The Two-Higgs Doublet Model beyond tree-level : A gauge-invariant formalism

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The 2HDM content & motivations

- Extension of the Standard Model with two Higgs doublets φ_1, φ_2

$$\varphi_1 = \begin{pmatrix} \varphi_1^+ \\ \varphi_1^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1^1 + i\sigma_1^1 \\ \pi_1^2 + i\sigma_1^2 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_2^1 + i\sigma_2^1 \\ \pi_2^2 + i\sigma_2^2 \end{pmatrix}$$

- 8 scalar components :
  - 3 SM Goldstone bosons : G^0 and $G^\pm$
  - 3 neutral scalars : h_1, h_2 and h_3
  - 2 charged scalars : $H^\pm$
- Rich phenomenology :
  - CP violation
  - dark matter candidates
  - new EWSB scenarios
  - included in MSSM, GUTs, axions and others BSM theories

General tree-level potential

The most general tree-level potential reads

$$V^{(0)}(\varphi_1, \varphi_2) = m_{11}^2(\varphi_1^\dagger \varphi_1) + m_{22}^2(\varphi_2^\dagger \varphi_2) - \left[m_{12}^2(\varphi_1^\dagger \varphi_2) + \text{h.c.}\right]$$

$$+\frac{1}{2}\lambda_1(\varphi_1^\dagger \varphi_1)^2 + \frac{1}{2}\lambda_2(\varphi_2^\dagger \varphi_2)^2 + \lambda_3(\varphi_1^\dagger \varphi_1)(\varphi_2^\dagger \varphi_2) + \lambda_4(\varphi_1^\dagger \varphi_2)(\varphi_2^\dagger \varphi_1)$$

$$+\frac{1}{2}\left[\lambda_5(\varphi_1^\dagger \varphi_2)^2 + \text{h.c.}\right] + \left[\lambda_6(\varphi_1^\dagger \varphi_1)(\varphi_1^\dagger \varphi_2) + \lambda_7(\varphi_2^\dagger \varphi_2)(\varphi_1^\dagger \varphi_2) + \text{h.c.}\right]$$

with 14 parameters (6 reals + 4 complex)

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with 14 parameters (6 reals + 4 complex)

φ_1 and φ_2 are gauge-dependent :

- hide symmetries
- make calculations difficult

Gauge-invariant formalism

Definition of gauge-invariant bilinears

$$K_0 = \varphi_1^\dagger\varphi_1 + \varphi_2^\dagger\varphi_2 \ , \qquad K_1 = \varphi_1^\dagger\varphi_2 + \varphi_2^\dagger\varphi_1 \ ,$$
$$K_2 = i(\varphi_2^\dagger\varphi_1 - \varphi_1^\dagger\varphi_2) \ , \qquad K_3 = \varphi_1^\dagger\varphi_1 - \varphi_2^\dagger\varphi_2$$

with a one-to-one correspondence between the K's and the physical d.o.f.

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with a one-to-one correspondence between the K's and the physical d.o.f.

We note $\widetilde{\boldsymbol{K}} = (K_0, K_1, K_2, K_3)^\intercal = (K_0, \boldsymbol{K})^\intercal$ such that the potential is

$$V^{(0)}(\widetilde{\boldsymbol{K}}) = \widetilde{\boldsymbol{K}}^\intercal \widetilde{\boldsymbol{\xi}} + \widetilde{\boldsymbol{K}}^\intercal \widetilde{\boldsymbol{E}} \widetilde{\boldsymbol{K}}$$

- Conditions : $K_0 \geq 0$ & $K_0^2 \geq \boldsymbol{K}^2$

Vacuum structure

We can decompose the vacuum structure in 3 types :

- Unbroken electroweak symmetry if $K_0 = 0$
- Charge-breaking minimum if $K_0^2 > \boldsymbol{K}^2$

Charge-conserving minimum ($\equiv$ SM EWSB)

$$K_0 > 0 \quad \& \quad K_0^2 = \boldsymbol{K}^2 \quad \& \quad \partial_\mu V = 2u \left(\widetilde{g}\widetilde{\boldsymbol{K}}\right)_\mu$$

with u the Lagrange multiplier and $\widetilde{g} = \mathrm{diag}\,(1, -\mathbb{1}_3)$

Mass expressions

- We need a connection between the physical scalars and the bilinears

We define $\phi^i := \left(\pi_1^1, \pi_1^2, \sigma_1^1, \sigma_1^2, \pi_2^1, \pi_2^2, \sigma_2^1, \sigma_2^2\right)^\mathsf{T}$ recalling that

$$\varphi_1 = \begin{pmatrix} \varphi_1^+ \\ \varphi_1^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_1^1 + i\sigma_1^1 \\ \pi_1^2 + i\sigma_1^2 \end{pmatrix}, \qquad \varphi_2 = \begin{pmatrix} \varphi_2^+ \\ \varphi_2^0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \pi_2^1 + i\sigma_2^1 \\ \pi_2^2 + i\sigma_2^2 \end{pmatrix}$$

and $K^\mu = (K_0, K_1, K_2, K_3)^\mathsf{T}$

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and $K^\mu = (K_0, K_1, K_2, K_3)^\mathsf{T}$

Bilinears/scalar components connection

$$K^\mu := \frac{1}{2}\Delta^\mu_{ij}\phi^i\phi^j \quad \text{and} \quad \Gamma^\mu_i := \frac{\partial K^\mu}{\partial \phi^i} = \partial_i K^\mu = \Delta^\mu_{ij}\phi^j$$

Mass expressions

The scalar masses are obtained from

$$\left(M_S^2\right)_{ij} = \partial_i \partial_j V$$

The analogous expression with bilinear is

$$\left(\mathcal{M}^2\right)_{\mu\nu} = \partial_\mu \partial_\nu V$$

Therefore we can write

$$\left(M_S^2\right)_{ij} = \partial_i \left(\Gamma_j^\mu \partial_\nu V\right) = \Delta_{ij}^\mu \partial_\mu V + \Gamma_i^\mu \Gamma_j^\nu \partial_\mu \partial_\nu V$$

or simply

$$\boxed{M_S^2 = \Delta^\mu \partial_\mu V + \Gamma \mathcal{M} \Gamma^\intercal}$$

Mass expressions

It's useful to define the canonical basis such that

$$\hat{\Gamma} = U_c \Gamma = \begin{pmatrix} 0_{5\times 4} \\ \gamma_{3\times 4} \end{pmatrix}$$

and thus we have

$$\widehat{M_S^2} = U_c M_S^2 U_c^\intercal = \begin{pmatrix} 0_{3\times 3} & & \\ & \widehat{M^2_{\mathrm{charged}}} & \\ & & \widehat{M^2_{\mathrm{neutral}}} \end{pmatrix}$$

- Gauge-invariant formalism leads to clear separation of massive scalars and Goldstones bosons

Mass expressions

Finally we can obtain the complete diagonal form with

$$\overline{M_S^2} = \bar{U}\widehat{M_S^2}\bar{U}^\intercal \quad \text{where} \quad \bar{U} = \mathrm{diag}\left(1, R'_{3\times 3}, 1, R_{3\times 3}\right)$$

We define

$$\bar{\boldsymbol{K}} = R\boldsymbol{K}, \quad \bar{\boldsymbol{\xi}} = R\boldsymbol{\xi}, \quad \bar{\boldsymbol{\eta}} = R\boldsymbol{\eta}, \quad \bar{\boldsymbol{E}} = R\boldsymbol{E}R^\intercal, \quad k_a = \frac{K_a}{K_0}$$

Gauge-invariant mass expressions (CC case)

- Charged sector : $m_{H^\pm}^2 = 4uK_0$ for any potential ($\equiv$ at all orders)
- Neutral sector : $m_a^2 = 4K_0\left[(\eta_{00} - u)\,\bar{k}_a^2 + 2\bar{\eta}_a\bar{k}_a + \bar{E}_{aa} + u\right]$

The $\hbar$-expansion principle

Principle

Corrections of the tree-level potential can be obtained from a perturbative expansion in $\hbar$ of the scalar potential with a Taylor expansion around the tree-level vacuum where $\hbar \to 0$ corresponds to the classical limit.

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Corrections of the tree-level potential can be obtained from a perturbative expansion in $\hbar$ of the scalar potential with a Taylor expansion around the tree-level vacuum where $\hbar \to 0$ corresponds to the classical limit.

- The effective potential is

$$V_{\text{eff}} = V^{(0)} + \kappa V^{(1)} + \kappa^2 V^{(2)} + \dots \qquad \text{with} \quad \kappa = \frac{\hbar}{16\pi^2}$$

  where V_0 is the tree-level potential and $V^{(1)}$ corresponds to one-loop quantum corrections

- We assume here that

$$\phi = \phi^{(0)} + \kappa\phi^{(1)} + \kappa^2\phi^{(2)} + \dots$$

Effective potential in terms of bilinears

Quantum corrections modify the potential as

$$V_{\text{eff}}(\widetilde{\boldsymbol{K}}) = V^{(0)}(\widetilde{\boldsymbol{K}}) + \kappa V^{(1)}(\widetilde{\boldsymbol{K}}, \mu) + \ldots$$

with

$$\widetilde{\boldsymbol{K}} = \widetilde{\boldsymbol{K}}^{(0)} + \kappa \widetilde{\boldsymbol{K}}^{(1)} + \ldots$$

The Coleman-Weinberg one-loop correction is given by

$$\boxed{V_1(\widetilde{\boldsymbol{K}}, \mu) = \frac{1}{4} \sum_i n_i m_i^4(\widetilde{\boldsymbol{K}}) \left[\ln\left(\frac{m_i^2(\widetilde{\boldsymbol{K}})}{\mu^2} \right) - c_i \right]}$$

where n_i and c_i are known constants (which depend on particule-type and renormalization scheme).

Motivations

- The gauge-invariance is preserved at each order in the expansion
- Precise predictions for mass and vacuum position with
  - $m = m^{(0)} + \kappa m^{(1)}$
  - $\widetilde{\boldsymbol{K}} = \widetilde{\boldsymbol{K}}^{(0)} + \kappa \widetilde{\boldsymbol{K}}^{(1)}$
  - $u = u^{(0)} + \kappa u^{(1)}$
- Study potential stability, vacuum structure and new possible phenomenology at loop-level ?

Vacuum structure

The minimization of the potential requires

$$\begin{aligned}
\partial_\alpha V(\widetilde{\boldsymbol{K}}) &= 2u\left(\widetilde{g}\widetilde{\boldsymbol{K}}\right)_\alpha \\
&= 2u^{(0)}\left(\widetilde{g}\widetilde{\boldsymbol{K}}^{(0)}\right)_\alpha + \kappa\left[2u^{(0)}\left(\widetilde{g}\widetilde{\boldsymbol{K}}^{(1)}\right)_\alpha + 2u^{(1)}\left(\widetilde{g}\widetilde{\boldsymbol{K}}^{(0)}\right)_\alpha\right] + \dots \\
&= \partial_\alpha V^{(0)} + \kappa\left[\partial_\alpha V^{(1)} + K_\mu^{(1)}\partial_\mu\partial_\alpha V^{(0)}\right] + \dots
\end{aligned}$$

Vacuum structure

The minimization of the potential requires

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Therefore we have the relations :

$$2u^{(1)}\left(\widetilde{g}\widetilde{\boldsymbol{K}}^{(0)}\right)_\alpha = \partial_\alpha V^{(1)} + \left(\mathcal{M}^{(0)} - 2u^{(0)}\widetilde{g}\right)^{\alpha\mu} K^{(1)}_\mu$$

$$\widetilde{\boldsymbol{K}}^\intercal \widetilde{g} \widetilde{\boldsymbol{K}} = 0 \quad \Longrightarrow \quad 2\widetilde{\boldsymbol{K}}^{(0)\intercal}\widetilde{g}\widetilde{\boldsymbol{K}}^{(1)} = 0$$

Effective minimum and masses

We can define

$$D^a := \gamma^{a\mu}\partial_\mu$$

Finally, we obtain

$$\boxed{\begin{aligned} \bar{K}_a^{(1)} &= -\frac{\sqrt{2K_0^{(0)}}\,\bar{D}_a V^{(1)}}{m_a^{2\,(0)}} \\ u^{(1)} &= \frac{1}{4K_0^{(0)}}\left[2\partial_0 V^{(1)} + \frac{1}{2}\boldsymbol{f}_\pm^\intercal \boldsymbol{k}^{(1)}\right] \\ \left(m_a^2\right)^{(1)} &= \bar{D}_a\bar{D}_a V^{(1)} + \frac{K_0^{(1)}}{K_0^{(0)}}\left(m_a^2\right)^{(0)} + \bar{f}_\pm^a \bar{\delta}^{(1),\ a} \end{aligned}}$$

with $\boldsymbol{f}_\pm = 8K_0^{(0)}\left[\left(\eta_{00} - u^{(0)}\right)\boldsymbol{k}^{(0)} + \boldsymbol{\eta}\right]$, $\boldsymbol{k}^{(1)} = \frac{\boldsymbol{K}^{(1)}}{K_0^{(0)}}$ and $\bar{\boldsymbol{\delta}}^{(1)} = \bar{\boldsymbol{k}}^{(1)} - \frac{K_0^{(1)}}{K_0^{(0)}}\bar{\boldsymbol{k}}^{(0)}$.

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$$u^{(1)} = \frac{1}{4K_0^{(0)}}\left[2\,\partial_0 V^{(1)} + \frac{1}{2}\boldsymbol{f}_\pm^\intercal \boldsymbol{k}^{(1)}\right]$$

$$\left(m_a^2\right)^{(1)} = \bar{D}_a\bar{D}_a V^{(1)} + \frac{K_0^{(1)}}{K_0^{(0)}}\left(m_a^2\right)^{(0)} + \bar{f}_\pm^a\bar{\delta}^{(1),\ a}$$

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Last step

Recalling that

$$V_1(\widetilde{\boldsymbol{K}}, \mu) = \frac{1}{4} \sum_i n_i m_i^4(\widetilde{\boldsymbol{K}}) \left[\ln \left(\frac{m_i^2(\widetilde{\boldsymbol{K}})}{\mu^2} \right) - c_i \right]$$

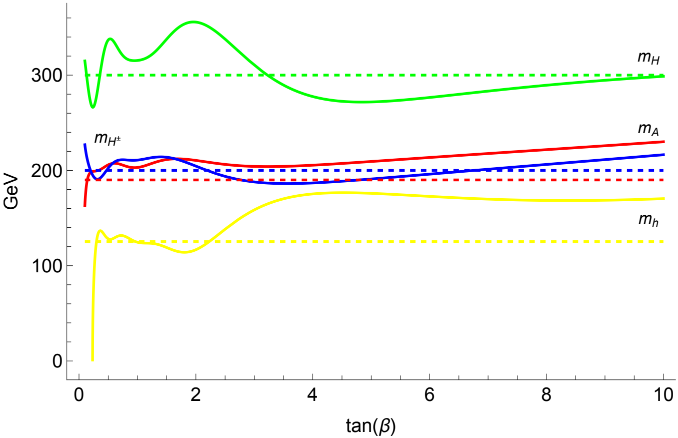
We obtain

$$\partial_\mu V_i^{(1)} = \frac{n_i}{2} \sum_{I=1}^{N} \left(\overline{\partial_\mu M^2} \right)^{II} A_i \left(\lambda_I \right)$$

where A_i are one-loop tadpole functions.

- We need to obtain bilinear derivatives of all masses.

Example of corrected masses (CP conserved type I case)



One-loop corrections to scalar masses, dashed lines are tree-level masses.

Conclusion

- Gauge invariance is manifest at each order of $\hbar$
- Simplifies vacuum stability studies and spontaneous symmetry breaking analysis at loop level
- Perspectives :
  - Implement these results (and other) in a ready to use code
  - Study of the Gildener-Weinberg mechanism
  - Other phenomenological applications

Gauge-invariant bilinears

We construct 4 real gauge-invariant bilinears

$$K_0 = \varphi_1^\dagger \varphi_1 + \varphi_2^\dagger \varphi_2 \ , \qquad K_1 = \varphi_1^\dagger \varphi_2 + \varphi_2^\dagger \varphi_1 \ ,$$
$$K_2 = i(\varphi_2^\dagger \varphi_1 - \varphi_1^\dagger \varphi_2) \ , \qquad K_3 = \varphi_1^\dagger \varphi_1 - \varphi_2^\dagger \varphi_2$$

with a one-to-one correspondence between the K's and the physical d.o.f.

We can define our new parameters as

$$\xi_0 = \frac{1}{2}\left(m_{11}^2 + m_{22}^2\right) , \quad \boldsymbol{\xi} = (\xi_a) = \frac{1}{2}\left(-2\mathrm{Re}(m_{12}^2),\ 2\mathrm{Im}(m_{12}^2),\ m_{11}^2 - m_{22}^2\right)^\mathsf{T} ,$$
$$\eta_{00} = \frac{1}{4}\left(\frac{1}{2}(\lambda_1 + \lambda_2) + \lambda_3\right) , \quad \boldsymbol{\eta} = (\eta_a) = \frac{1}{4}\left(\mathrm{Re}(\lambda_6 + \lambda_7),\ -\mathrm{Im}(\lambda_6 + \lambda_7),\ \frac{1}{2}(\lambda_1 - \lambda_2)\right)^\mathsf{T} ,$$
$$\boldsymbol{E} = (E_{ab}) = \frac{1}{4}\begin{pmatrix} \lambda_4 + \mathrm{Re}(\lambda_5) & -\mathrm{Im}(\lambda_5) & \mathrm{Re}(\lambda_6 - \lambda_7) \\ -\mathrm{Im}(\lambda_5) & \lambda_4 - \mathrm{Re}(\lambda_5) & -\mathrm{Im}(\lambda_6 - \lambda_7) \\ \mathrm{Re}(\lambda_6 - \lambda_7) & -\mathrm{Im}(\lambda_6 - \lambda_7) & \frac{1}{2}(\lambda_1 + \lambda_2) - \lambda_3 \end{pmatrix} .$$

THDM potential in bilinears

The tree-level potential is then given as

$$V^{(0)}(\widetilde{\boldsymbol{K}}) = K_0\xi_0 + K_a\xi_a + K_0^2\eta_{00} + 2K_0K_a\eta_a + K_aK_bE_{ab}$$

One can combine the bilinears and parameters as

$$\widetilde{\boldsymbol{K}} = \begin{pmatrix} K_0 \\ \boldsymbol{K} \end{pmatrix}, \quad \widetilde{\boldsymbol{\xi}} = \begin{pmatrix} \xi_0 \\ \boldsymbol{\xi} \end{pmatrix}, \quad \widetilde{\boldsymbol{E}} = \begin{pmatrix} \eta_{00} & \boldsymbol{\eta}^\intercal \\ \boldsymbol{\eta} & \boldsymbol{E} \end{pmatrix},$$

so that the potential is written

$$V^{(0)}(\widetilde{\boldsymbol{K}}) = \widetilde{\boldsymbol{K}}^\intercal\widetilde{\boldsymbol{\xi}} + \widetilde{\boldsymbol{K}}^\intercal\widetilde{\boldsymbol{E}}\widetilde{\boldsymbol{K}}$$

- Conditions : $K_0 \geq 0$ & $K_0^2 \geq \boldsymbol{K}^2$

Vacuum structure

- Unbroken electroweak symmetry if $K_0 = 0$

$$K_0 = 0$$

- Charge-breaking minimum if $K_0^2 > \boldsymbol{K}^2$

$$K_0 > 0 \quad \& \quad K_0^2 > \boldsymbol{K}^2 \quad \& \quad \frac{\partial V}{\partial K^\mu} \equiv \partial_\mu V = 0$$

- Charge-conserving minimum if

$$K_0 > 0 \quad \& \quad K_0^2 = \boldsymbol{K}^2 \quad \& \quad \partial_\mu V = 2u \left(\widetilde{g} \widetilde{\boldsymbol{K}} \right)_\mu$$

with u the Lagrange multiplier and $\widetilde{g} = \mathrm{diag}\,(1, -\mathbb{1}_3)$

Mass expressions

In fact we are really interested in the diagonal congruent matrix of

$$M_S^2 = \Delta^\mu \partial_\mu V + \Gamma \mathcal{M} \Gamma^\intercal = 2u \left(\widetilde{g}\widetilde{\boldsymbol{K}}\right)_\mu \Delta^\mu + \Gamma \mathcal{M} \Gamma^\intercal$$

It's useful to define the canonical basis such that

$$\hat{\Gamma} = U_c \Gamma = \begin{pmatrix} 0_{5\times 4} \\ \gamma_{3\times 4} \end{pmatrix} \implies \hat{\Gamma}\mathcal{M}\hat{\Gamma}^\intercal = \begin{pmatrix} 0_{5\times 5} & 0_{5\times 3} \\ 0_{3\times 5} & \gamma \mathcal{M} \gamma^\intercal \end{pmatrix}$$

and thus we have

$$\widehat{M_S^2} = U_c M_S^2 U_c^\intercal = \begin{pmatrix} 0_{3\times 3} & & \\ & \widehat{M^2_{\text{charged}}} & \\ & & \widehat{M^2_{\text{neutral}}} \end{pmatrix}$$

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Effective potential in terms of bilinears

Quantum corrections modify the potential as

$$V_{\text{eff}}(\widetilde{\boldsymbol{K}}) = V^{(0)}(\widetilde{\boldsymbol{K}}) + \kappa V^{(1)}(\widetilde{\boldsymbol{K}}, \mu) + \dots$$

with

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The Coleman-Weinberg one-loop correction is given by

$$\boxed{V_1(\widetilde{\boldsymbol{K}}, \mu) = \frac{1}{4} \sum_i n_i m_i^4(\widetilde{\boldsymbol{K}}) \left[\ln\left(\frac{m_i^2(\widetilde{\boldsymbol{K}})}{\mu^2} \right) - c_i \right]}$$

where

$$n_i = (-1)^{2s_i}(2s_i + 1) \implies n_s = 1, \quad n_f = -2, \quad n_g = 3$$

$$\overline{\text{MS}} \text{ scheme} \implies c_s = \frac{3}{2}, \quad c_f = \frac{3}{2}, \quad c_g = \frac{5}{6}$$

Last step

We have

$$\partial_\mu V_i^{(1)} = \frac{n_i}{2} \sum_{I=1}^{N} \left(\overline{\partial_\mu M^2}\right)^{II} A_i\left(\lambda_I\right)$$

$$\partial_\mu \partial_\nu V_i^{(1)} = \frac{n_i}{2} \left[\sum_{I=1}^{N} \left(\overline{\partial_\mu \partial_\nu M^2}\right)^{II} A_i\left(\lambda_I\right) + \sum_{I=1}^{N} \sum_{J=1}^{N} \left(\overline{\partial_\mu M^2}\right)^{IJ} \left(\overline{\partial_\nu M^2}\right)^{JI} B_i\left(\lambda_I, \lambda_J\right) \right]$$

where the functions A_i and B_i are given by

$$A_s(x) = A_f(x) \equiv A(x) = x \left[\log\left(\frac{x}{\mu^2}\right) - 1\right], \qquad A_g(x) = x \left[\log\left(\frac{x}{\mu^2}\right) - \frac{1}{3}\right],$$

$$B_s(x,y) = B_f(x,y) \equiv B(x,y) = \frac{A(x) - A(y)}{x - y}, \qquad B_g(x,y) = \frac{A_g(x) - A_g(y)}{x - y},$$

$$B(x,x) = \frac{dA}{dx}(x) = \log\left(\frac{x}{\mu^2}\right), \qquad B_g(x,x) = \frac{dA_g}{dx}(x) = \log\left(\frac{x}{\mu^2}\right) + \frac{2}{3}$$

Example : gauge contribution

Gauge masses are given from

$$M_W^2 = \frac{1}{2} K_0 \left(g_+^2 - g_-^2 \right), \qquad M_{Z,\gamma}^2 = \frac{1}{2} \left[K_0 g_+^2 \pm \sqrt{g_+^4 \boldsymbol{K}^2 + \left(K_0^2 - \boldsymbol{K}^2 \right) g_-^4} \right]$$

Therefore we have

$$\partial_\mu V_g^{(1)} = \frac{3}{2} \Big\{ 2 g_{\mu WW} \, A_g \left(M_W^2 \right) + g_{\mu ZZ} \, A_g \left(M_Z^2 \right) \Big\}$$

$$g_{\mu WW} = \frac{M_W^2}{K_0} \delta^{\mu 0}, \quad g_{\mu ZZ} = \frac{M_Z^2}{2 K_0^2} \left[K^\mu + \cos^2(2\theta_W) \big(\widetilde{g} \widetilde{\boldsymbol{K}} \big)^\mu \right]$$

And in the good basis

$$\bar{D}_a V_g^{(1)} = \frac{3}{2} \Big\{ 2 \bar{g}_{aWW} \, A_g \left(M_W^2 \right) + \bar{g}_{aZZ} \, A_g \left(M_Z^2 \right) \Big\}$$

$$\bar{g}_{iAB} = \bar{\Gamma}^\mu_{\ i} \, g_{\mu AB} \quad \Longrightarrow \quad \bar{g}_{aWW} = \sqrt{\frac{2}{K_0}} M_W^2 \bar{k}^a \text{ and } \bar{g}_{aZZ} = \sqrt{\frac{2}{K_0}} M_Z^2 \bar{k}^a$$