Exercises GRASPA
Astroparticle physics and Astrophysics

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1 Detection of an astrophysical source

1. The ESA astrometry mission, Gaia, is able to measure the parallax of remote stars up to 10 micro-arcsec ($10^{-5\prime\prime}$). What is the corresponding distance? How does it compare with the radius of the Galaxy?
   Answer: 100 kpc vs $\sim$ 20 kpc

2. Suppose a sun-like star is sitting far from us, at a distance of 1/10 of the previously computed one. Astronomers use the magnitude to measure the brightness of an object, usually in a given bandwidth. The apparent magnitude of a star $m_\star$ in the V(visible) band can be defined with respect to the Sun for which $m_\odot = -27$, it reads:
$$m_\star - m_\odot = -2.5 \log_{10}\left(\frac{\mathcal{F}_\star}{\mathcal{F}_\odot}\right) , \tag{1}$$
   with $\mathcal{F}_\odot$ and $\mathcal{F}_\star$ the flux of the sun and of the star measured at Earth. You can notice that the dimmer the star, the larger its magnitude. Given that, on a relatively clear sky, the limiting visibility is about 6th magnitude with the naked eye ($m_{\rm lim} \simeq +6$), is it possible to distinguish this star?
   Answer: $m_\star = +19.5 > +6$ so it is not visible

3. Now imagine that, instead of the star, there is a supernova at this specific distance. A typical supernova releases gravitational energy of 10^{53} erg, with $\sim$99% carried by neutrinos, about $\sim$1% released as kinetic energy of the ejecta, and $\sim$0.01% into photons. Assuming that this energy is released within the first several months (say 100 days) of its life, estimate the photon flux at Earth. Would such a supernova be visible with naked eye during a night sky? How does it compare with the magnitude of Jupiter of -2.7?
   (The solar flux outside the atmosphere, so-called *solar constant* is $\mathcal{F}_\odot = 1372\ \mathrm{W\,m^{-2}}$.)
   Answer: $F_{\rm SN} \simeq 10^{-4}\,\mathrm{erg\,s^{-1}\,cm^{-2}} = 10^{-7}\,\mathrm{W\,m^{-2}}$ and $m_{\rm SN} = -1.7$ namely slightly less bright than Jupiter

4. (**Bonus**) Cosmic rays (CRs) may be accelerated in supernova shocks, converting $\sim$10% of the kinetic energy of the SN. When accelerated, CRs interact with the ambient medium and produce secondary particles, charged pions $\pi^+\pi^-$ which subsequently decay into leptons and neutrinos, and neutral pions π^0 that decay into γ rays. γ rays and neutrinos are among the secondary particles that are stable and propagate over large distances. Assume that the luminosity in γ rays is constant during 10 kyr and that they carry approximately 10^{-7} of the energy of the accelerated CRs. Would Fermi-LAT be able to detect a γ-ray flux at 1 GeV within the first 100 days of a Galactic SN at 10 kpc? Use the following Fig. 1 to compare your result with the Fermi-LAT point source sensitivity, assuming that the threshold flux scales as:
$$F(t) = F_0\left(\frac{\Delta t}{T_0}\right)^{-1/2} , \tag{2}$$
   where F is the threshold flux for an event of duration Δt. T_0 is the full data taking period which is according to Fig. 1, 10 years. In fact, the γ-rays luminosity of an SN is probably several orders of magnitude higher at such an early stage.
   Answer: for $\Delta T = 100$ d and $T_0 = 10$ yr the threshold flux is a factor of 0.2 less compared to the one given in the plot (in other words the sensitivity curve of the plot has increased by a factor of 5), and the flux from the SN is $10^{-14}\,\mathrm{erg\,s^{-1}\,cm^{-2}}$ and the Fermi sensitivity $\sim 10^{-12}\,\mathrm{erg\,cm^{-2}\,s^{-1}}$, so it will not be detected within the first 100 d

5. (**Bonus**) Assuming that the neutrinos produced by inelastic collisions carry an equal amount of power as the γ rays, would IceCube be able to detect any at neutrino energies of 1 TeV? Use the following graph (Fig. 2).
   Answer: the neutrino flux is $10^{-5}\,\mathrm{erg\,s^{-1}\,cm^{-2}}$ at 1 GeV, and $6\times10^{-18}\,\mathrm{TeV^{-1}\,s^{-1}\,cm^{-2}}$ at 1 TeV assuming a flux dependence of E^{-2}

2 Measuring the Hubble constant H_0

For the following exercise, download the jupyter notebook H0-estimation-exercise.ipynb from the detailed timetable of the school. Follow the instructions therein to calculate the Hubble constant using the data provided of SN Ia supernovae. You will need to download and use the auxiliary file SCPUnion2.1_AllSNe.txt that contains the observational data of the supernovae.

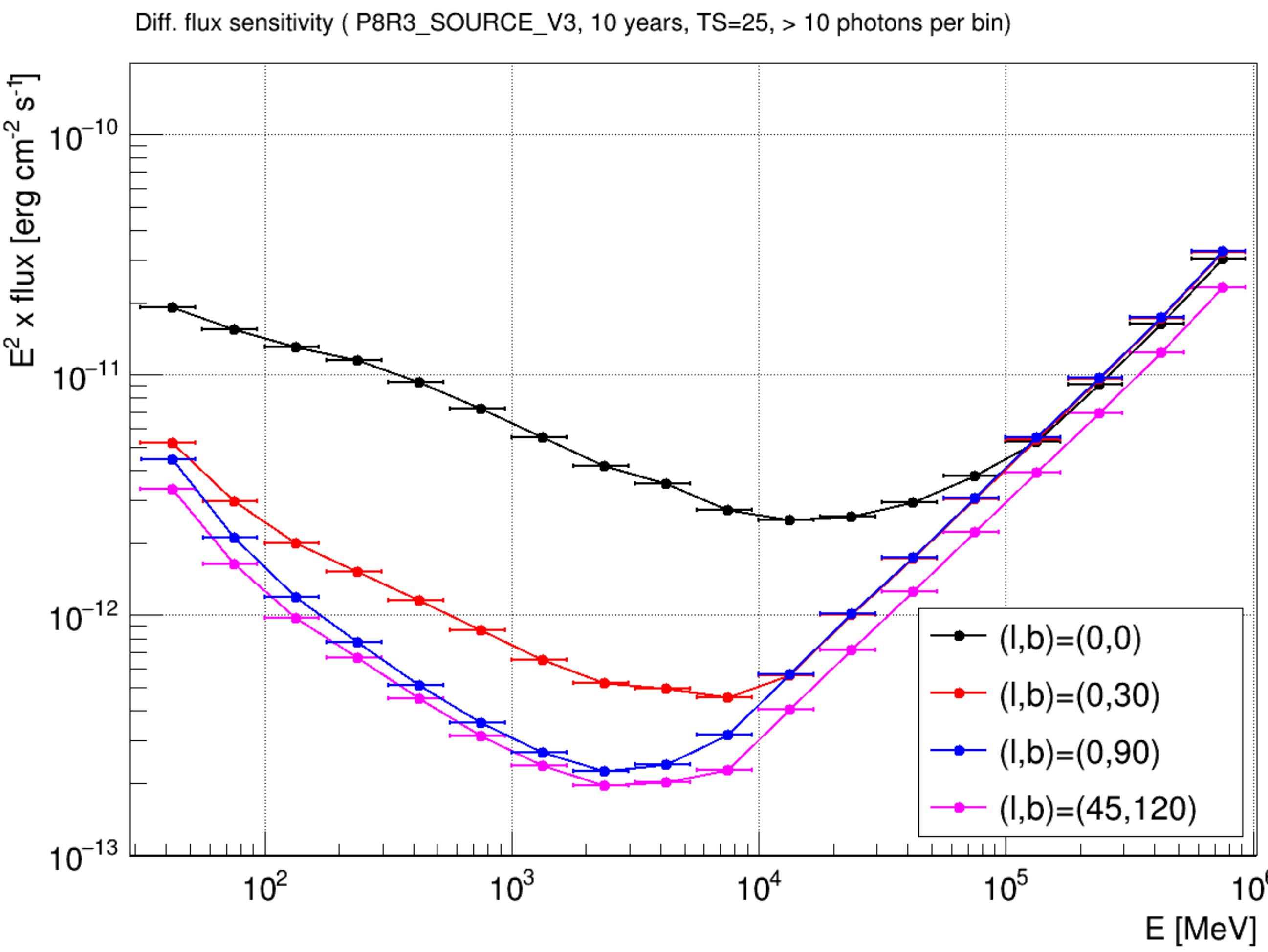

Figure 1: The Fermi-LAT point source sensitivity after 10 years of observations.
Adapted from https://www.slac.stanford.edu/exp/glast/groups/canda/lat_Performance.htm

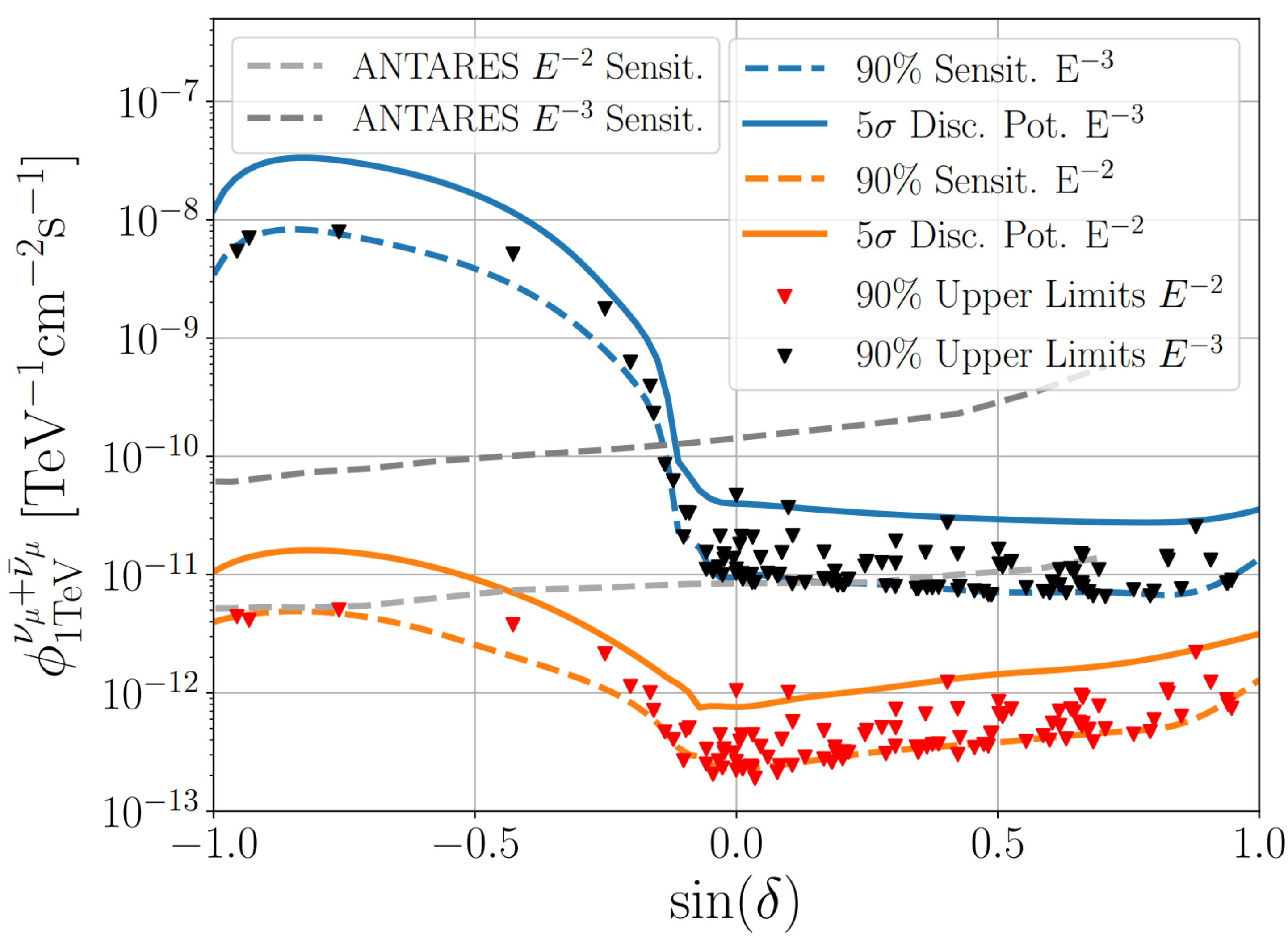

Figure 2: The IceCube point source sensitivity at neutrino energy of 1 TeV after 10 years of operations. Extracted from https://arxiv.org/pdf/1910.08488.pdf

3 Newtonian toy cosmology

1. **First Friedman equation**

   Consider a uniform sphere of pressureless dust expanding radially outwards, where dust refers to matter whose pressure is negligible with respect to its energy density. At time t, the sphere has density $\rho(t)$ and an expanding radius $R(t) = a(t)R_0$, where $a(t)$ describes the expansion and is called *scale factor*.

   Starting from the total conserved energy for a point of mass m on the surface of the sphere

   $$T + U \equiv -\frac{mc^2kR_0^2}{2},$$

   where the constant is written like so for dimensional reasons, T is the kinetic energy, and U is the gravitational potential, obtain a relation for $H^2 \equiv \left(\frac{\dot{a}}{a}\right)^2$. On the right-hand side, c is the speed of light and k is a constant related to the curvature, which can be positive, null, or negative. The equation you will obtain is commonly known as the first Friedman equation, and it describes the kinematics of the Universe.

   Answer: The kinetic energy and the potential for the test particle are:

   $$T = \frac{1}{2}m\dot{a}^2R_0^2, \tag{3}$$

   $$U = -G\frac{mM}{aR_0}, \tag{4}$$

   where $M = \frac{4}{3}\pi\rho a^3R_0^3$ is the total mass enclosed in the sphere.

   By substituting these two expressions, we obtain:

   $$\frac{1}{2}m\dot{a}^2R_0^2 - \frac{4}{3}\pi G\rho ma^2R_0^2 + \frac{1}{2}mc^2kR_0^2 = 0, \tag{5}$$

   then we finally derived:

   $$\left(\frac{\dot{a}}{a}\right)^2 + \frac{c^2k}{a^2} = \frac{8}{3}\pi G\rho. \tag{6}$$

2. **Second Friedman equation**

   Now, considering that the sphere is adiabatically expanding, differentiate the first law of thermodynamics

   $$\mathrm{d}U = -p\,\mathrm{d}V$$

   where the total internal energy is $U = V\rho c^2$ with respect to a and write down a relation for $\frac{\mathrm{d}\rho}{\mathrm{d}a}$. This equation is equivalent to the second Friedman equation and relates the expansion of the Universe to its content. If you want to write the second Friedman equation, differentiate with respect to t the first equation, substitute the equation you have just derived to find a relation for $\frac{\ddot{a}}{a}$.

   Answer:

   We rewrite the differential equation into

   $$\rho c^2dV + c^2Vd\rho = -pdV,$$

   $$\left(\rho + \frac{p}{c^2}\right)dV + Vd\rho = 0, \tag{7}$$

   $$\frac{4}{3}\pi a^3d\rho + \left(\rho + \frac{p}{c^2}\right)\frac{4}{3}\pi 3a^2da = 0,$$

   $$\frac{d\rho}{da} + \left(\rho + \frac{p}{c^2}\right)\frac{3}{a} = 0. \tag{8}$$

   In addition, to get the time dependence of this second Friedman equation, we first use the chain rule to get

   $$\frac{d\rho}{dt}\frac{1}{\dot{a}} + \left(\rho + \frac{p}{c^2}\right)\frac{3}{a} = 0. \tag{9}$$

   From Eq. (6), we have

   $$\frac{8\pi G}{3}\frac{d\rho}{dt}\frac{1}{\dot{a}} = \frac{2}{a}\left(\frac{\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{c^2k}{a^2}\right), \tag{10}$$

   $$= \frac{2}{a}\left(\frac{\ddot{a}}{a} - \frac{8}{3}\pi G\rho\right), \tag{11}$$

   where we use again the first Friedman equation on the second step.

   Substituting Eq. (11) into Eq. (9), we can finally have

   $$\frac{\ddot{a}}{a} + \frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right) = 0. \tag{12}$$

Component	EoS (w value)	$\rho(a)$	$a(t)$
Matter	0		
Radiation	1/3		
Cosmological constant	-1		

Table 1: Solution to the toy cosmology equations

3. **Evolution of the components of the Universe**

Assuming the general equation of state $p = w\,\rho\,c^2$ (with $w \neq w(a)$), and using the method of variable separation, solve the differential equation for $\frac{\mathrm{d}\rho}{\mathrm{d}a}$ to find how the density depends on the scale factor. Then, assuming a flat universe ($k = 0$), substitute in the first Friedman equation and solve it to find how the scale factor depends on time.

Finally, fill in Table 1.

Answer:

From Eq. (8), we have

$$\frac{d\rho}{da} + (1+w)\,\rho\frac{3}{a} = 0. \tag{13}$$

The solution to this differential equation can be obtained by variable separation and is

$$\rho = \rho_0 a^{-3(1+w)}, \tag{14}$$

where the ρ_0 is a constant.

For different energy components, we have

$$\rho_{\text{matter}}(a) \propto a^{-3}, \tag{15}$$
$$\rho_{\text{rad}}(a) \propto a^{-4}, \tag{16}$$
$$\rho_{\text{const}}(a) \propto 1., \tag{17}$$

As for the time dependence, we replace the above relation into Eq. (6), and get the following differential equation

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{8}{3}\pi G\rho_0\right)^{\frac{1}{2}} a^{-\frac{1+3w}{2}}, \tag{18}$$

which we can again solve by variable separation and obtain these solutions

$$a_{\text{matter}}(t) \propto t^{2/3}, \tag{19}$$
$$a_{\text{rad}}(t) \propto a^{1/2}, \tag{20}$$
$$a_{\text{const}}(t) \propto e^{H_0 t}, \tag{21}$$

where H_0 is the Hubble constant.