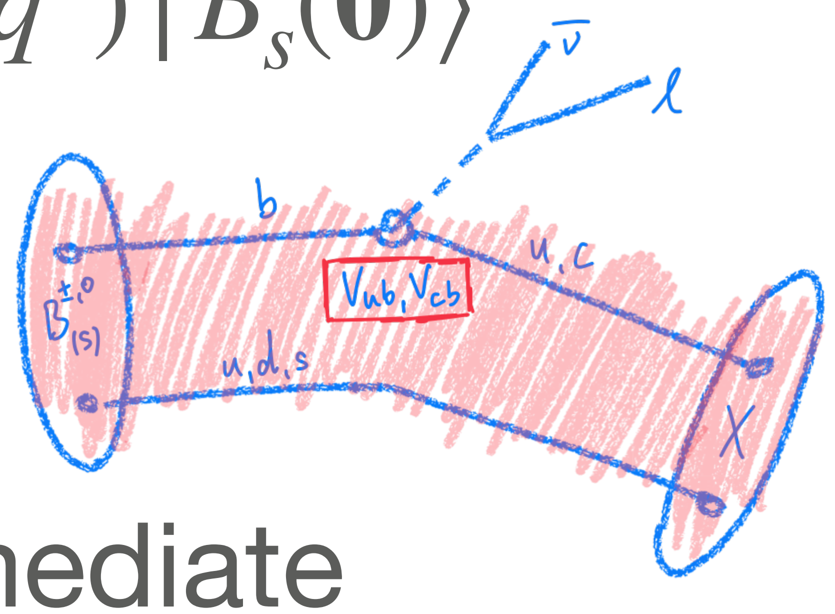
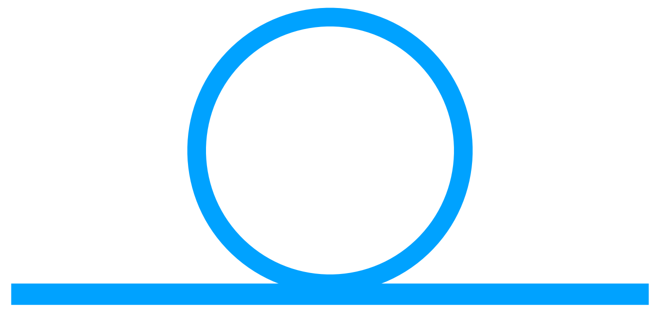
Let's consider the case $B_s \to X_c \ell \nu$ (B_s rest frame):

$$W^{\mu\nu}(p_{B_s}, q) = \frac{1}{2E_{B_s}} \sum_{X_c} (2\pi)^3 \delta^{(4)}(p_{B_s} - q - p_{X_c}) \langle B_s(\mathbf{0}) | (\tilde{J}^\mu(q^2))^\dagger | X_c(p_{X_c}) \rangle \langle X_c(p_{X_c}) | \tilde{J}^\nu(q^2) | B_s(\mathbf{0}) \rangle$$



We now have a lattice method for integrating inclusively over all intermediate states contributing to the corresponding spectral density

A regularisation…



dimensional regularisation, cutoff, mass-term, …, or

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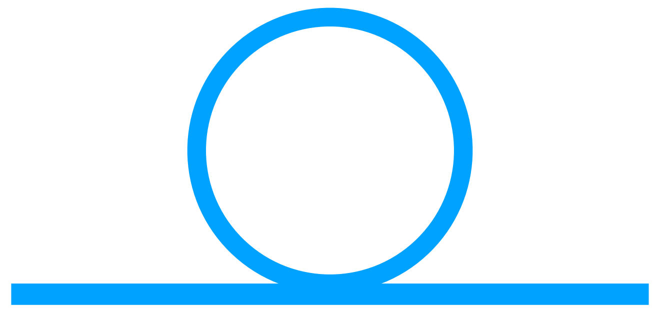

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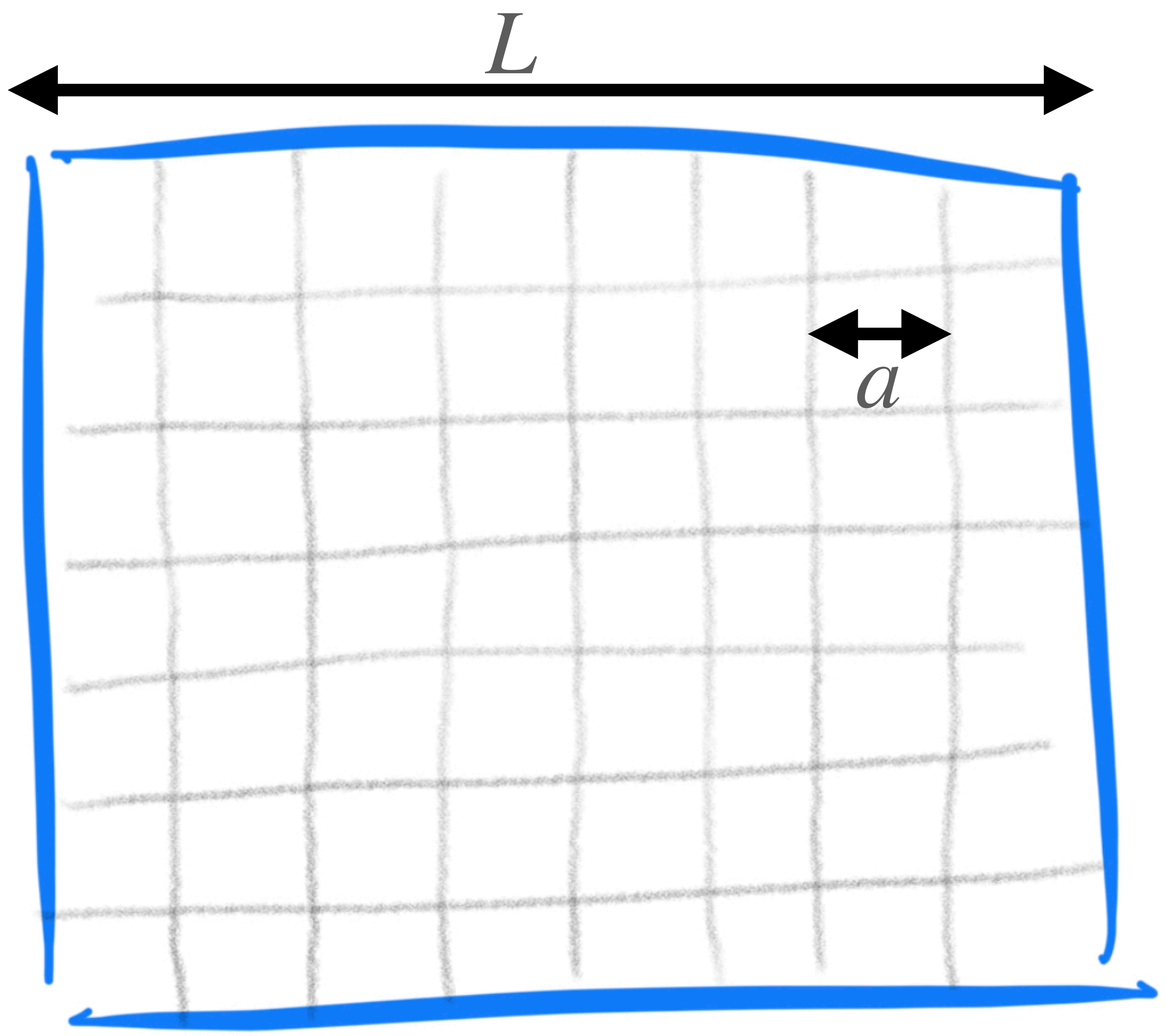

$$\frac{1}{L} \ll \Lambda_{\text{physics}} \ll \frac{1}{a}$$

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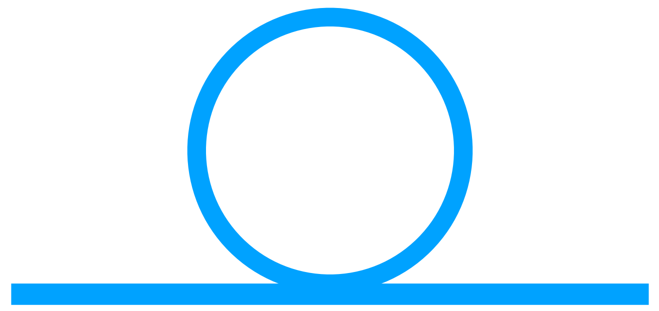

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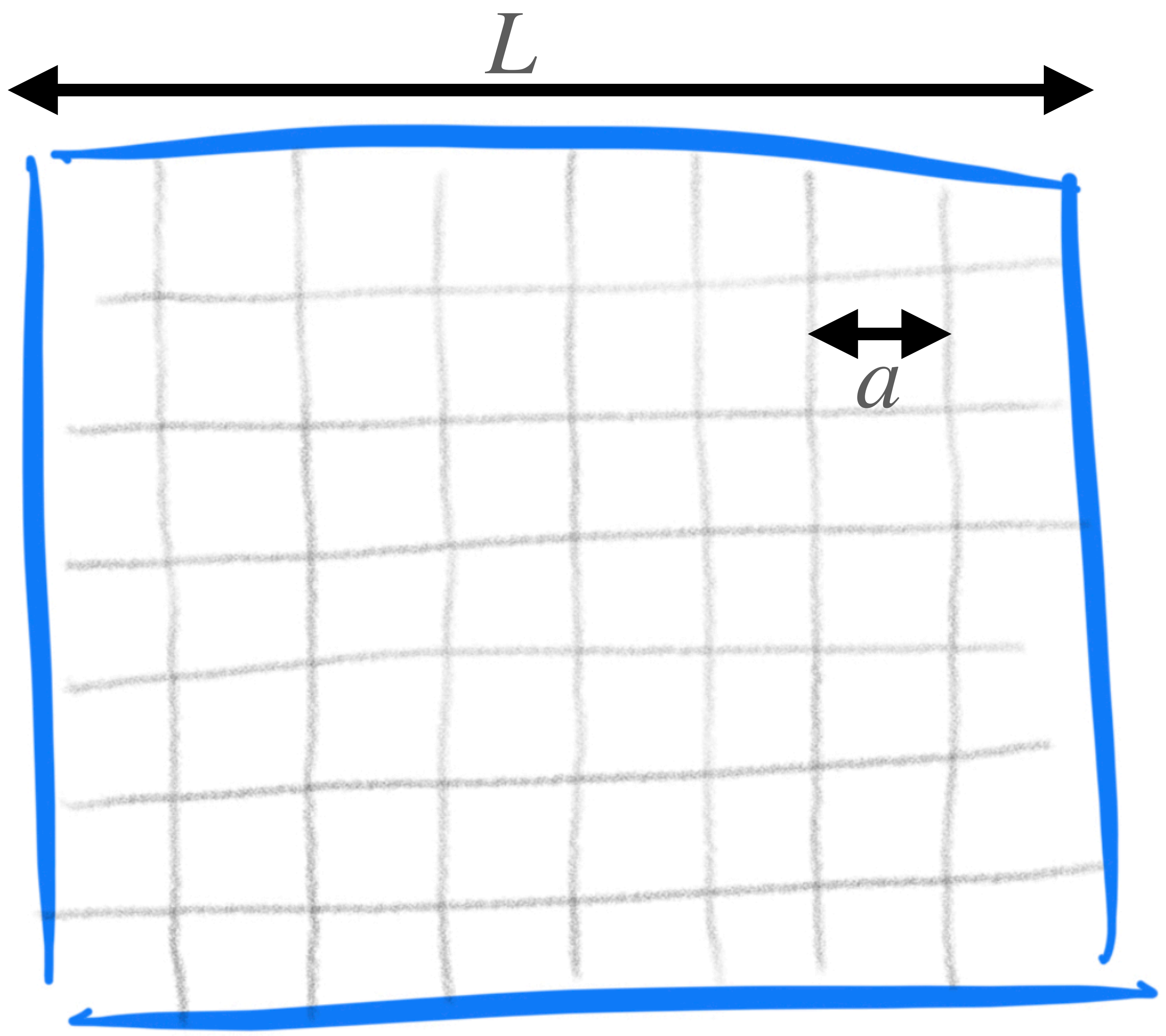

$$\frac{1}{L} \ll \Lambda_{\text{physics}} \ll \frac{1}{a}$$

$$\int dx\, f(x) \to a \sum_x f(x) \qquad \partial_\mu f(x) \to \frac{f(x+a) - f(x)}{a}$$

What to do with it?

- can do nonPT calculations

$$\langle 0|O|0\rangle = \frac{1}{Z}\int \mathcal{D}[U,\psi,\bar{\psi}]Oe^{-S_{\text{lat}}[U,\psi,\bar{\psi}]}$$

Euclidean space-time
Boltzmann factor

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Euclidean space-time Boltzmann factor

Free parameters:

- gauge coupling $g \to \alpha_s = g^2/4\pi$
- quark masses $m_f = u,d,s,c,b,t$

$$\mathcal{L}_{\mathrm{QCD}} = -\frac{1}{4}F^a_{\mu\nu}F^{a\,\mu\nu} + \sum_f \bar{\psi}_f\left(i\gamma^\mu D_\mu - m_f\right)\psi_f$$

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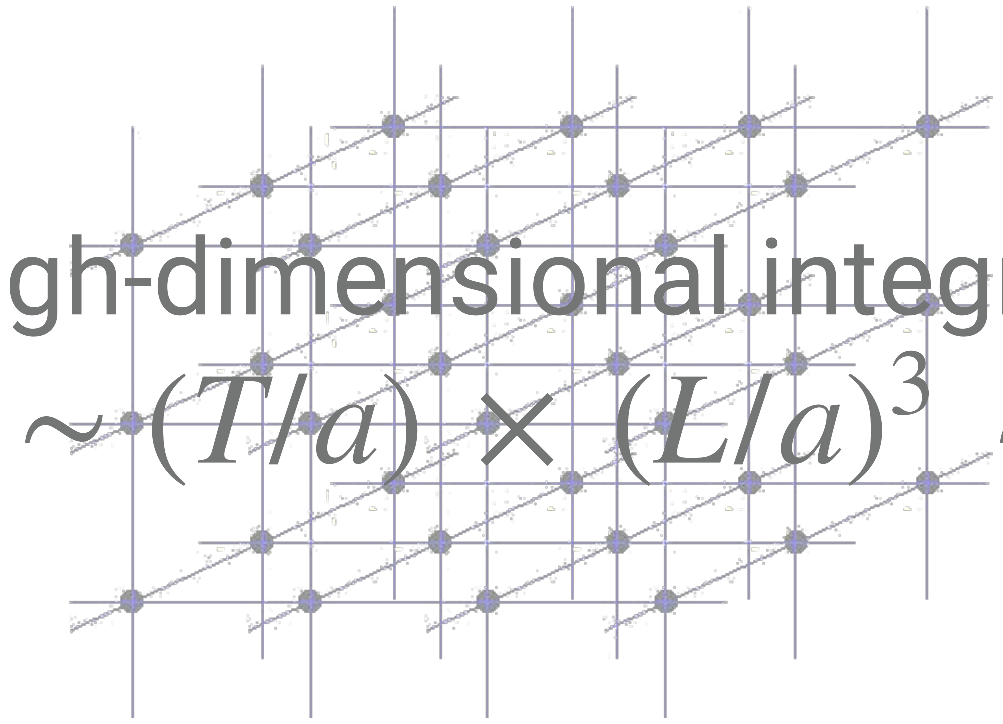
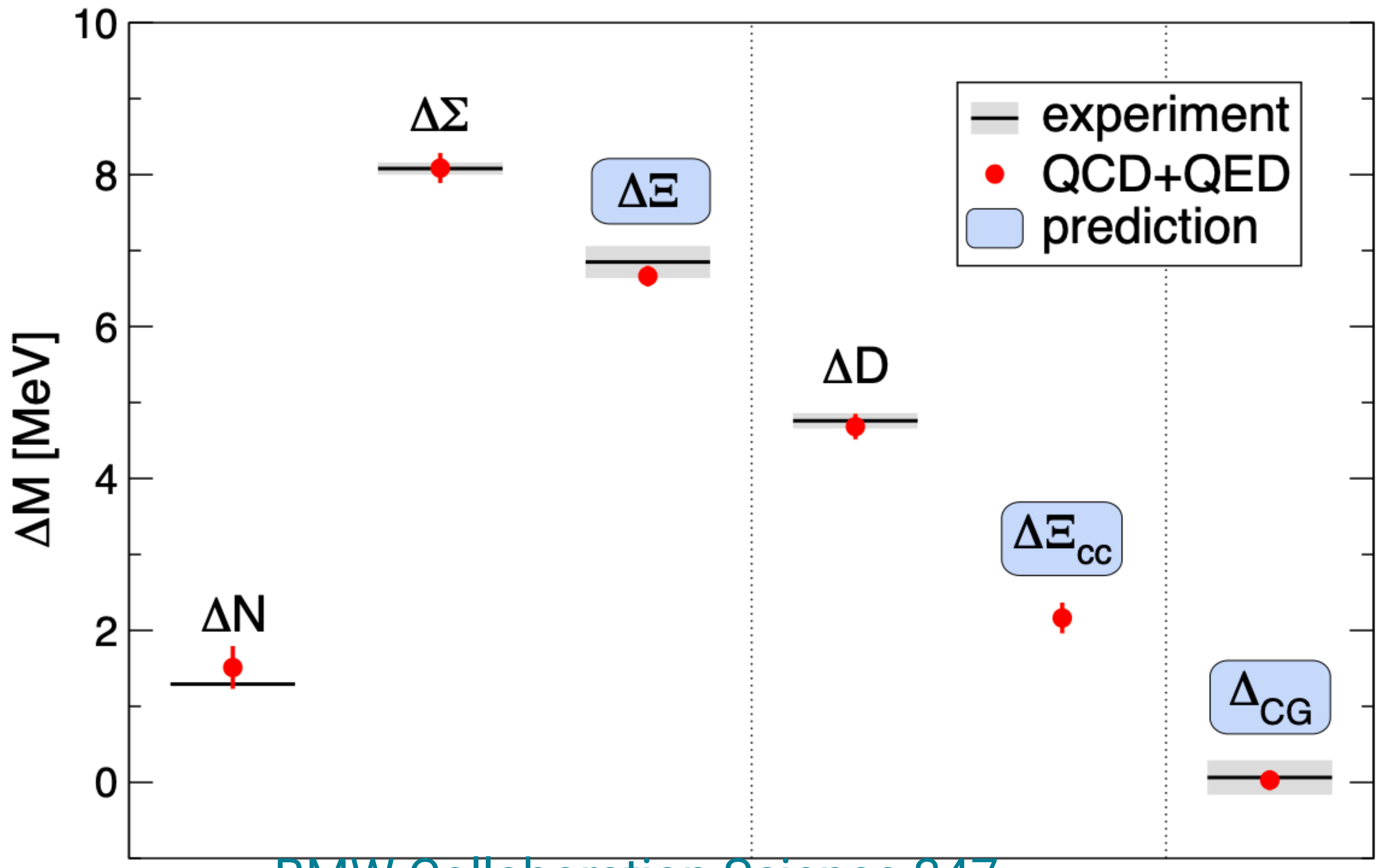


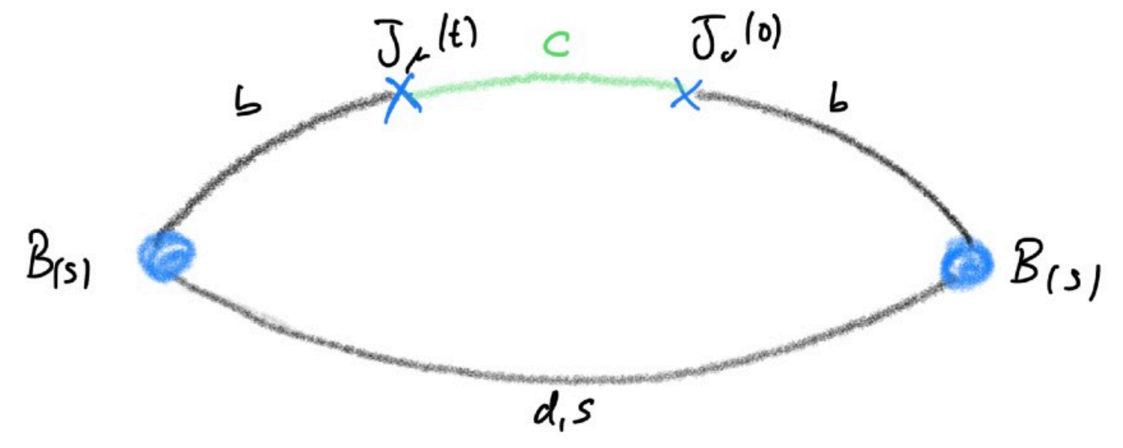
high-dimensional integral:

$$N_{\mathrm{sim}} \sim (T/a) \times (L/a)^3 \sim 10^9$$



BMW Collaboration Science 347 (2015) 1452-1455

Inclusive decays on the lattice



$$C_{\mu\nu}(t,\mathbf{q}) = \frac{C^{\mu\nu}_{4\mathrm{pt}}(t_2,t_1)}{C_{2\mathrm{pt}}(t_2)C_{2\mathrm{pt}}(t_1)} = \sum_{\mathbf{x}} \frac{e^{i\mathbf{q}\cdot\mathbf{x}}}{2M_{B_s}} \langle B_s \,|\, J^\dagger_\mu(\mathbf{x},t)\, J_\nu(\mathbf{0},0) \,|\, B_s \rangle \qquad (t = t_2 - t_1 \geq 0)$$

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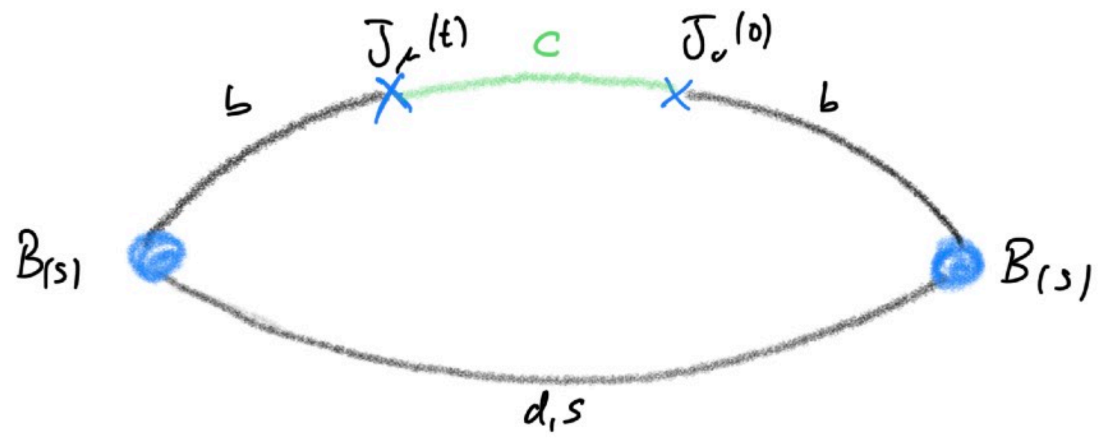

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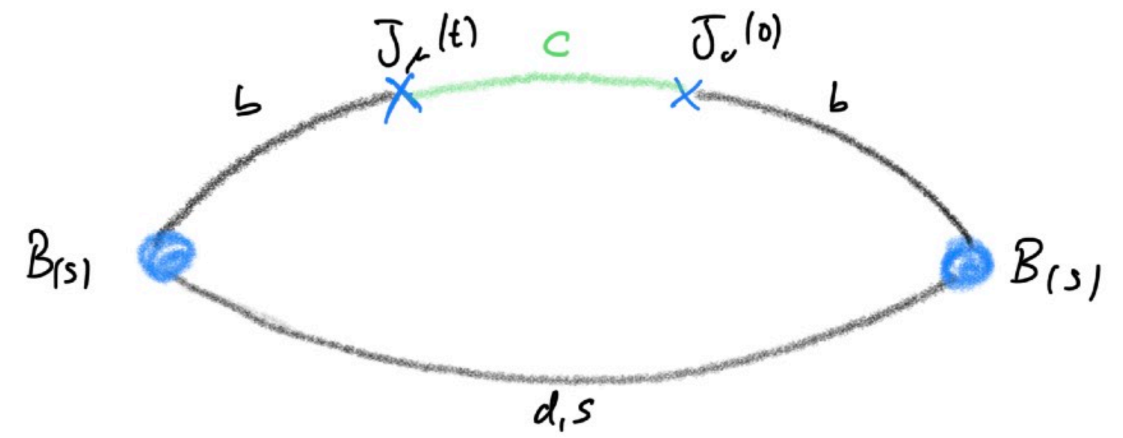

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$$C_{\mu\nu}(t,\mathbf{q}) = \int_{\omega_{\min}}^{\infty} d\omega\, W_{\mu\nu}(\omega,\mathbf{q})\, e^{-\omega t}$$

$\omega_{\min}$ is mass of lightest final state

Euclidean 4pt function is Laplace transform of hadronic tensor

$\bar{X}(\mathbf{q})$ from Euclidean correlation

$$C_{\mu\nu}(t, \mathbf{q}) = \int_{\omega_0}^{\infty} d\omega \, W_{\mu\nu}(\omega, \mathbf{q}) \, e^{-\omega t} \quad \longleftrightarrow \quad \bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega \, W_{\mu\nu}(\omega, \mathbf{q}) \, K^{\mu\nu}(\mathbf{q}, \omega)$$

$\bar{X}(\mathbf{q})$ from Euclidean correlation

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Expand the kernel K (analytically known) in powers of $e^{-a\omega}$:

$$\begin{aligned}\bar{X}(\mathbf{q}) &\approx c_{\mu\nu,0}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q}) + c_{\mu\nu,1}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q})\, e^{-a\omega} + c_{\mu\nu,2}(\mathbf{q}) \int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q})\, e^{-2a\omega} + \ldots \\ &= c_{\mu\nu,0}(\mathbf{q})\;\; C_{\mu\nu}(0,\mathbf{q}) + c_{\mu\nu,1}(\mathbf{q})\;\; C_{\mu\nu}(a,\mathbf{q}) + c_{\mu\nu,2}(\mathbf{q})\;\; C_{\mu\nu}(2a,\mathbf{q}) + \ldots\end{aligned}$$

$\bar{X}(\mathbf{q})$ from Euclidean correlation

$$C_{\mu\nu}(t,\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega,\mathbf{q})\, e^{-\omega t} \quad \longleftrightarrow \quad \bar{X}(\mathbf{q}) = \int_{\omega_0}^{\infty} d\omega\, W_{\mu\nu}(\omega,\mathbf{q})\, K^{\mu\nu}(\mathbf{q},\omega)$$

Expand the kernel K (analytically known) in powers of $e^{-a\omega}$:

$$\begin{aligned}\bar{X}(\mathbf{q}) &\approx c_{\mu\nu,0}(\mathbf{q})\int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q}) + c_{\mu\nu,1}(\mathbf{q})\int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q})\, e^{-a\omega} + c_{\mu\nu,2}(\mathbf{q})\int_{\omega_0}^{\infty} d\omega W_{\mu\nu}(\omega,\mathbf{q})\, e^{-2a\omega} + \ldots \\ &= \tilde{c}_{\mu\nu,0}(\mathbf{q})\ \langle\tilde{T}\rangle_{\mu\nu}(0,\mathbf{q}) + \tilde{c}_{\mu\nu,1}(\mathbf{q})\ \langle\tilde{T}\rangle_{\mu\nu}(a,\mathbf{q}) + \tilde{c}_{\mu\nu,2}(\mathbf{q})\ \langle\tilde{T}\rangle_{\mu\nu}(2a,\mathbf{q}) + \ldots\end{aligned}$$

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$$= \tilde{c}_{\mu\nu,0}(\mathbf{q})\ \langle \tilde{T} \rangle_{\mu\nu}(0,\mathbf{q}) \quad + \tilde{c}_{\mu\nu,1}(\mathbf{q})\ \langle \tilde{T} \rangle_{\mu\nu}(a,\mathbf{q}) \quad + \tilde{c}_{\mu\nu,2}(\mathbf{q})\ \langle \tilde{T} \rangle_{\mu\nu}(2a,\mathbf{q}) \quad + \ldots$$

$$\bar{X}(\mathbf{q}) = \sum_k \tilde{c}_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega,\mathbf{q})\, \tilde{T}_k(\omega)$$

$$= \sum_k \tilde{c}_{\mu\nu,k}(\mathbf{q})\, \langle \tilde{T}_k \rangle_{\mu\nu}$$

- $\tilde{c}_{\mu\nu,k}(\mathrm{q})$ known analytically
- $\langle \tilde{T} \rangle_{\mu\nu}$ from lattice data
- max order ↔ computed 4pt-fn. time slices

Kernel approximation

$$K_{\mu\nu}(\omega, \mathbf{q}) \approx \frac{\tilde{c}_{\mu\nu,0}}{2} + \sum_{k=1}^{N} \tilde{c}_{\mu\nu,k} \tilde{T}_k(\omega)$$

kernel expanded in Chebyshevs

$$\tilde{c}_{\mu\nu,k} = \int_{\omega_0}^{\infty} d\omega \, K_{\mu\nu}(\omega, \mathbf{q}) \tilde{T}_k(\omega) \tilde{\Omega}(\omega)$$

orthogonal projection determine expansion coefficients

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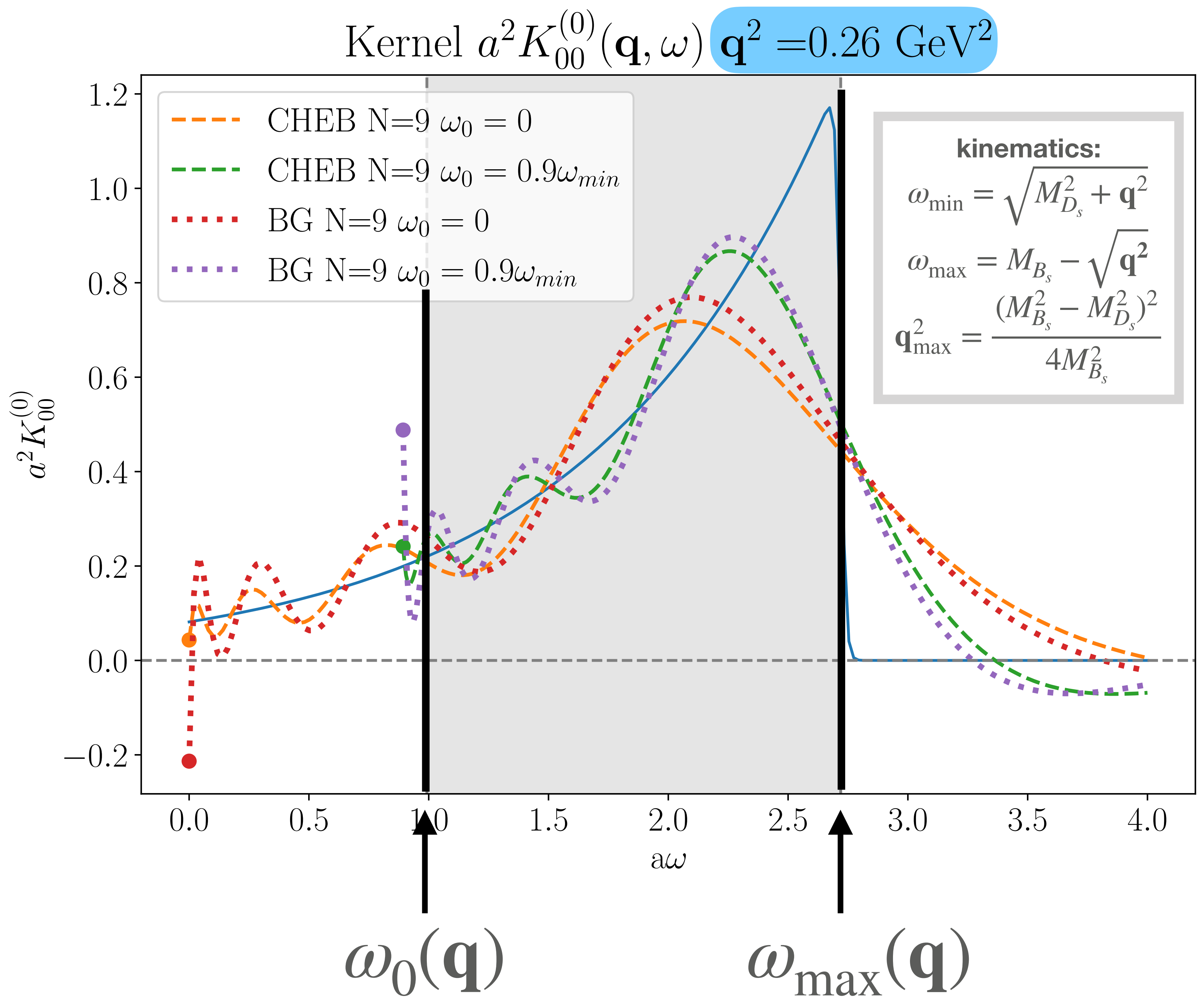
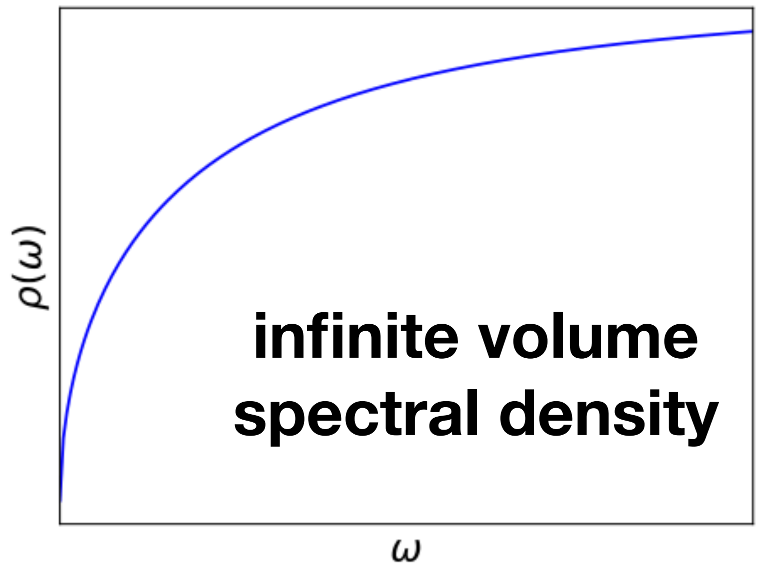
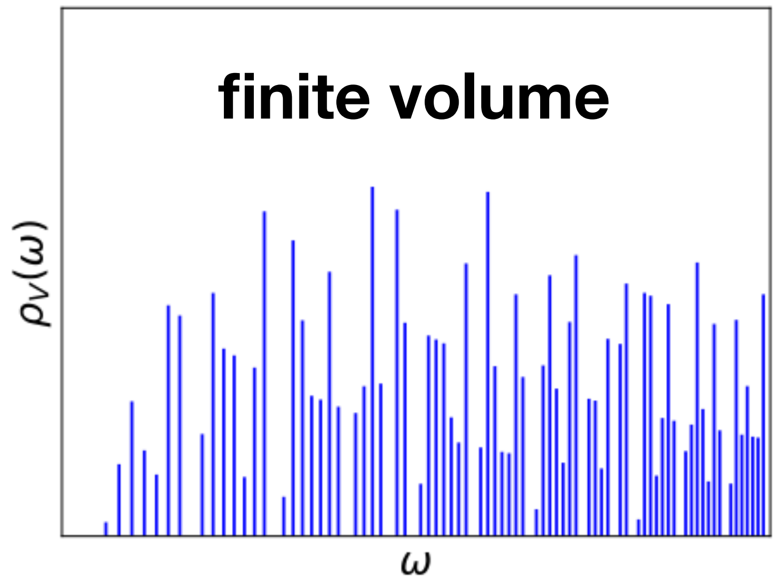
orthogonal projection determine expansion coefficients



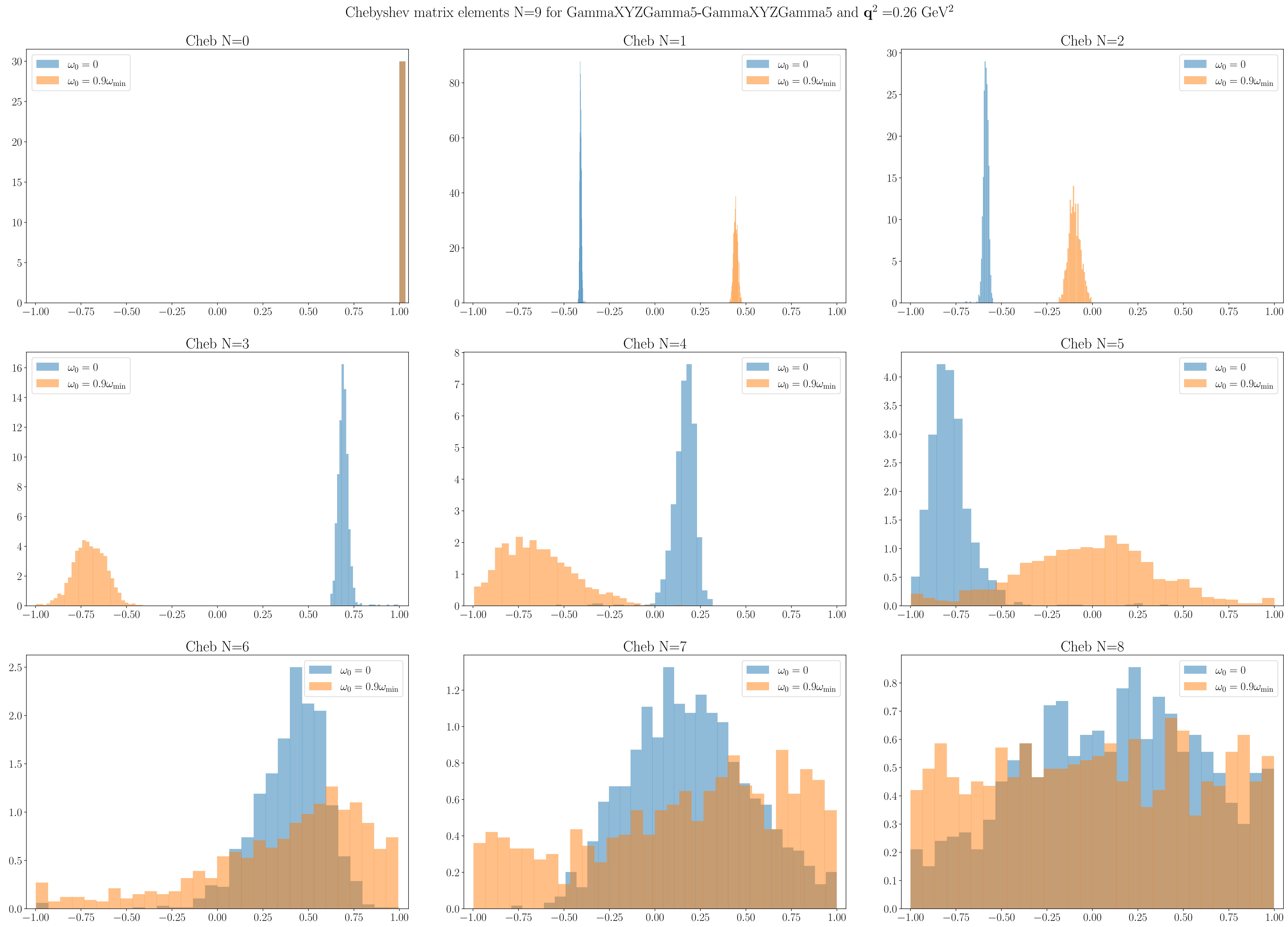
- this analysis stage independent of data
- smearing σ





- order of approximation $N \leftrightarrow C_{\mu\nu}(t)$
- we can play with ω_0

Lattice determination of $\langle \tilde{T}_k \rangle_{\mu\nu}$

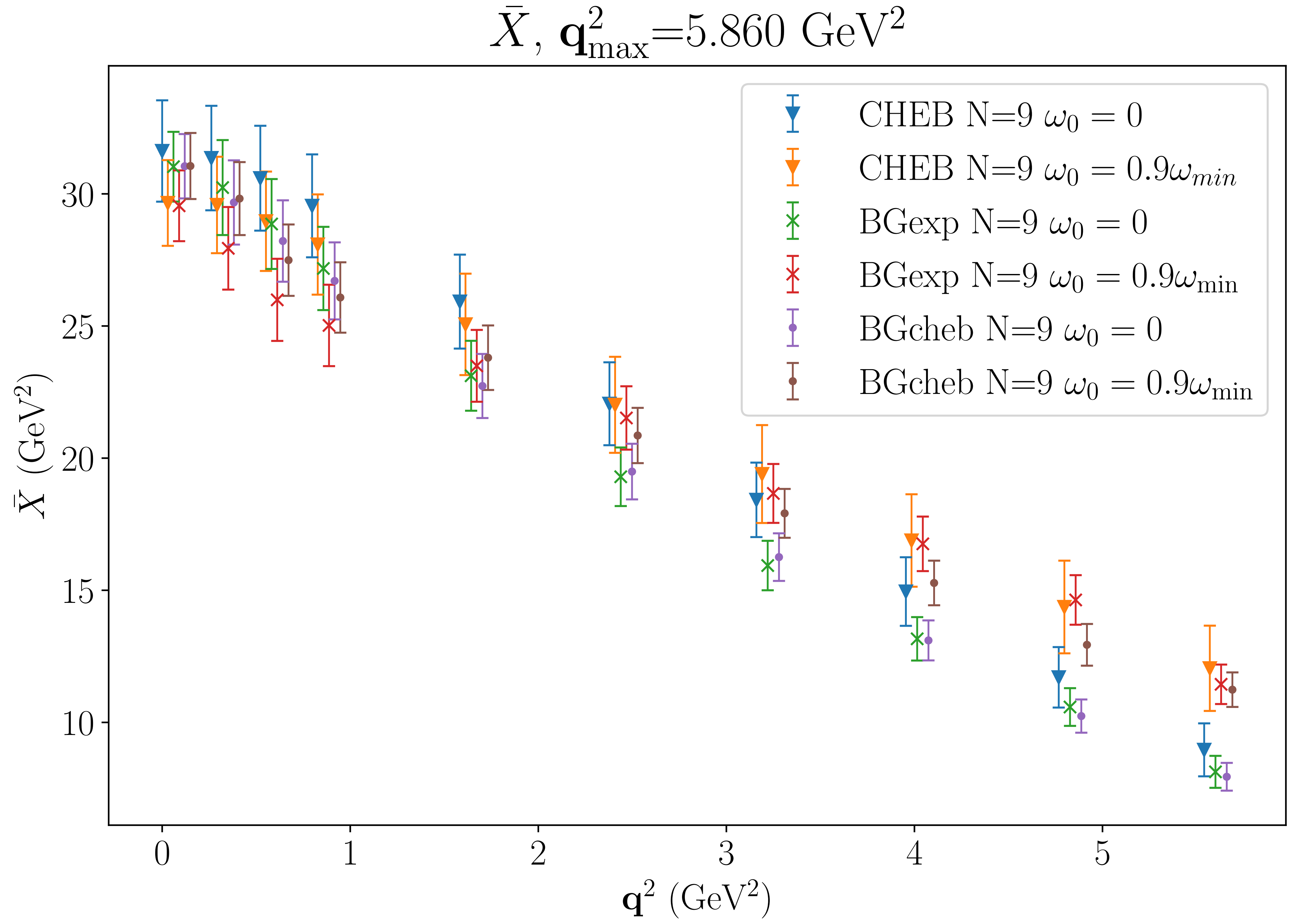


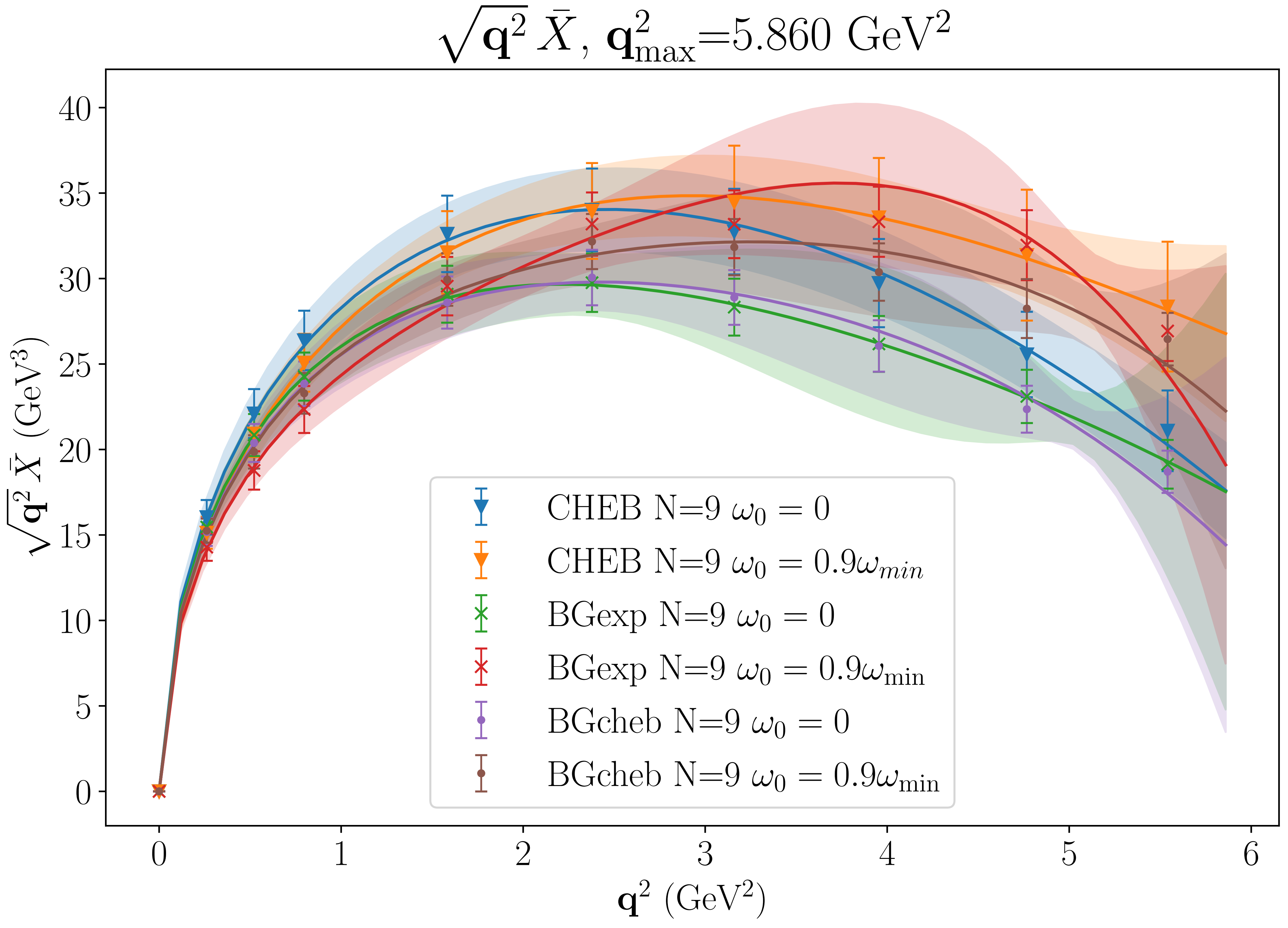
$$\langle \tilde{T}_k \rangle_{\mu\nu} = \frac{\sum_{j=0}^{k} \tilde{t}_j^{(k)} C_{\mu\nu}(j+2t_0)}{C_{\mu\nu}(2t_0)}$$

$$\bar{X}(\mathbf{q}) = \sum_k \tilde{c}_{\mu\nu,k}(\mathbf{q}) \int d\omega W_{\mu\nu}(\omega,\mathbf{q})\, \tilde{T}_k(\omega)$$
$$= \sum_{k,j} \tilde{c}_{\mu\nu,k}(\mathbf{q}) \langle \tilde{T}_k \rangle_{\mu\nu}$$

- $\omega_0 = 0$ and $\omega_0 = 0.9\,\omega_{\min}$
- Bayesian prior $-1 \leq \langle \tilde{T}_k \rangle_{\mu\nu} \leq +1$ acts as regulator for noise induced by higher orders

Results for $\bar{X}(\mathrm{q})$

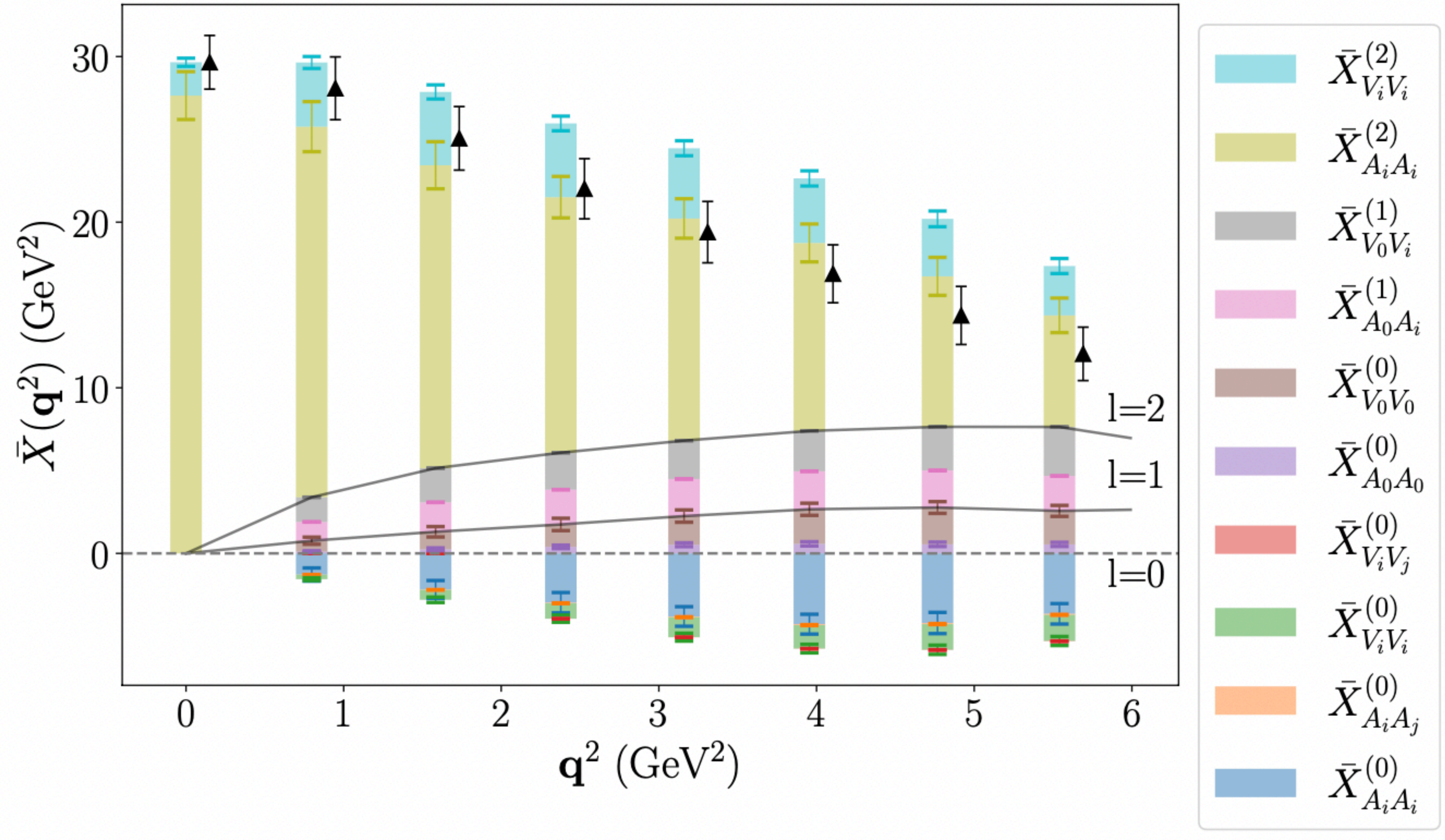




variations of analysis techniques largely consistent — tension at larger $\mathbf{q}^2$ visible

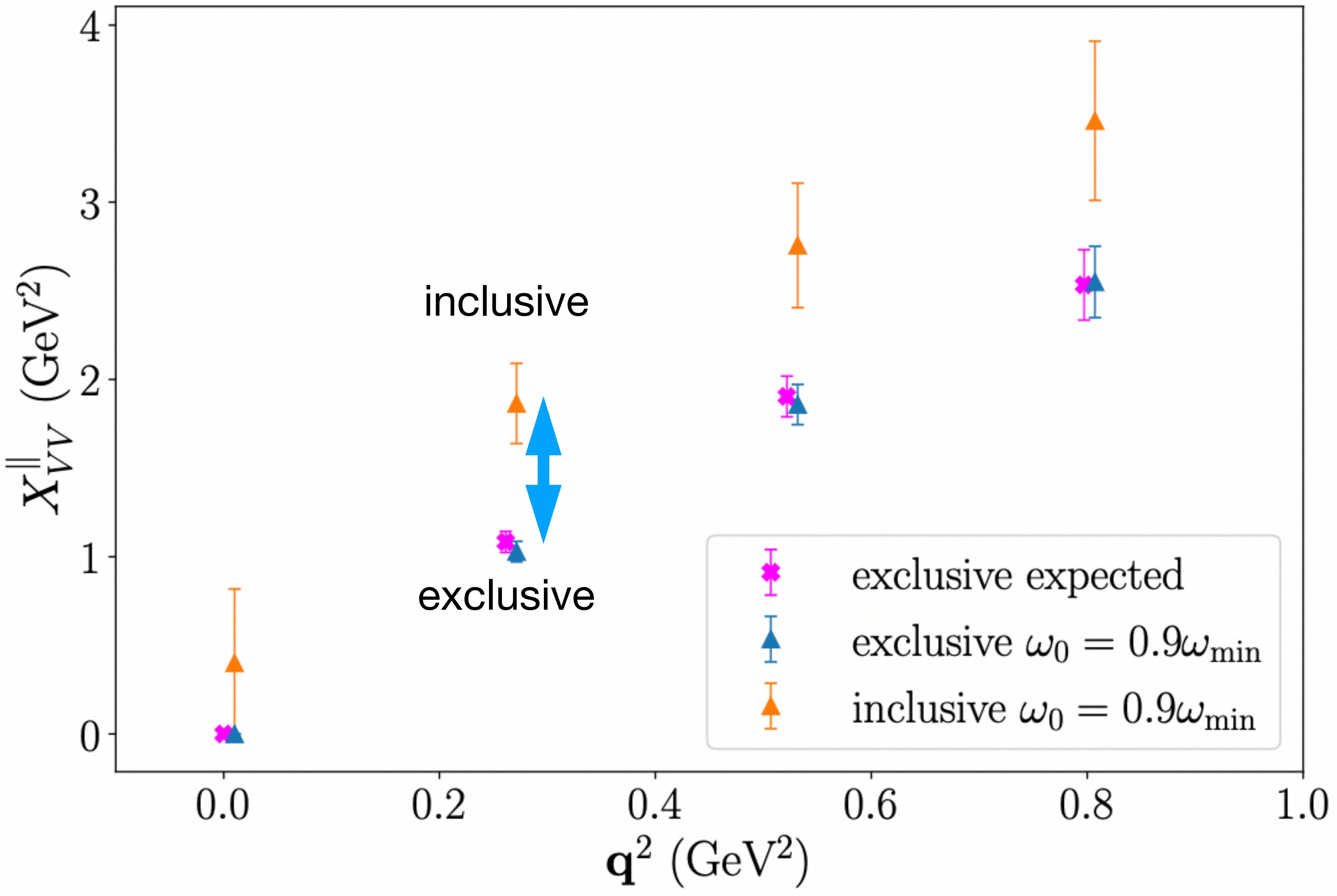
Integral of $\sqrt{\mathbf{q}^2}\,\bar{X}(\mathbf{q}^2)$ proportional to Γ;

We can disect contributions from different channels



Approach provides for nice laboratory to understand and probe contributions to inclusive decay from various sources

Ground-state limit



- lattice determination (exclusive) of decay into ground state straightforward:

$$\langle D_s | V_\mu | B_s \rangle = f_+(q^2)(p_{B_s} + p_{D_s})_\mu + f_-(q^2)(p_{B_s} - p_{D_s})_\mu$$

$$\bar{X}^{\parallel}_{VV} \to \frac{M_{B_s}}{E_{D_s}} \mathbf{q}^2 |f_+(q^2)|^2$$

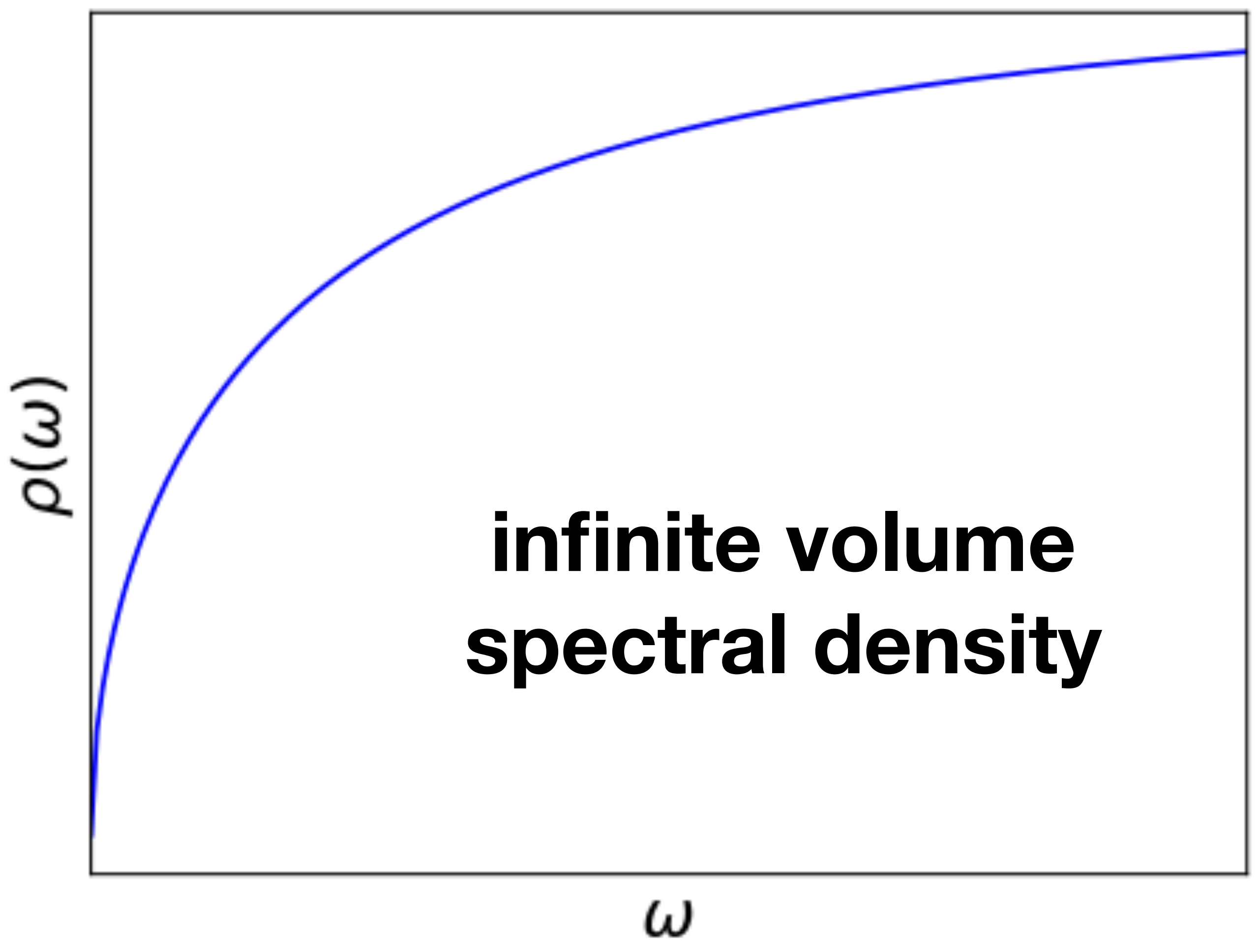
- clear distinction between ground-state and full inclusive determination
- we are also working on inclusive $D_s \to X\ell\bar{\nu}$ where very little excited-state contributions

[see also De Santis and Gross arXiv:2502.15519]

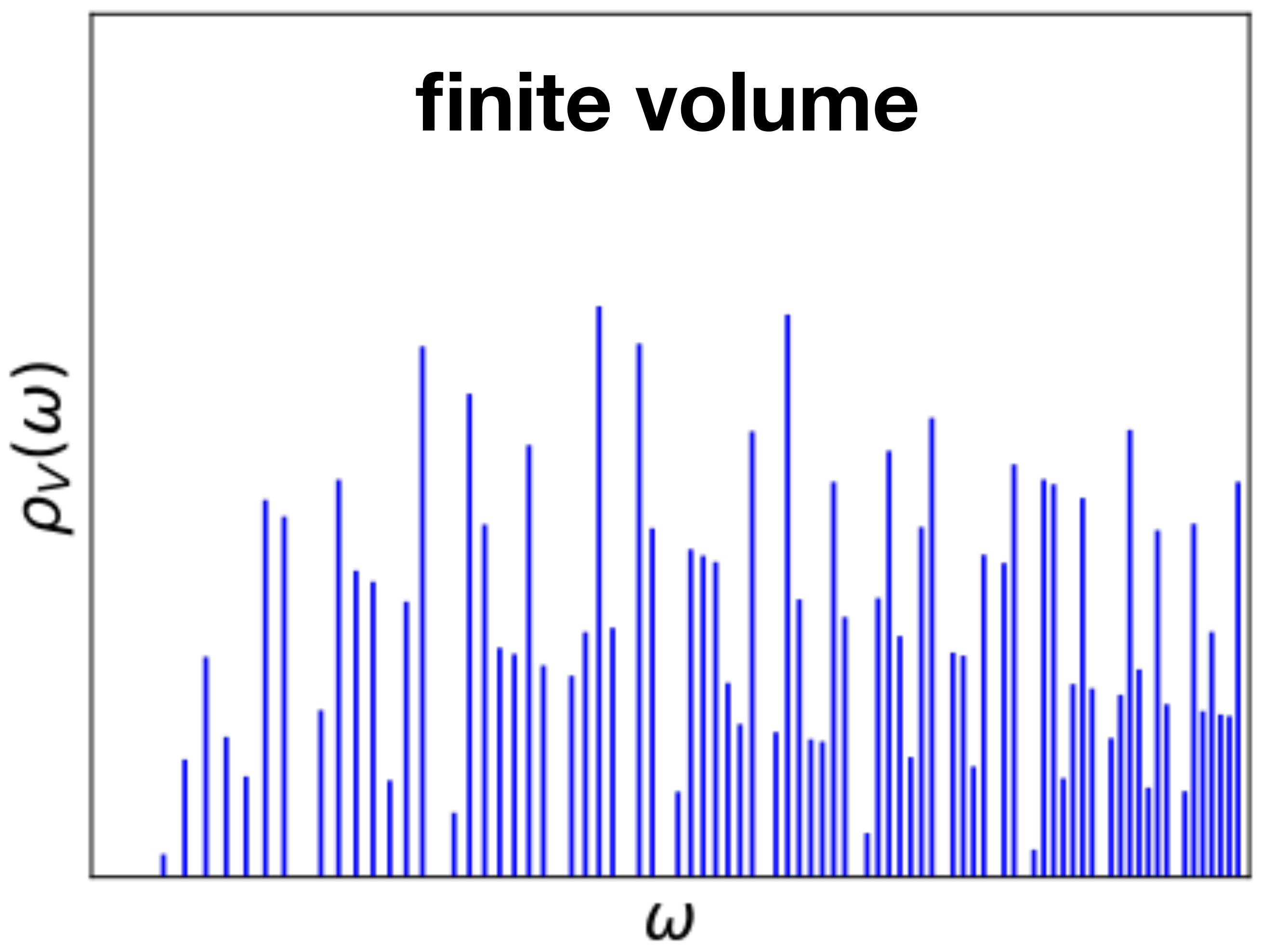
Systematics — finite volume

- Order of limits: $\lim_{\sigma\to 0}\lim_{V\to\infty}$
- in practice lattice simulations in finite volume
- need to find ways for estimating effects reliably
- Here: model finite-size effects with spectral density of two non-interacting particles

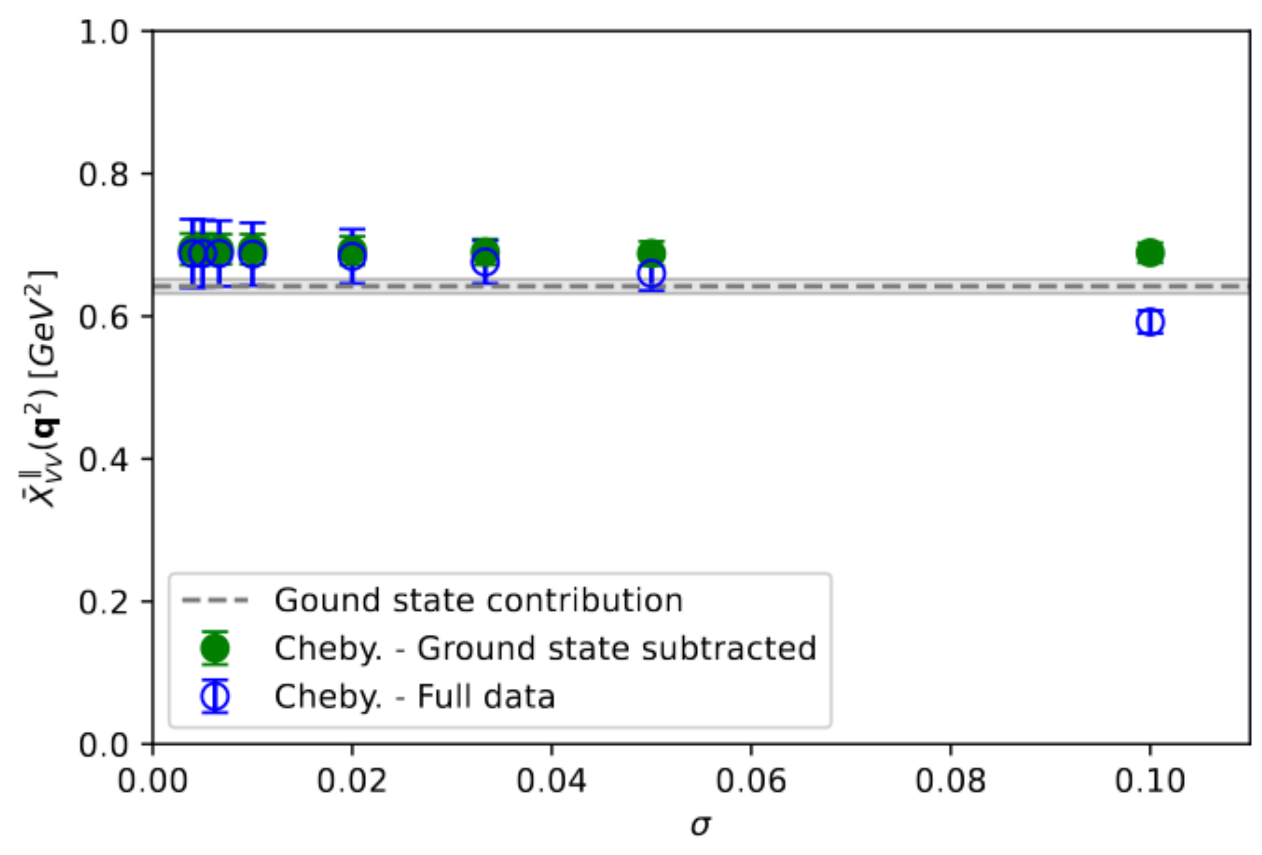
$$\rho(\omega) = \frac{1}{2\pi}\int_0^\infty dq \frac{q^2}{4(q^2+M^2)}\delta(\omega - 2\sqrt{q^2+M^2})$$



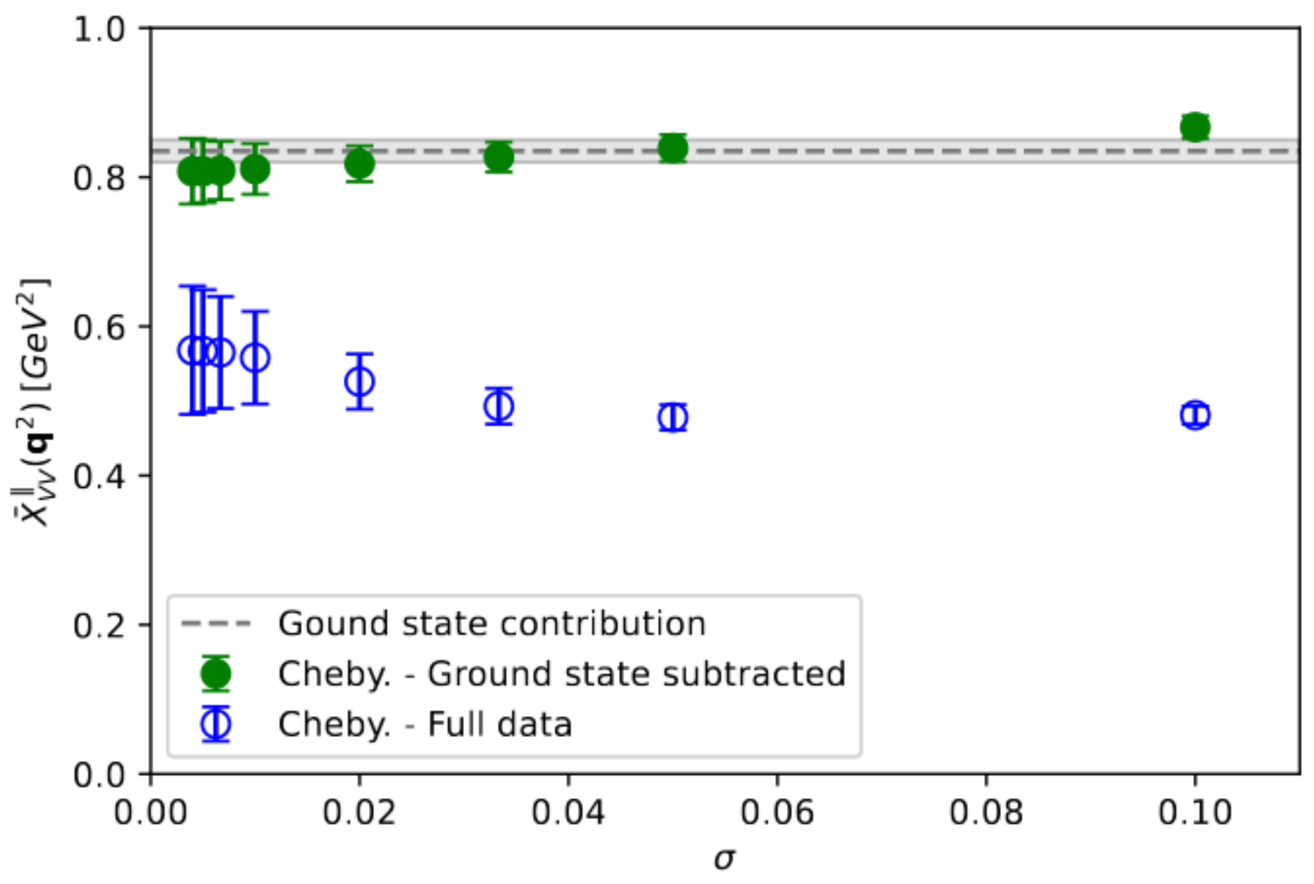
$$\rho_V(\omega) = \frac{\pi}{V}\sum_{\mathbf{q}} \frac{\mathbf{q}^2}{4(\mathbf{q}^2+M^2)}\delta\left(\omega - 2\sqrt{\mathbf{q}^2+M^2}\right)$$



Systematics – $\sigma \to 0$

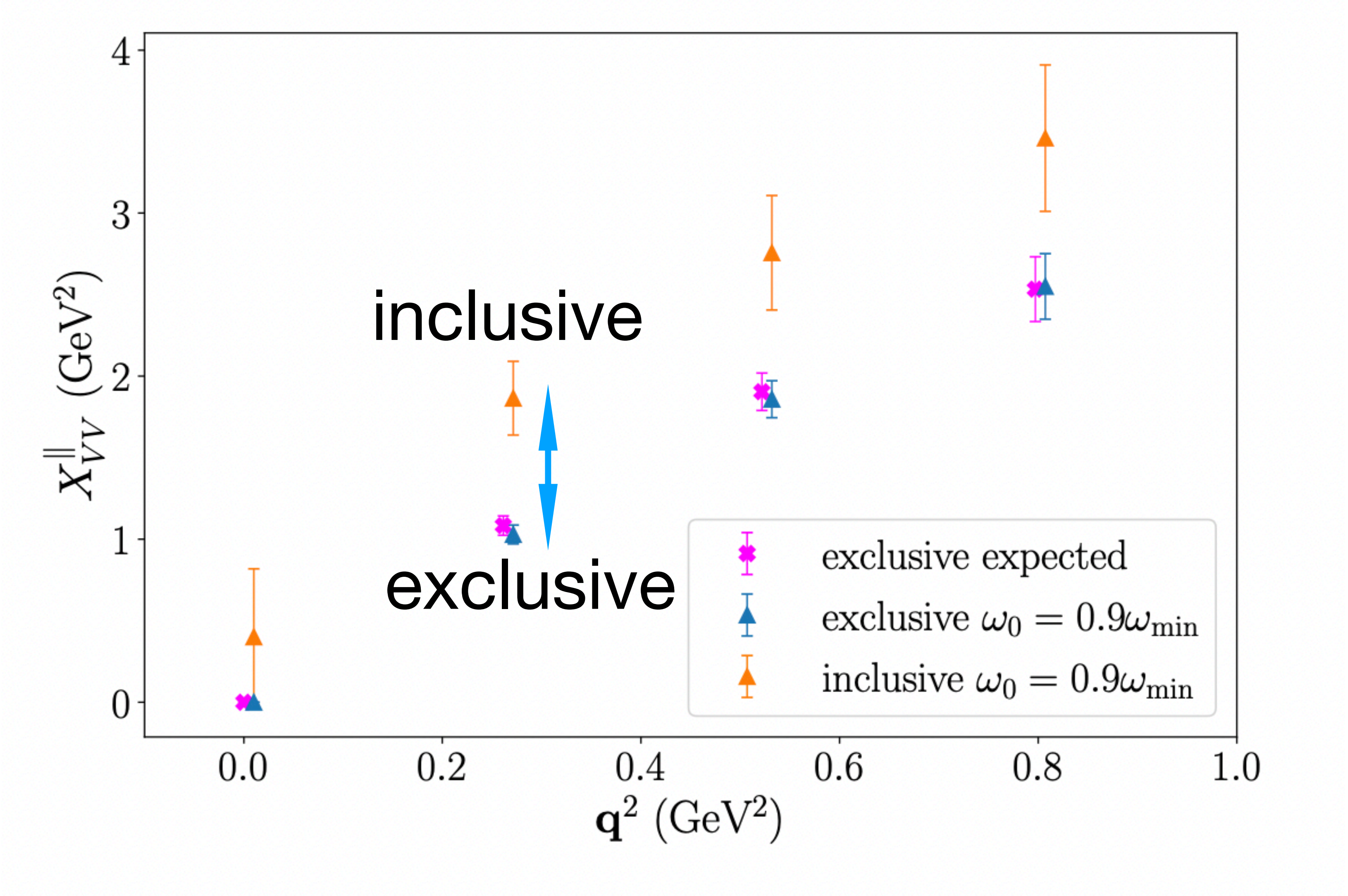


(b) $\boldsymbol{q} = (0, 0, 1)$



(d) $\boldsymbol{q} = (1, 1, 1)$

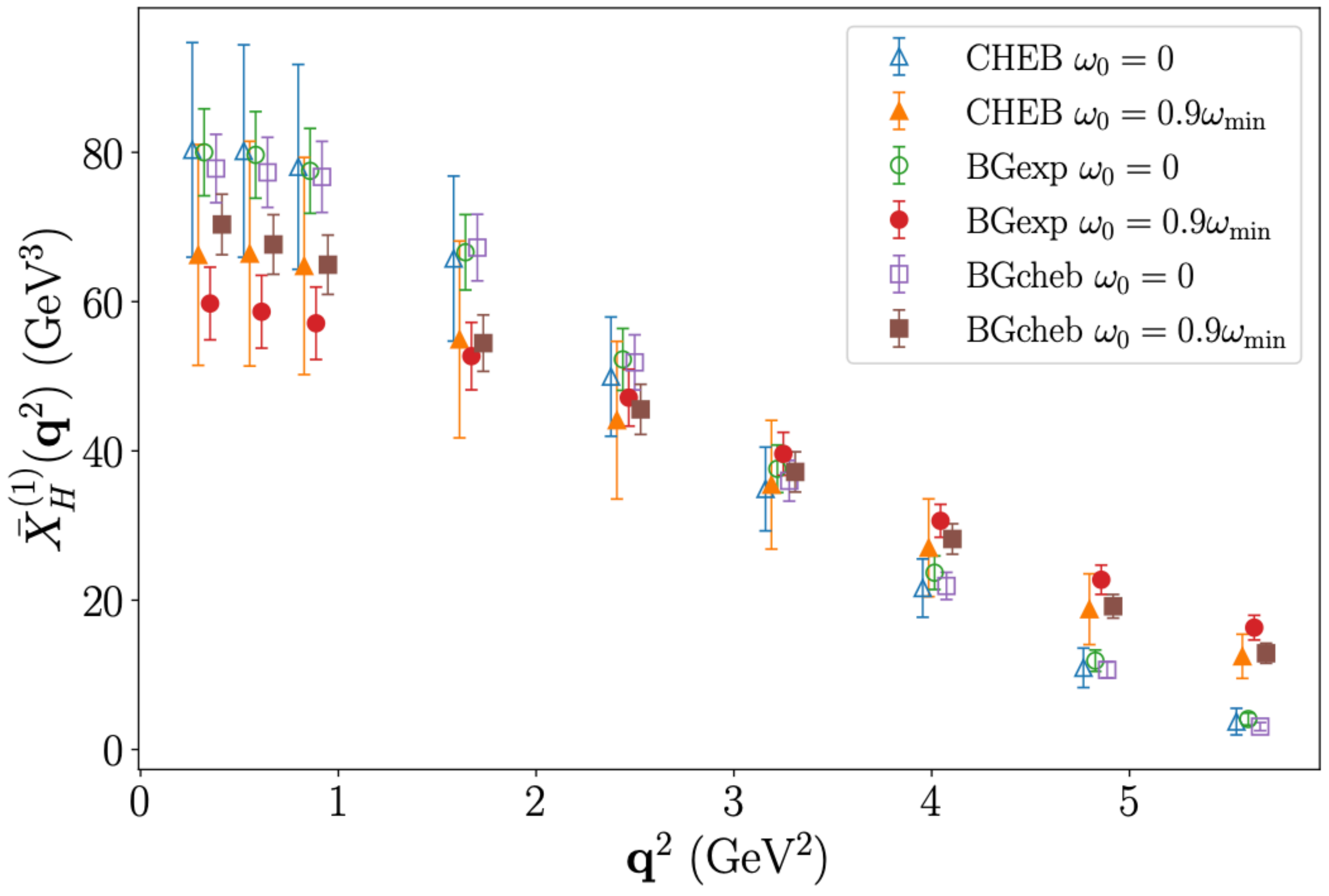
- After the infinite-volume limit also the $\sigma \to 0$ limit of vanishing smearing has to be taken
- Here, we assume finite-volume effects are under control
- current thinking: compute ground-state explicitly; apply the inclusive analysis only to the remainder substantially reduces sensitivity to finite smearing width/volume

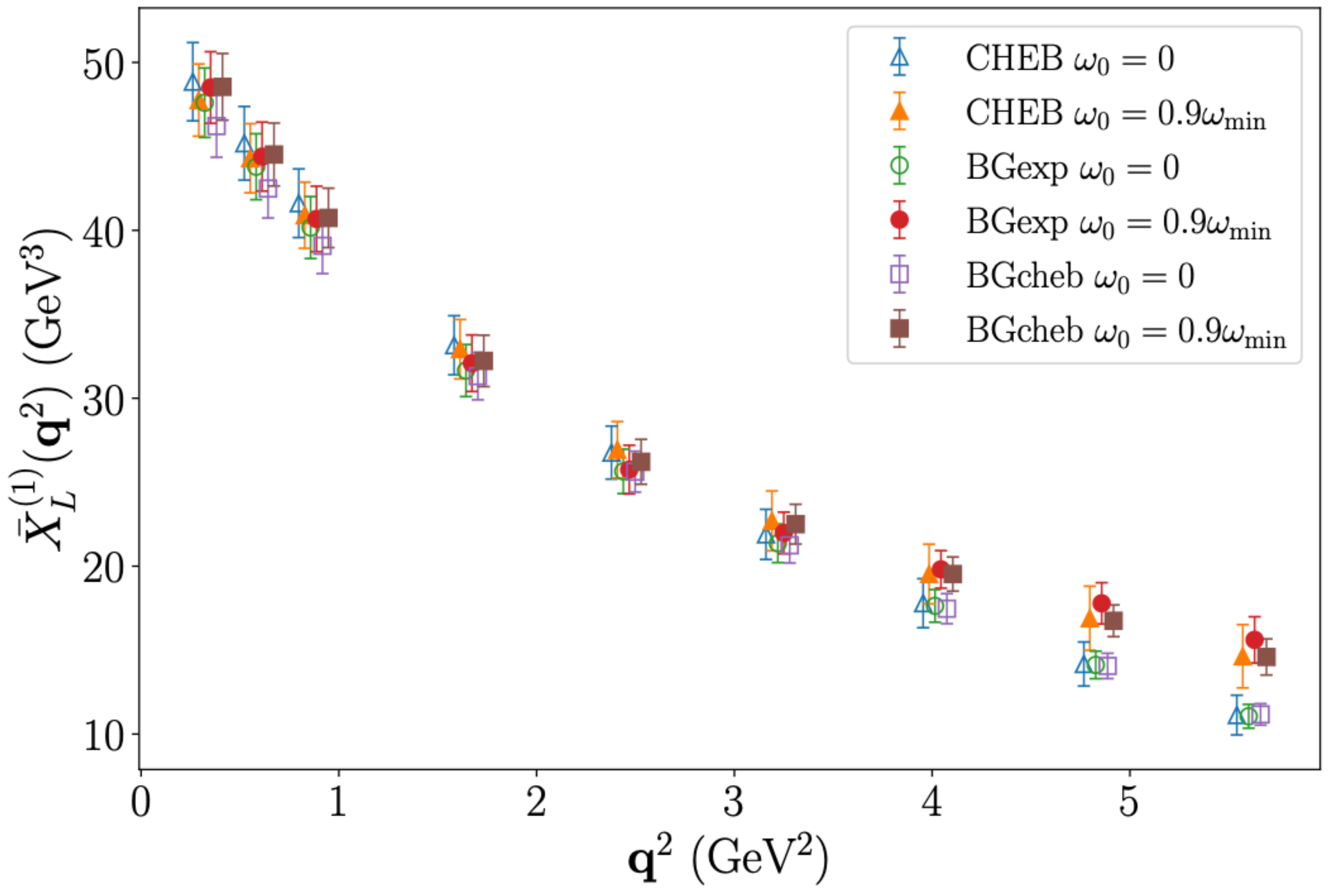


[Kellerman et al. @ Lattice 2024 and in preparation]

Moments

- hadronic or leptonic moments are essential building block of OPE analysis of inclusive decays
- they can be computed from the lattice data and allow for mutually scrutinising continuum and lattice computations

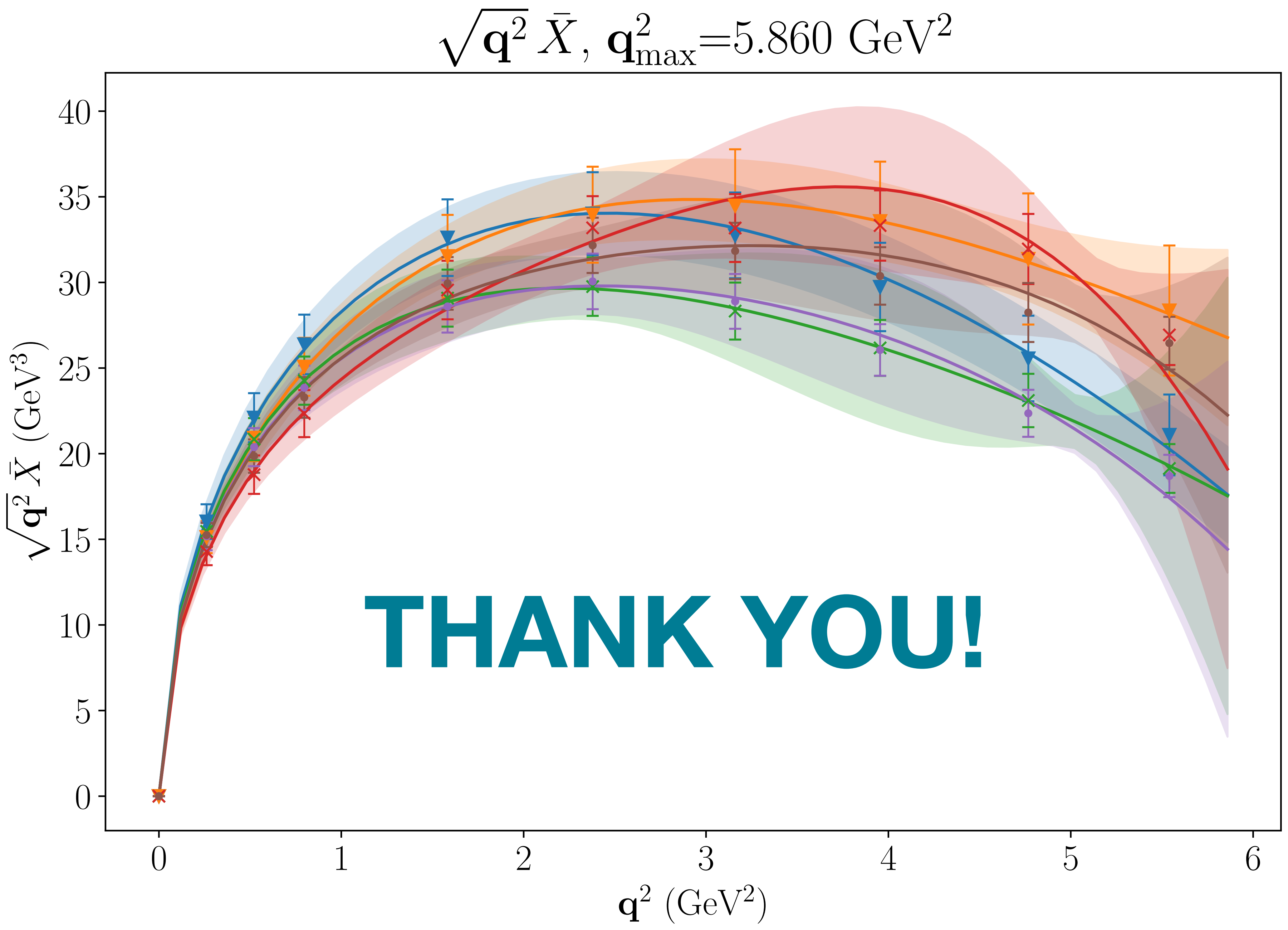




[Barone@Lattice2023]

Conclusions

- An independent calculation of $|V_{ub}|$ or $|V_{cb}|$ from inclusive decays has become within reach
- We are working on a fully comprehensive analysis of both $B_{(s)}$ and $D_{(s)}$ inclusive decays
- We can also compute building blocks of OPE analysis
- Why not think about new smeared set of observables for experiment and theory
- Should be possible to extend approach to inclusive rare decays
- Maybe a phenomenologically relevant prediction for CKM for inclusive decay on the lattice isn't that far off?


$\sqrt{\mathbf{q}^2}\,\bar{X}$, $\mathbf{q}^2_{\rm max}$=5.860 GeV^2
$\sqrt{\mathbf{q}^2}\,\bar{X}$ (GeV^3)
$\mathbf{q}^2$ (GeV^2)
0
5
10
15
20
25
30
35
40
0
1
2
3
4
5
6
THANK YOU!