Adam Falkowski

ON-SHELL APPROACH TO GRAVITATIONAL RADIATION BEYOND GR

Talk at the TUG 2025 conference, IPhT Saclay

16 October 2025

Plan:

1. Quantum Amplitudes and Gravitational Waves

2. Radiation in Scalar-Tensor Theories

AA, Marinellis
2407.16457
2411.12909

``````
````````

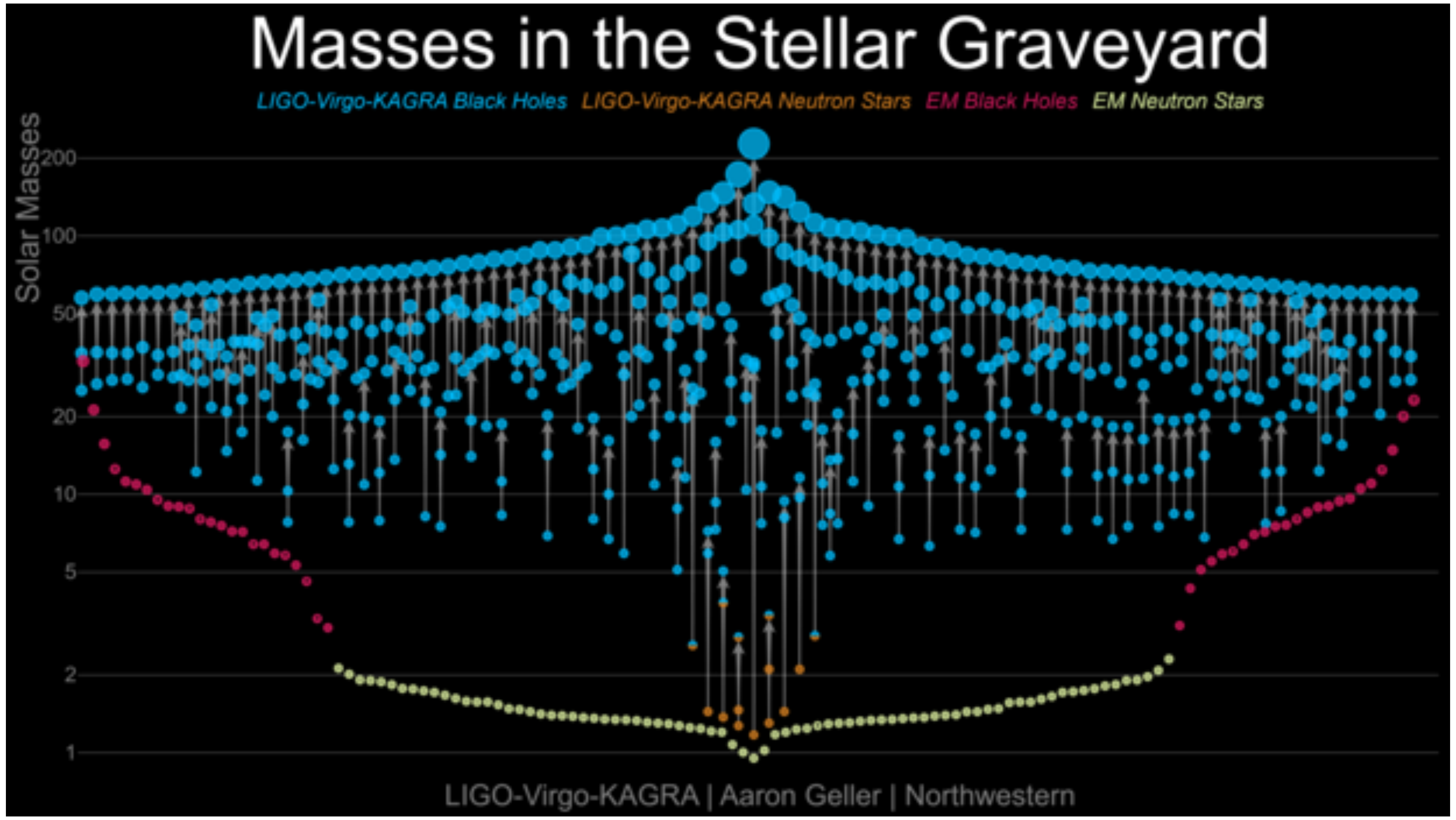
....
....
,,,,,,
,,,,,,
....
......

Introduction: gravitational waves detection

Image Credit: EGO*

LIGO: First detection of **Gravitational Waves** (GWs) in **2015**

Many more black hole and neutron star mergers discovered by the LIGO-VIRGO-KAGRA collaboration



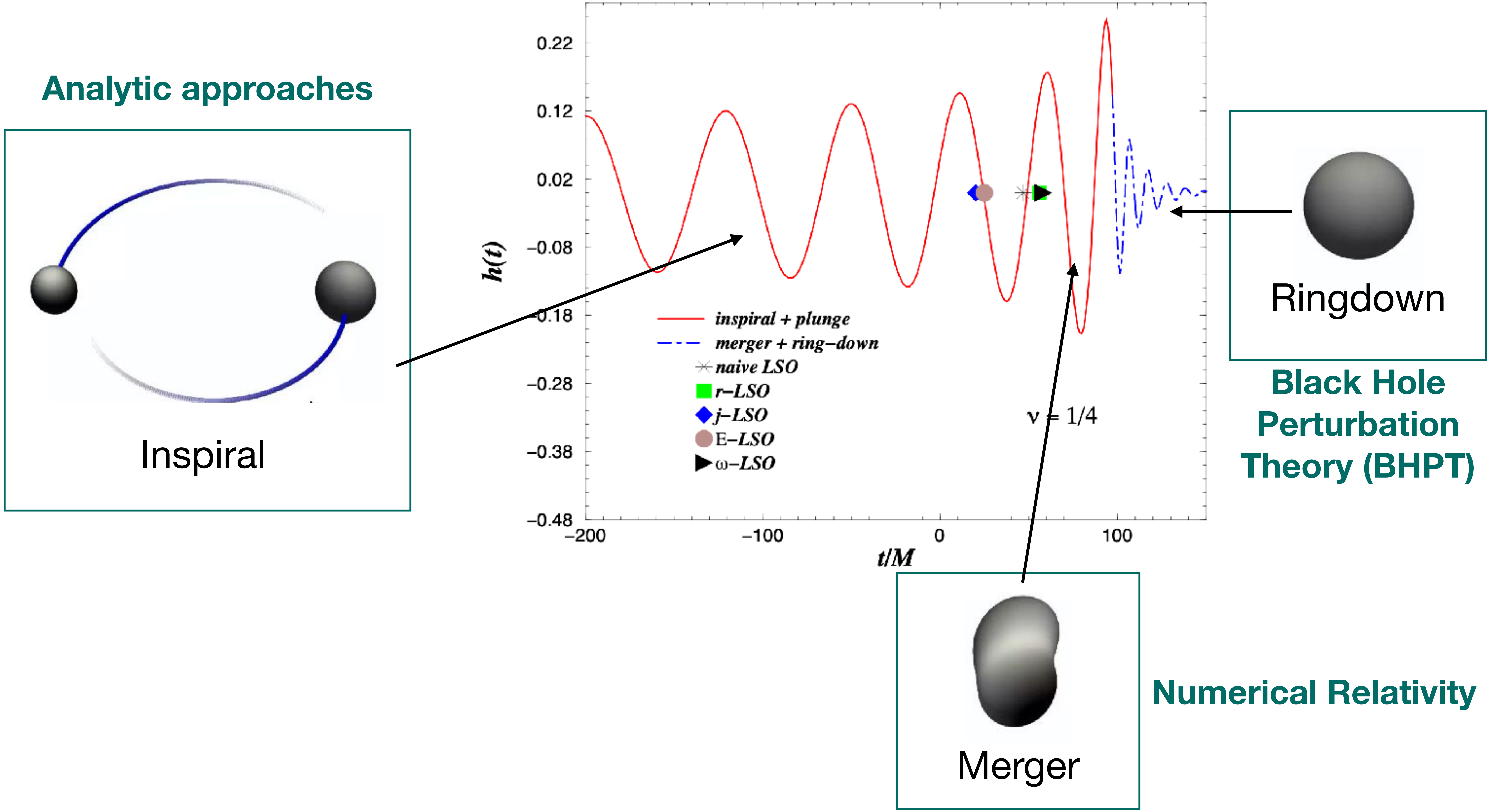
Credit:LIGO-Virgo/Aaron Geller/Northwestern University*

New era of precision measurements of GWs

Need for highly accurate GW templates, especially in view of the future upgrades of the LIGO/VIRGO/KAGRA network and future missions such as LISA, Einstein Telescope, Cosmic Explorer, Decigo, Tian-Qin, GEO-HF

New window to test physics beyond general relativity (GR)!

Introduction: gravitational waves detection



Analytic approaches

Inspiral

Ringdown

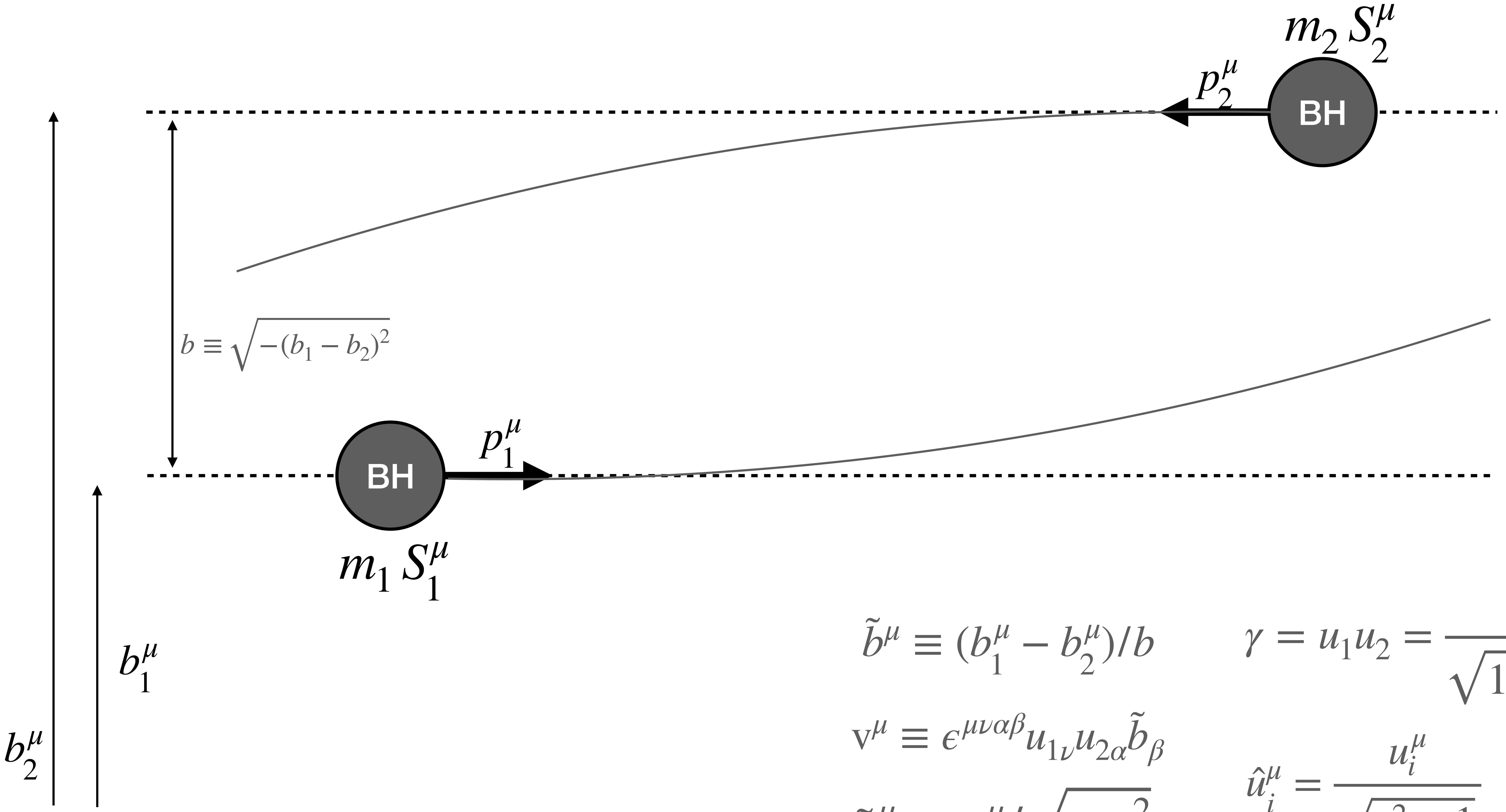
Black Hole Perturbation Theory (BHPT)

Merger

Numerical Relativity

Illustration Credit: Max-Planck-Gesellschaft*

Quantum Amplitudes and Classical Observables



$$u_i^\mu = p_i^\mu / m_i$$
$$a_i^\mu = S_i^\mu / m_i$$

$$\tilde{b}^\mu \equiv (b_1^\mu - b_2^\mu)/b$$
$$\mathrm{v}^\mu \equiv \epsilon^{\mu\nu\alpha\beta} u_{1\nu} u_{2\alpha} \tilde{b}_\beta$$
$$\tilde{\mathrm{v}}^\mu \equiv \mathrm{v}^\mu / \sqrt{-\mathrm{v}^2}$$

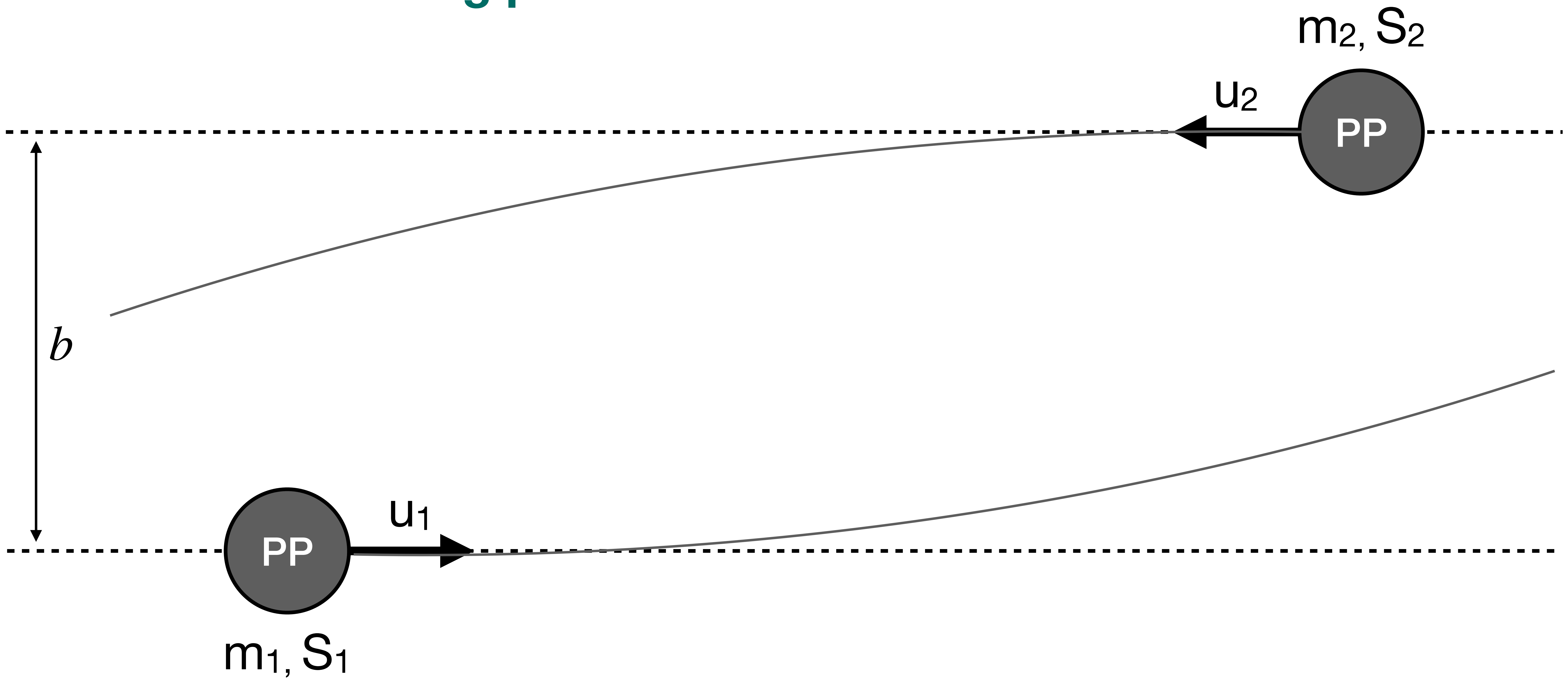
$$\gamma = u_1 u_2 = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$
$$\hat{u}_i^\mu = \frac{u_i^\mu}{\sqrt{\gamma^2 - 1}}$$

$\hbar = c = 1$

Quantum Amplitudes
and Gravitational Radiation

Quantum Amplitudes and Classical Observables

- **Classical scattering problem in GR**



Large separation:
$R_s \ll b$

Arkani-Hamed, Huang, O'Connell
arXiv:1906.10100

Black holes represented by point particles
whose interactions are those of a spin-S particle minimally coupled to gravity
Observables in the scattering processes can be extracted from classical limits of S matrix elements

Quantum Amplitudes and Classical Observables

Why employ quantum amplitudes for classical calculations?

- Computations organized in perturbative expansion with Lorentz covariance preserved at each step
- Often, analytic results in places where only numerical results previously available
- Can exploit many modern techniques used in particle physics to simplify calculation
- Easy to include spin of the colliding objects
- Can be easily applied to relevant theories such as QED or GR
- **Can be straightforwardly extended to beyond GR predictions**

One downside is that amplitudes naturally give scattering observables, while phenomenologically bound systems are more relevant.
The problem of bound-to-boundary continuation is not fully solved yet
Alternative formalism of worldline EFT where this particular problem is absent

KMOC formalism

One can define GR radiation observables as vacuum expectation values of metric field operators and its derivatives

$$\mathscr{R}_h \equiv {}_{\mathrm{out}}\langle\psi|\, h^{\mu\nu}(x)\,|\psi\rangle_{\mathrm{out}}\epsilon^-_{\mu\nu}$$

strain

Given radiation observable $\mathscr{R}_h$ one defines **waveform** W_h as

$$\mathscr{R}_h(x) = \frac{W_h(t)}{|\boldsymbol{x}|} \qquad |\boldsymbol{x}| \to \infty \qquad t \equiv x^0 - |\boldsymbol{x}|$$

retarded time

Furthermore one defines **spectral waveform** or waveshape f_h as Fourier transform:

$$f_h(\omega) = \int dt e^{i\omega t} W_h(t)$$

KMOC formalism

Kosower, Maybee, O'Connell
[arXiv:1811.10950]

Cristofoli, R. Gonzo, D. A. Kosower, D. O'Connell
[arXiv:2107.10193]

$$\boxed{\mathcal{R}_h \equiv {}_{\text{out}}\langle\psi|\,h^{\mu\nu}(x)\,|\psi\rangle_{\text{out}}\epsilon^{-}_{\mu\nu}}$$

$$|\psi\rangle_{\text{out}} = S\,|\psi\rangle_{\text{in}} \quad \Rightarrow \quad \mathcal{R}_h = {}_{\text{in}}\langle\psi|\,S^{\dagger}h^{\mu\nu}(x)S\,|\psi\rangle_{\text{in}}\epsilon^{-}_{\mu\nu}$$

"in-in observable"

$$|\psi\rangle_{\text{in}} = \Pi_{i=1,2}\Big[\int d\Phi(p_i) f_i(p_i) e^{ip_ib_i}\Big]\,|p_1p_2\rangle_{\text{in}}$$

The radiation observables depends on an amplitude-like object

$$h_{\mu\nu} \sim \int d\Phi_k\, a_{\text{in}}(k)\, e^{-ikx} + \text{h.c} \quad \Rightarrow \quad \mathcal{R}_h \supset {}_{\text{in}}\langle\psi|\,S^{\dagger}a_{\text{in}}(k)S\,|\psi\rangle_{\text{in}} \equiv \text{Exp}_k$$

"generalized amplitude"
distinct from ordinary S matrix elements

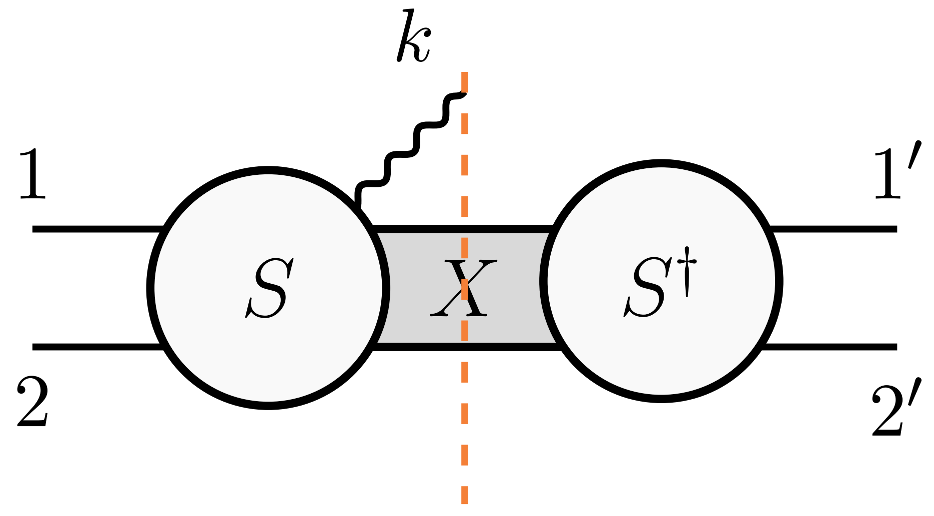
$${}_{\text{out}}\langle\psi'|\psi\rangle_{\text{in}} = {}_{\text{in}}\langle\psi'|\,S^{\dagger}\,|\psi\rangle_{\text{in}}$$

Caron-Huot, Giroux, Hannesdottir, Mizera
arXiv:2310.12199

It can be related to usual S matrix elements for the 2-to-3 process $12 \rightarrow 12k_h$

$${}_{\text{in}}\langle\psi|\,S^{\dagger}a_{\text{in}}(k)S\,|\psi\rangle_{\text{in}} = \int d\Phi_X \;\; {}_{\text{in}}\langle\psi|\,S^{\dagger}\,|X\rangle_{\text{in}}\;{}_{\text{in}}\langle X|\,a_{\text{in}}(k)S\,|\psi\rangle_{\text{in}} = \int d\Phi_X \;\; {}_{\text{in}}\langle\psi|\,S^{\dagger}\,|X\rangle_{\text{in}}\;{}_{\text{in}}\langle Xk|\,S\,|\psi\rangle_{\text{in}}$$

$\text{Exp}_k =$



In the following the "in" label dropped to reduce clutter

Quantum Amplitudes and Radiation Observables

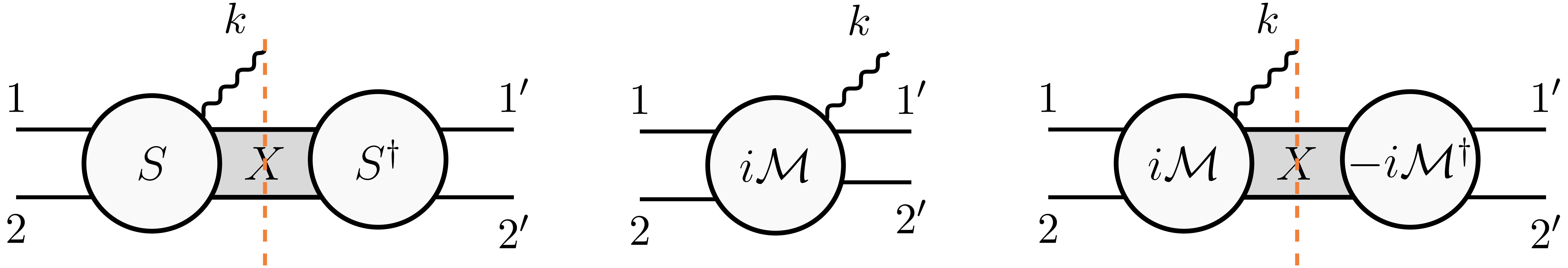
$$R_h \sim \int d\Phi_X \;\; {}_{\text{in}}\langle \psi | S^\dagger | X \rangle_{\text{in}} \; {}_{\text{in}}\langle X k | S | \psi \rangle_{\text{in}}$$

In terms of usual amplitudes $S = 1 + i\delta^4(p)\mathscr{M}$

5-point amplitude

cut terms

$$R_h \sim \langle \psi k | \mathscr{M} | \psi \rangle + \int d\Phi_X \;\; \langle \psi | \mathscr{M}^\dagger | X \rangle \langle X k | \mathscr{M} | \psi \rangle$$



$$\text{Exp}_k \;=\; i\mathcal{M} \;+\; \text{Cut}_{1'2'}\,.$$

Well defined
object in classical limit

5-point amplitude
contains superclassical
terms $\mathscr{M} \sim e^{iS/\hbar}$

Cut terms cancel
superclassical
terms

Waveforms from amplitudes

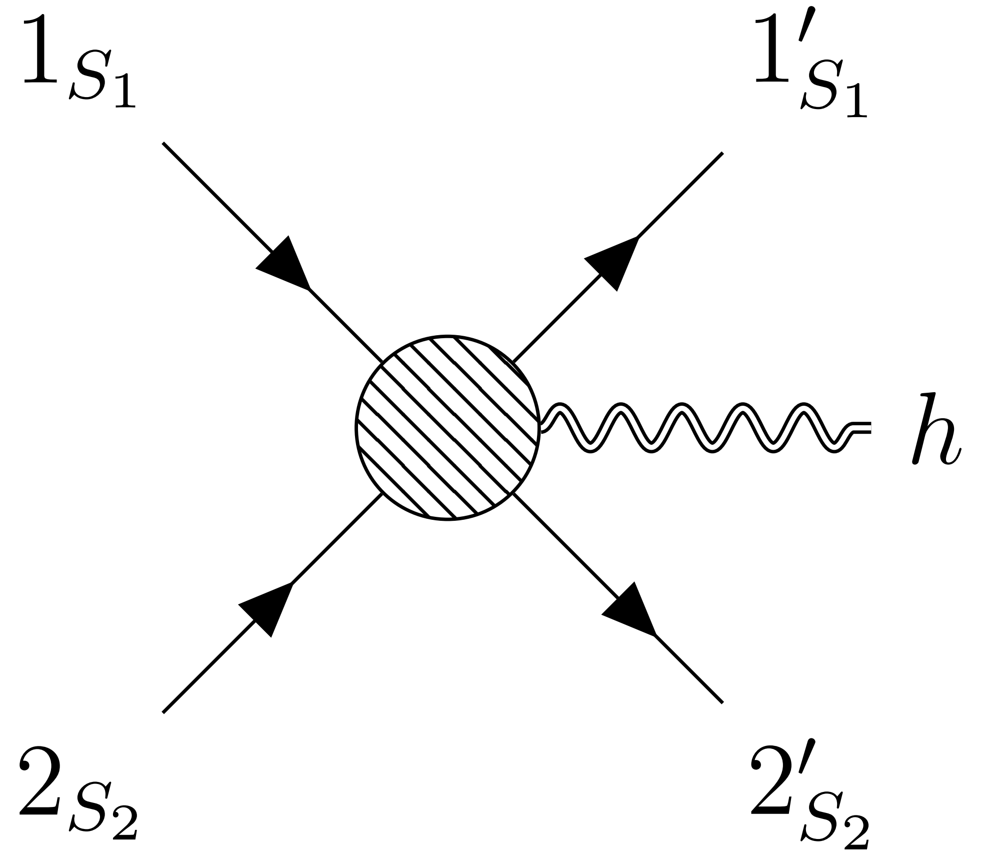
The 5-point amplitude can be calculated in loop expansion,
which corresponds to the post-minkowskian (PM) expansion of the radiation observable

At leading PM order KMOC formalism relates spectral waveform to integral of the tree-level 5-point amplitude.

$$f_h(\omega) = \frac{1}{64\pi^3 m_1 m_2} \int d\mu \mathscr{M}^{\mathrm{cl}}_{\mathrm{tree}}[p_1 + w_1, p_2 + w_2 \rightarrow p_1, p_2, k]_{|k=\omega n}$$

$$d\mu \equiv \delta^4(w_1 + w_2 - k)\Pi_{i=1,2}[e^{ib_i w_i} d^4 w_i \delta(u_i w_i)]$$

Integration measure



In fact, not full 5-point amplitude but just its certain residues are needed to calculate the integral!

$$\mathrm{R} \equiv \mathrm{Res}_{w_2^2 \rightarrow 0} \mathscr{M}^{\mathrm{cl}}_{\mathrm{tree}}[p_1 + w_1, p_2 + w_2 \rightarrow p_1, p_2, k_h]$$

De Angelis, Novichkov, Gonzo
[arXiv:2309.17429]

$$f_h(\omega) = -\frac{1}{64\pi^2 m_1 m_2 \sqrt{\gamma^2 - 1}} \int_{-\infty}^{\infty} \frac{dz e^{ib_1 k + iz(\hat{u}_1 k)b}}{\sqrt{z^2+1}} \frac{1}{2} \Big\{ R\big(w_2 \rightarrow (\hat{u}_1 k)\big[\gamma\hat{u}_2 - \hat{u}_1 + z\tilde{b} + i\sqrt{z^2+1}\tilde{\mathrm{v}}\big]\big) + R\big(w_2 \rightarrow (\hat{u}_1 k)\big[\gamma\hat{u}_2 - \hat{u}_1 + z\tilde{b} - i\sqrt{z^2+1}\tilde{\mathrm{v}}\big]\big) \Big\}$$

$$+(1 \leftrightarrow 2)$$

Quantum Amplitudes and Radiation Observables

Classical limit of R_h is the leading term
under the classical (soft) scaling

momenta of matter particles representing classical object	$p_i \to \hbar^0 p_i$
masses of matter particles	$m_i \to \hbar^0 m_i$
momentum transfer for matter particles $q_i = p'_i - p_i$	$q_i \to \hbar^1 q_i$
momenta of radiation quanta	$k_n \to \hbar^1 k_n$
spins of matter particles	$S_i \to \hbar^{-1} S_i$

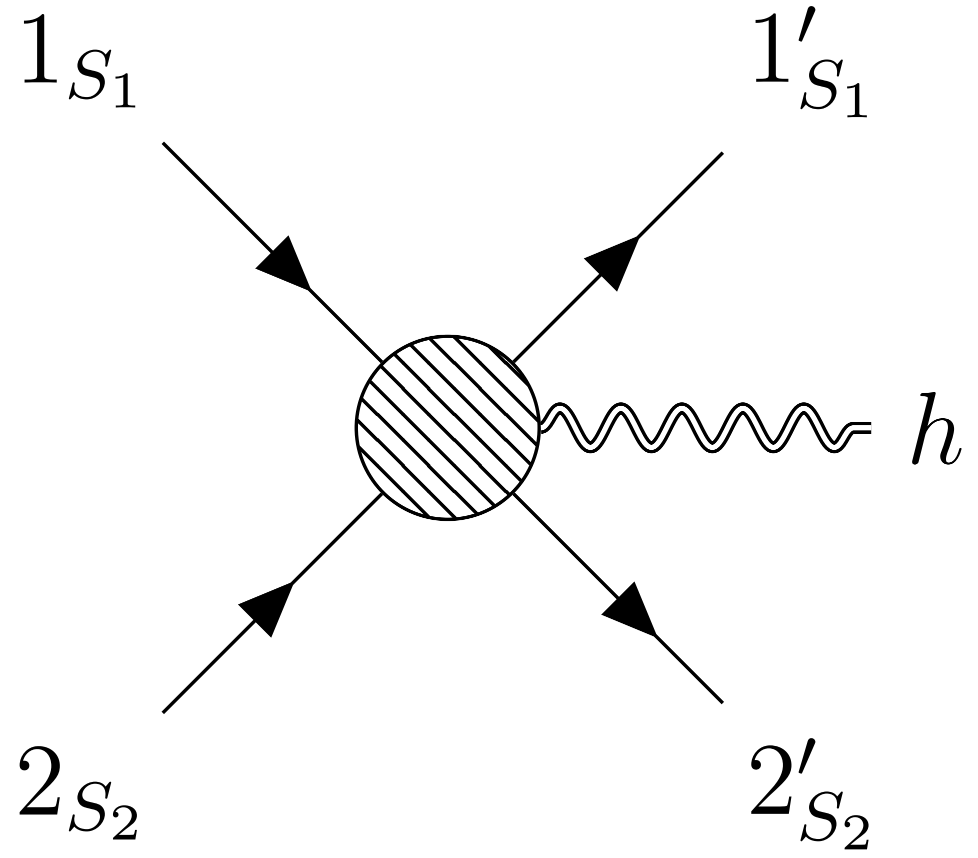
Waveforms from amplitudes

The rest is an exercise in wave function integration and extracting the leading 1/|x| behaviour

$$|\psi\rangle_{\rm in} = \int \Pi_{i=1,2}[d\Phi(p_i)f_i(p_i)e^{ip_ib_i}]\,|p_1p_2\rangle_{\rm in}$$

At leading PM order KMOC formalism relates spectral waveform to integral of 5-point amplitude.

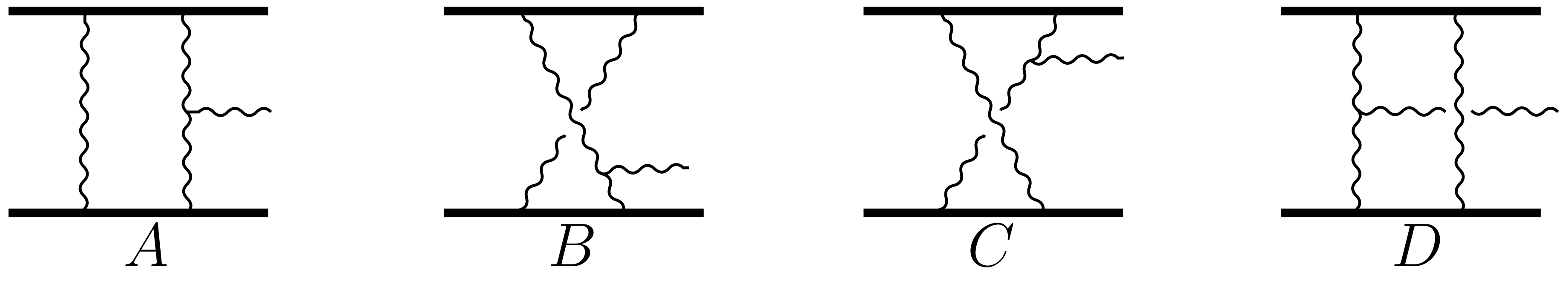
$$f_h(\omega) = \frac{1}{64\pi^3 m_1 m_2}\int d\mu\,\mathcal{M}^{\rm cl}_{\rm tree}[p_1+w_1, p_2+w_2 \to p_1, p_2, k]_{|k=\omega n}$$



$$d\mu \equiv \delta^4(w_1+w_2-k)\Pi_{i=1,2}[e^{ib_iw_i}d^4w_i\delta(u_iw_i)]$$

Integration measure

Calculation can be extended beyond tree level. Current state of the art is one loop



2303.06211
2303.06111
2303.06112
2303.07006
2310.12199

-

On-shell calculation of gravitational amplitudes

Quantum Amplitudes and Classical Observables: On-shell methods

Quantum Mechanics + Poincaré invariance, locality and unitarity

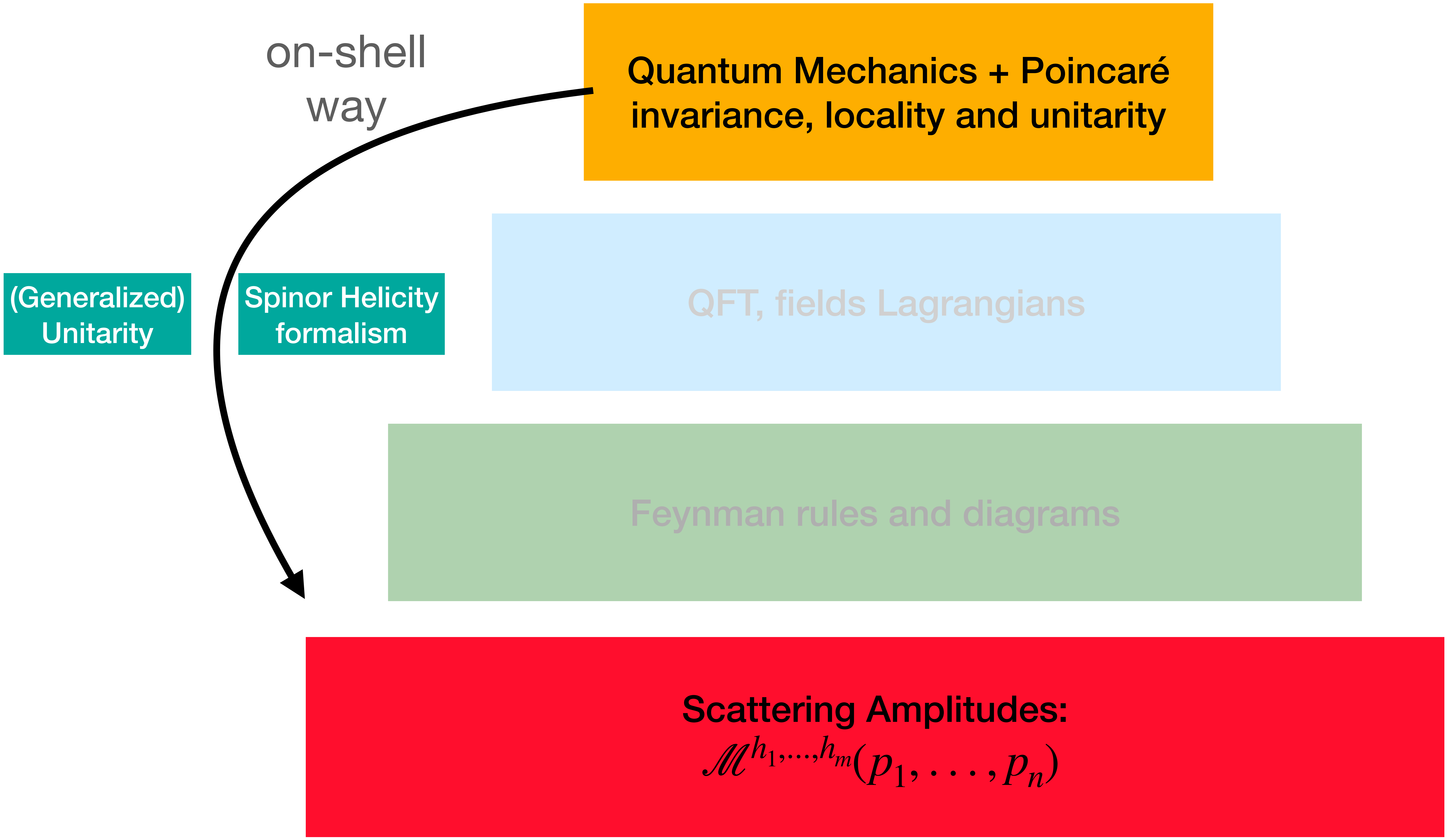
QFT, fields Lagrangians

Feynman rules and diagrams

Scattering Amplitudes:

$$\mathscr{M}^{h_1,\ldots,h_m}(p_1,\ldots,p_n)$$

Quantum Amplitudes and Classical Observables: On-shell methods



Quantum Mechanics + Poincaré invariance, locality and unitarity

on-shell way

(Generalized) Unitarity

Spinor Helicity formalism

QFT, fields Lagrangians

Feynman rules and diagrams

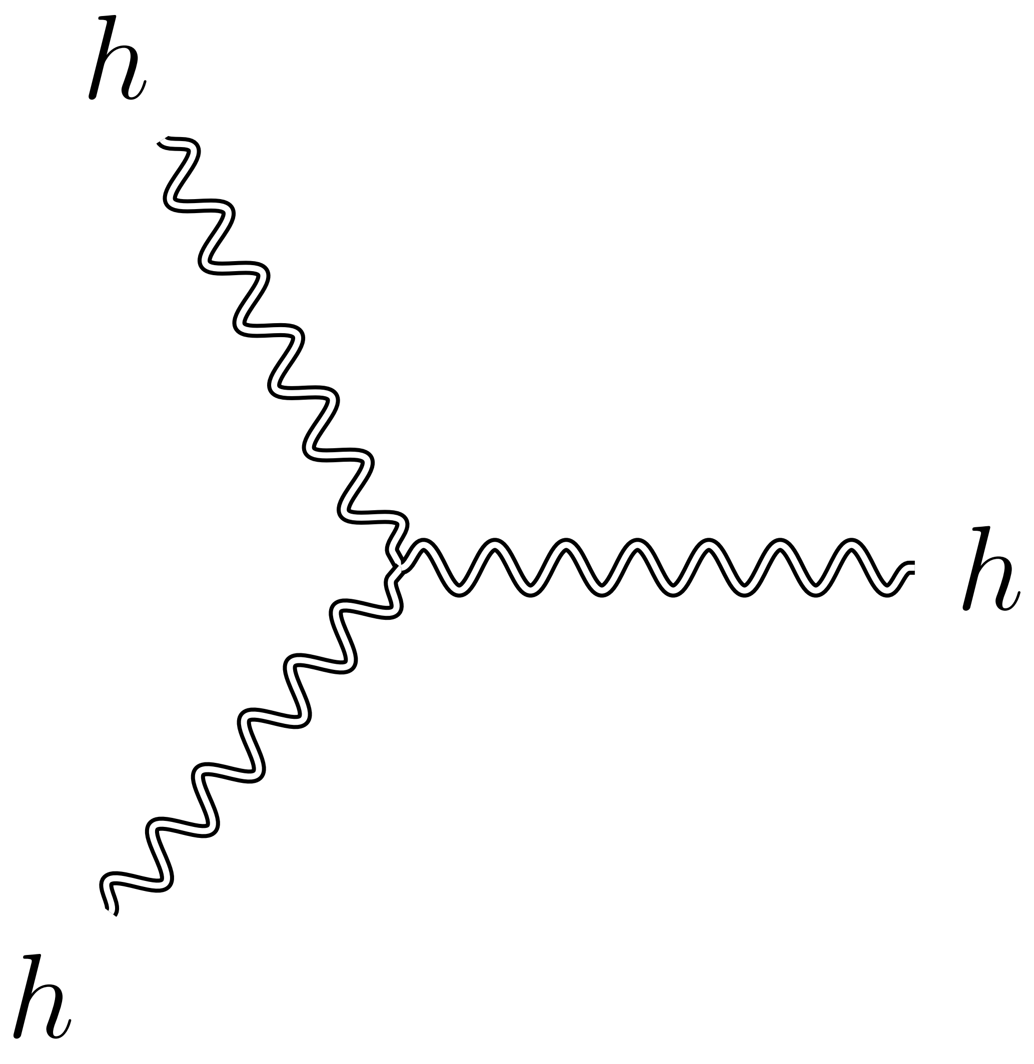
Scattering Amplitudes:

$$\mathscr{M}^{h_1,\ldots,h_m}(p_1,\ldots,p_n)$$

Quantum Amplitudes and Classical Observables: On-shell methods

Basis building block are **on-shell 3-point amplitudes**

Pure general relativity

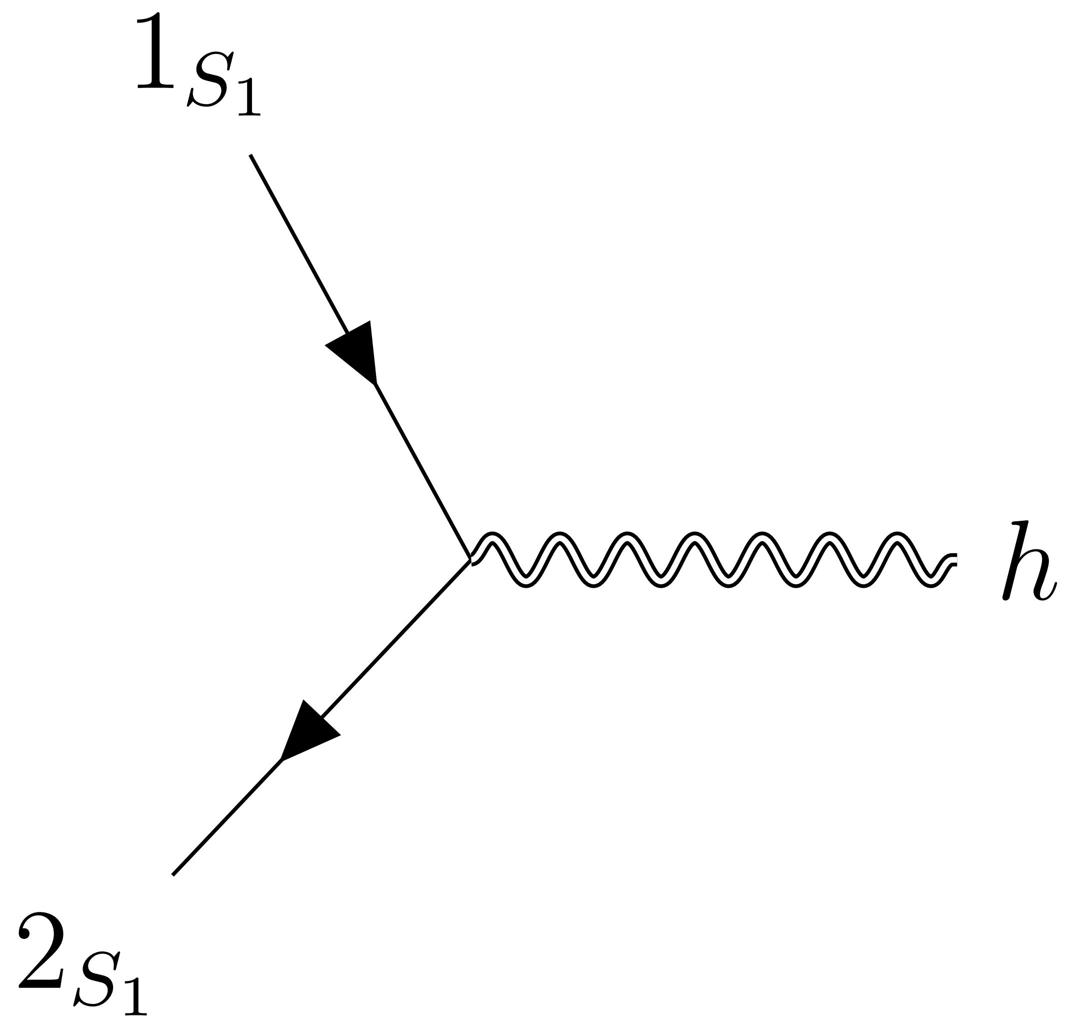


$$\mathscr{M}(1_h^-,2_h^-,3_h^+) = -\frac{1}{M_{Pl}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}$$

$$\mathscr{M}(1_h^+,2_h^+,3_h^-) = -\frac{1}{M_{Pl}}\frac{[12]^6}{[13]^2[23]^2}\,.$$

Matter coupling to gravity

,



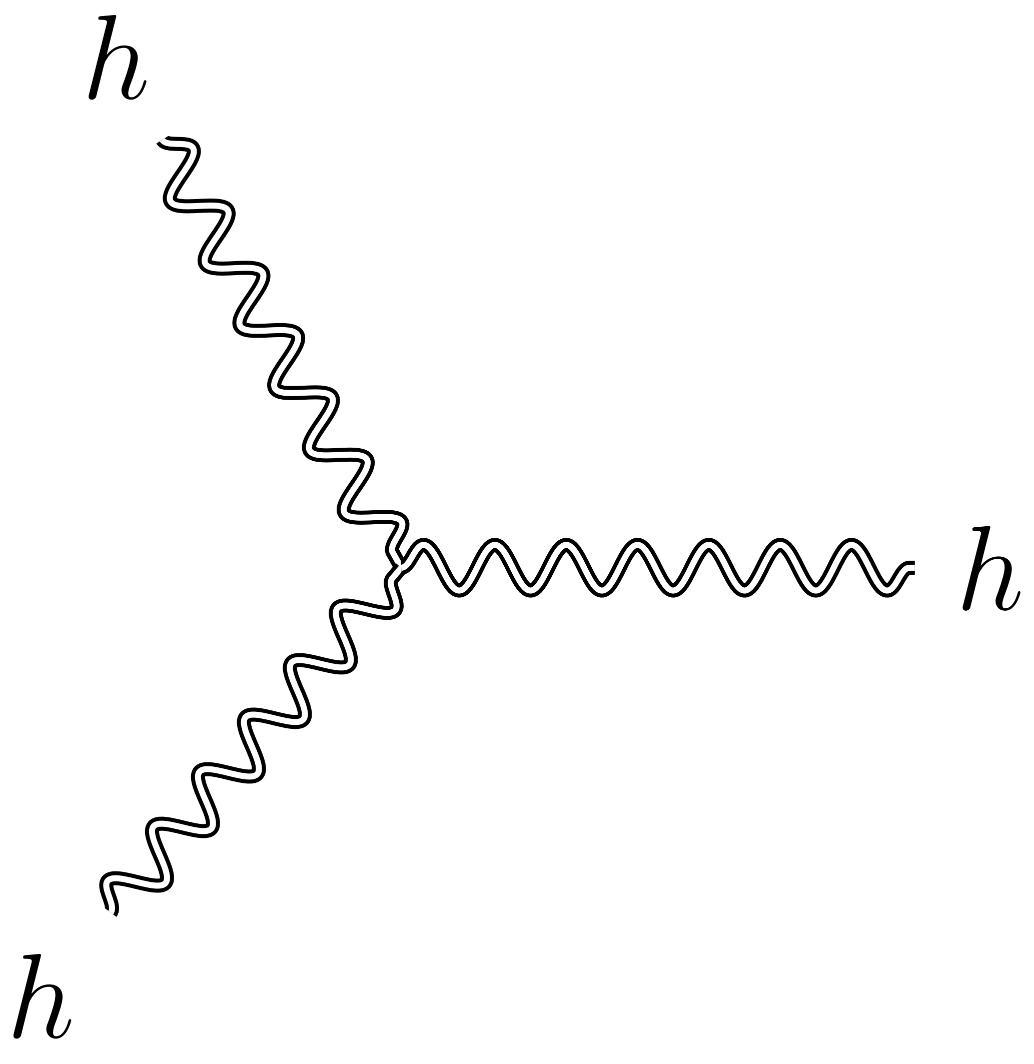
$$\mathscr{M}\left[1_\Phi 2_{\bar{\Phi}} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde{\zeta}]^2}{M_{Pl}[3\tilde{\zeta}]^2}\frac{[\mathbf{21}]^{2S}}{m^{2S}},$$

$$\mathscr{M}\left[1_\Phi 2_{\bar{\Phi}} 3_h^+\right] = -\frac{\langle \zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\frac{\langle\mathbf{21}\rangle^{2S}}{m^{2S}},$$

Quantum Amplitudes and Classical Observables: On-shell methods

Basis building block are **on-shell 3-point amplitudes**

Pure general relativity

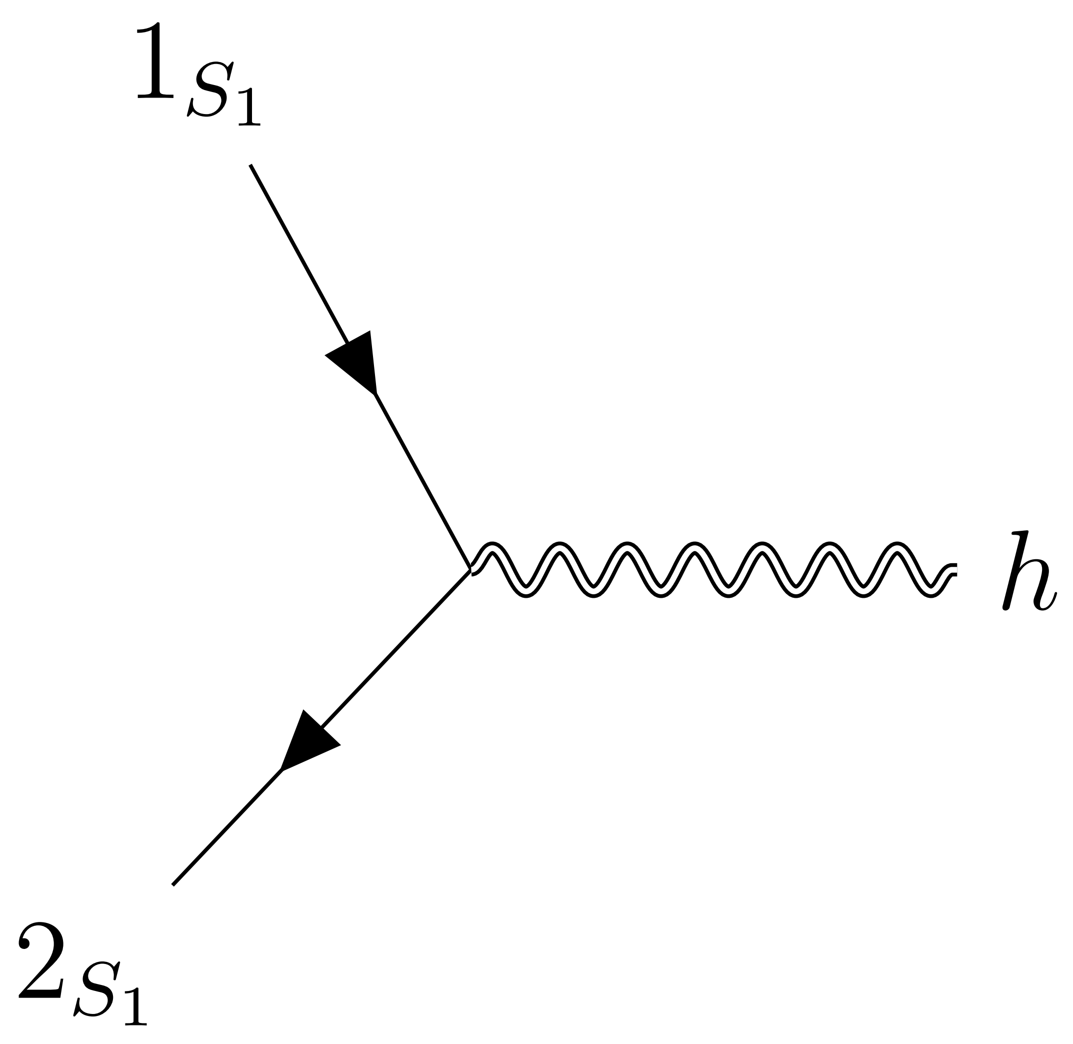


$$\mathcal{M}(1_h^-,2_h^-,3_h^+) = -\frac{1}{M_{Pl}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}$$

$$\mathcal{M}(1_h^+,2_h^+,3_h^-) = -\frac{1}{M_{Pl}}\frac{[12]^6}{[13]^2[23]^2}\,.$$

Matter coupling to gravity

,



Spin vector

$$\mathcal{M}^{\text{cl}}\left[1_\Phi 2_{\bar\Phi} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde\zeta]^2}{M_{Pl}[3\tilde\zeta]^2}\exp(+p_3a_1)$$

$$\mathcal{M}^{\text{cl}}\left[1_\Phi 2_{\bar\Phi} 3_h^+\right] = -\frac{\langle\zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\exp(-p_3a_1)$$

Quantum Amplitudes and Classical Observables: Compton amplitudes

$$\mathscr{M}^{\rm cl}\left[1_\Phi 2_{\bar\Phi} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde\zeta]^2}{M_{Pl}[3\tilde\zeta]^2}\exp(+p_3 a_1)$$

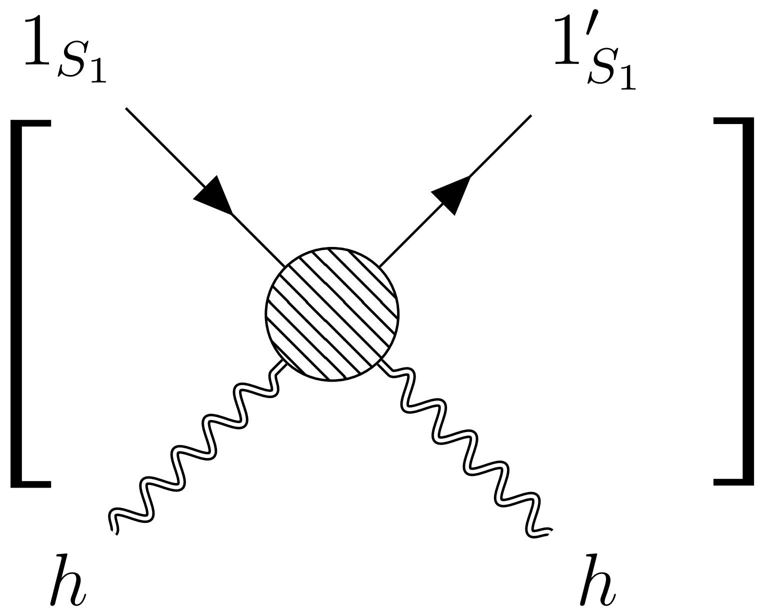
$$\mathscr{M}^{\rm cl}\left[1_\Phi 2_{\bar\Phi} 3_h^+\right] = -\frac{\langle \zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\exp(-p_3 a_1)$$

$$\mathscr{M}(1_h^-,2_h^-,3_h^+) = -\frac{1}{M_{Pl}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}$$

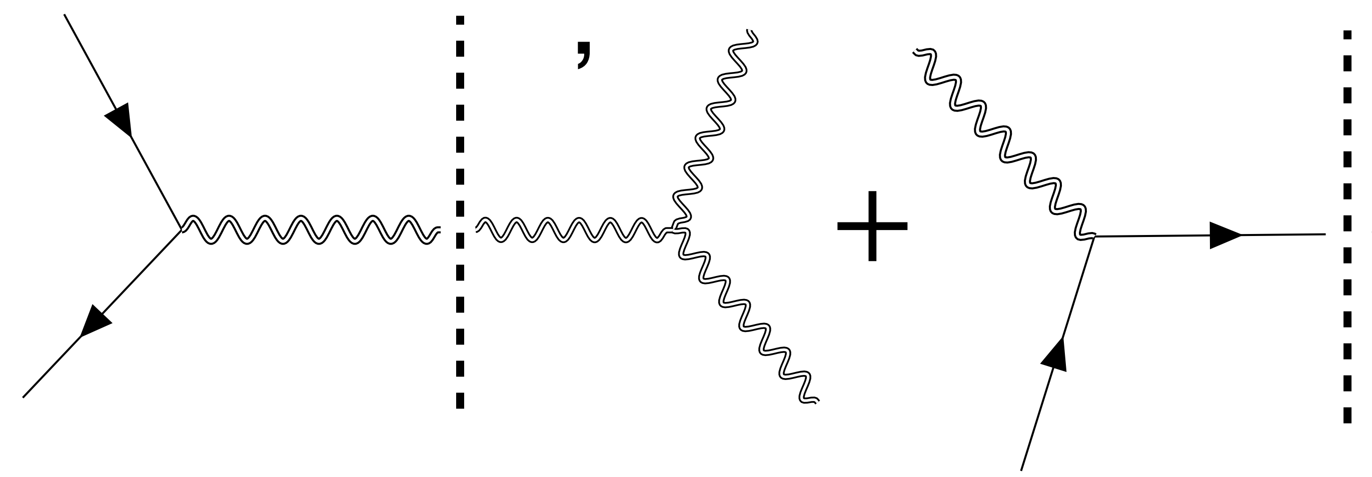
$$\mathscr{M}(1_h^+,2_h^+,3_h^-) = -\frac{1}{M_{Pl}}\frac{[12]^6}{[13]^2[23]^2}\,.$$

→ Build higher-point amplitudes (up to contact terms)
from their **residues** at kinematic poles in the **complex** plane

$$\sum_{i=2,3,4} \text{Res}_{(p_1+p_i)^2\to 0}\Bigg[$$



$$\Bigg]\Bigg|_{tree} = -\Bigg[$$



$$+(t \leftrightarrow u)\Bigg]$$

$$\mathscr{M}_U\left[1_h^- 2_h^+ 3_\Phi 4_{\bar\Phi}\right] = \frac{\langle 1|p_3|2]^4}{M_{\rm pl}^2 s(t-m^2)(u-m^2)}e^{a_3\tilde x} \qquad \mathscr{M}_U\left[1_h^- 2_h^- 3_\Phi 4_{\bar\Phi}\right] = \frac{m^4\langle 12\rangle^4}{M_{\rm Pl}^2 s(t-m^2)(u-m^2)}e^{a_3 q}$$

Contact terms start beyond quartic order in spin

$$q = p_1 + p_2$$

$$\tilde x^\mu \equiv \frac{i\epsilon^{\mu\nu\alpha\beta}(\lambda_1\sigma^\nu\tilde\lambda_2)p_3^\alpha p_4^\beta}{(\lambda_1 p_3\sigma\tilde\lambda_2)}$$

Quantum Amplitudes and Classical Observables: Compton amplitudes

$$\mathscr{M}^{\mathrm{cl}}\left[1_\Phi 2_{\bar\Phi} 3_h^-\right] = -\frac{\langle 3|p_1|\tilde\zeta]^2}{M_{Pl}[3\tilde\zeta]^2}\exp(+p_3 a_1)$$

$$\mathscr{M}^{\mathrm{cl}}\left[1_\Phi 2_{\bar\Phi} 3_h^+\right] = -\frac{\langle \zeta|p_1|3]^2}{M_{Pl}\langle 3\zeta\rangle^2}\exp(-p_3 a_1)$$

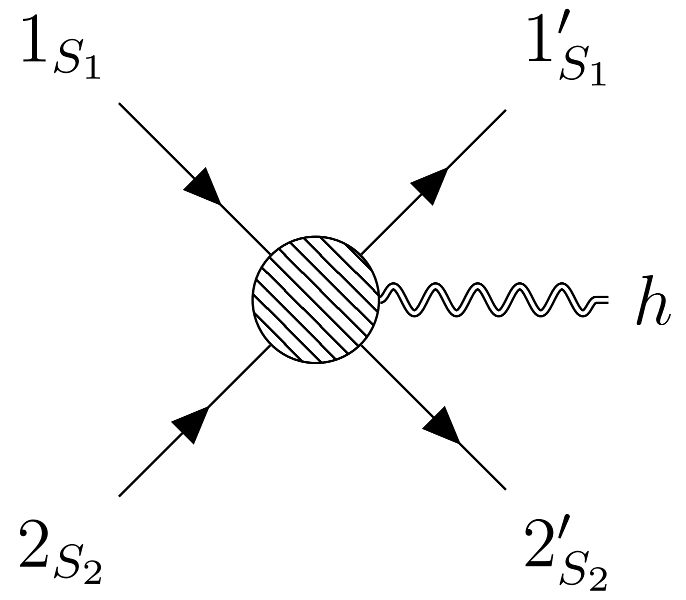
$$\mathscr{M}(1_h^-,2_h^-,3_h^+) = -\frac{1}{M_{Pl}}\frac{\langle 12\rangle^6}{\langle 13\rangle^2\langle 23\rangle^2}$$

$$\mathscr{M}(1_h^+,2_h^+,3_h^-) = -\frac{1}{M_{Pl}}\frac{[12]^6}{[13]^2[23]^2}\,.$$

$$\mathscr{M}_U\left[1_h^- 2_h^+ 3_\Phi 4_{\bar\Phi}\right] = \frac{(\lambda_1 p_3 \sigma \tilde\lambda_2)^4}{M_{\mathrm{pl}}^2 s(t-m^2)(u-m^2)} e^{a_3 \tilde x}$$

$$\mathscr{M}_U\left[1_h^- 2_h^- 3_\Phi 4_{\bar\Phi}\right] = \frac{m^4(\lambda_1 \lambda_2)^4}{M_{\mathrm{Pl}}^2 s(t-m^2)(u-m^2)} e^{a_3 q}$$

Build higher-point amplitudes (up to contact terms)
from their **residues** at kinematic poles in the **complex** plane



$$R^s \equiv \mathrm{Res}_{w_2^2\to 0}\mathscr{M}[(p_1+w_1)_{\Phi_1}(p_2+w_2)_{\Phi_2}(-p_1)_{\bar\Phi_1}(-p_2)_{\bar\Phi_2}(-k)_h^s]$$

$$= -\sum_{s'}\mathscr{M}\left[(-k)_h^s (w_2)_h^{s'}(p_1+w_1)_{\Phi_1}(-p_1)_{\bar\Phi_1}\right]\mathscr{M}\left[(-w_2)_h^{-s'}(p_2+w_2)_{\Phi_2}(-p_2)_{\bar\Phi_2}\right]$$

$$R^-_{(0)} = \frac{\left[2(p_1p_2)\Lambda[w_2,p_1] - m_1^2\Lambda[w_2,p_2] + 2(p_1k)\Lambda[p_1,p_2]\right]^2 + m_1^4\Lambda[w_2,p_2]^2}{8M_{\mathrm{Pl}}^3(w_2k)(p_1k)^2}$$

$$\Lambda[a,b,\ldots] \equiv (\lambda_k a\sigma b\bar\sigma\ldots\lambda_k)$$

$$R^-_{(1)} = -\frac{1}{8M_{\mathrm{Pl}}^3(w_2k)(p_1k)^2}\Big\{\left[a_1k + a_1w_2 + a_2w_2\right]\left[2(p_1p_2)\Lambda[w_2,p_1] - m_1^2\Lambda[w_2,p_2] + 2(p_1k)\Lambda[p_1,p_2]\right]^2$$
$$+m_1^4\left[a_1k - a_1w_2 - a_2w_2\right]\Lambda[w_2,p_2]^2 - 2(p_1k)\left[2(p_1p_2)\Lambda[w_2,p_1] - m_1^2\Lambda[w_2,p_2] + 2(p_1k)\Lambda[p_1,p_2]\right]\Lambda[p_1,p_2,w_2,a_1]\Big\}\,.$$

$$q = p_1 + p_2$$

$$\tilde x^\mu \equiv \frac{i\epsilon^{\mu\nu\alpha\beta}(\lambda_1\sigma^\nu\tilde\lambda_2)p_3^\alpha p_4^\beta}{(\lambda_1 p_3 \sigma\tilde\lambda_2)}$$

$$R^-_{(2)} = \ldots$$

Quantum Amplitudes and Classical Observables: gravitational waves

Waveform in GR calculated as expansion in spin

$$W_h = W_h^{(0)} + W_h^{(1)} + \ldots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b}$$

Kovacs Thorne
Astrophys. J. 217 (1977) 252
Astrophys. J. 224 (1978) 62.

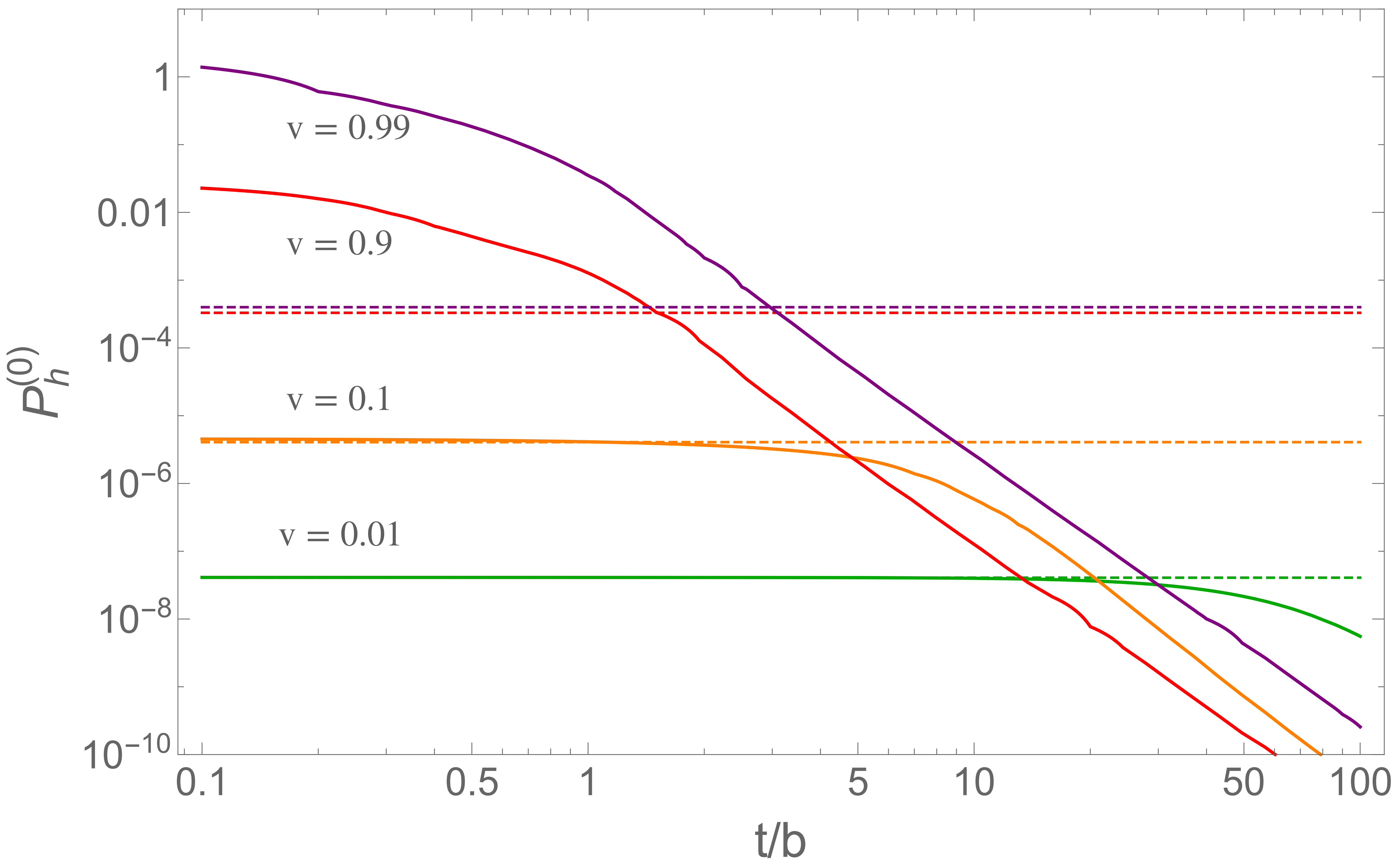
At leading order

$$W_h^{(0)} = -\frac{m_1 m_2}{512\pi^2 M_{\rm Pl}^3 b (\hat{u}_1 n)^2 \sqrt{\gamma^2 - 1}} \frac{1}{\sqrt{z^2+1}} \mathscr{R}\left\{ \frac{\left(\mathscr{F}_1^-(z)\right)^2 + \left(\mathscr{F}_2^-(z)\right)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b}n) + i\sqrt{z^2+1}(\tilde{v}n)} \right\}_{|z=T_1} + (1 \leftrightarrow 2)$$

$$\mathscr{F}_1^-(z) = \langle n | \left(\hat{u}_1\hat{u}_2 + 2\gamma z \hat{u}_1 \tilde{b} + z\tilde{b}\hat{u}_2 + 2i\gamma\sqrt{z^2+1}\,\hat{u}_1\tilde{v} + i\sqrt{z^2+1}\,\tilde{\mathrm{v}}\hat{u}_2\right) | n \rangle$$

$$\mathscr{F}_2^-(z) = \langle n | \left(\hat{u}_1 - z\tilde{b} - i\sqrt{z^2+1}\,\tilde{\mathrm{v}}\right)\hat{u}_2 | n \rangle$$

Emitted power in gravitational waves $\frac{dP_h}{d\Omega} = 2|\partial_t W_h|^2$



For $|t| \lesssim b/\mathrm{v}$ in the rest frame of particle 2

$$\partial_t W_h^{(0)} = -\,e^{-2i\phi}\frac{m_1 m_2}{16\pi^2 M_{\rm Pl}^3 b^2}\left\{(\hat{\mathbf{v}}\hat{\boldsymbol{n}})(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}}) + i(\hat{\mathbf{v}} \times \hat{\boldsymbol{b}}) \cdot \hat{\boldsymbol{n}}\right\}\mathrm{v} + O(\mathrm{v}^2)$$

velocity suppression

Emitted power

$$\frac{dP_h^{(0)}}{d\Omega} = \frac{m_1^2 m_2^2}{128\pi^4 M_{\rm Pl}^6 b^4}\left[(\hat{\mathbf{v}}\hat{\boldsymbol{n}})^2(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}})^2 + (\hat{\mathbf{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}})^2\right]\mathrm{v}^2 + \mathcal{O}(\mathrm{v}^3)$$

Total emitted power

$$P_h^{(0)} = \frac{m_1^2 m_2^2}{80\pi^3 M_{\rm Pl}^6 b^4}\mathrm{v}^2$$

Radiation in scalar-tensor theories

Scalar-Tensor Theories

- **Scalar-tensor theories** have long been a popular direction to study **extensions of GR**
- They consist of gravity theories with the introduction of an additional **massless scalar** degree of freedom

$$\boxed{S_{GR}[g_{\mu\nu}] \to S_{ST}[g_{\mu\nu}, \phi]}$$

Example: Scalar Gauss-Bonnet and Dynamical Chern Simons gravity

$$S = \int d^4x \frac{M_{Pl}^2}{2}\sqrt{-g}R + S_{SGB,DCS}[\phi, g_{\mu\nu}] + S_m[\Psi_m, \mathscr{A}(\phi)g_{\mu\nu}]$$

Gauss Bonnet invariant

$$\boxed{R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} - 4R^{\mu\nu}R_{\mu\nu} + R^2}$$

Chern Simons invariant

$$\boxed{R^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma} \quad , \quad \tilde{R}^{\mu}{}_{\nu\rho\sigma} = \frac{1}{2}\varepsilon_{\rho\sigma}{}^{\alpha\beta}R^{\mu}{}_{\nu\alpha\beta}}$$

- $$S_{SGB,DCS} = \int d^4x\sqrt{-g}\left[\frac{M_{\rm Pl}^2}{\Lambda^2}\left(f(\phi)\mathscr{G} + \tilde{f}(\phi)R\tilde{R}\right) + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi\right)\right]$$

$$f(\phi) = \text{const} + \alpha\frac{\phi}{M_{\rm Pl}} + O(\phi^2) \qquad \tilde{f}(\phi) = \text{const} + \tilde{\alpha}\frac{\phi}{M_{\rm Pl}} + O(\phi^2)$$

GW observations constrain them!

Phys.Rev.Lett. 126 (2021) 18, 181101 [Silva, Holgado, Cárdenas-Avendaño, Yunes]

Phys.Rev.D 107 (2023) 4, 044030 [Silva, Ghosh, Buonan

arXiv: 2406.13654 [Julié, Pompili, Buonanno]

Experimental window:

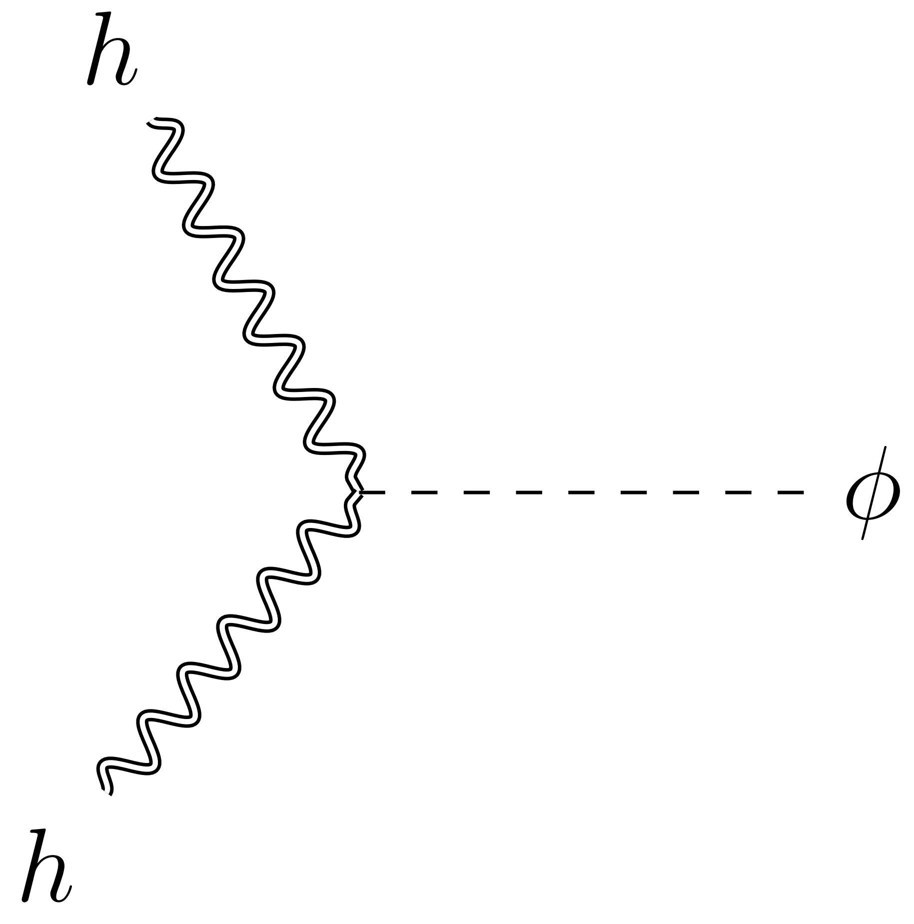
$$\frac{\sqrt{\alpha}}{\Lambda} \lesssim 0.22km \qquad , \qquad \frac{\sqrt{\tilde{\alpha}}}{\Lambda} \lesssim 9.5km$$

Scalar-Tensor Theories: shift-symmetric limit

- $$S_{SGB,DCS} = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{\Lambda^2}\left(f(\phi)\mathscr{G} + \tilde{f}(\phi)R\tilde{R}\right) + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi\right)\right]$$

$$f(\phi) = \text{const} + \alpha\frac{\phi}{M_{\text{Pl}}} + \mathcal{O}(\phi^2) \qquad \tilde{f}(\phi) = \text{const} + \tilde{\alpha}\frac{\phi}{M_{\text{Pl}}} + \mathcal{O}(\phi^2)$$

The linear term in ϕ induces a cubic interaction between scalar and gravity corresponding to 3-point amplitudes



$$\mathscr{M}\left[1_h^- 2_h^- 3_\phi\right] = \frac{2\hat{\alpha}}{\Lambda^2 M_{\text{pl}}}\langle 12\rangle^4$$

$$\mathscr{M}\left[1_h^+ 2_h^+ 3_\phi\right] = \frac{2\hat{\alpha}^*}{\Lambda^2 M_{\text{pl}}}[12]^4$$

$$\mathscr{M}\left[1_h^- 2_h^+ 3_\phi\right] = 0$$

$$\hat{\alpha} \equiv \alpha + i\tilde{\alpha}$$

However in large classes of scalar-tensor theories the leading corrections to gravitational radiation comes from a different interaction...

Scalar-Tensor Theories: scalar hair

Compact objects can acquire scalar hair in scalar-tensor theories
This is the case for black holes in SGB and DCS

Non-trivial configuration of the field $\phi \to$ Scalar hair

BH solution in ST theory

Far zone

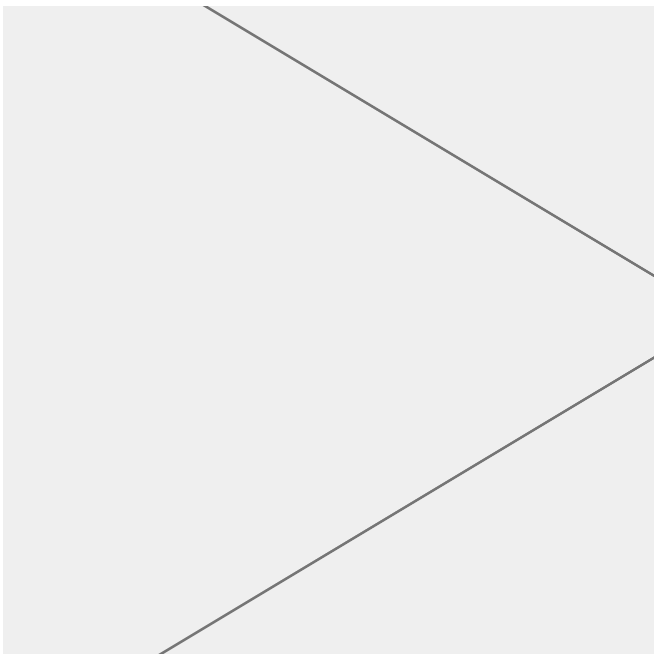
$x \to \infty$

"Monopole hair"

"Dipole hair"

$$\phi = \frac{c_i}{r} + \frac{d_i}{r^2} + \dots$$

$$c_n \sim \alpha \frac{M_{\rm Pl}^4}{\Lambda^2 m_n^2}$$



How can we model this behaviour with amplitudes?

Scalar-Tensor Theories: scalar hair on shell

AA, Marinellis
2411.12909

We model the black hole as a point-particle interacting with the scalar field via an effective metric (scalar conformal coupling)

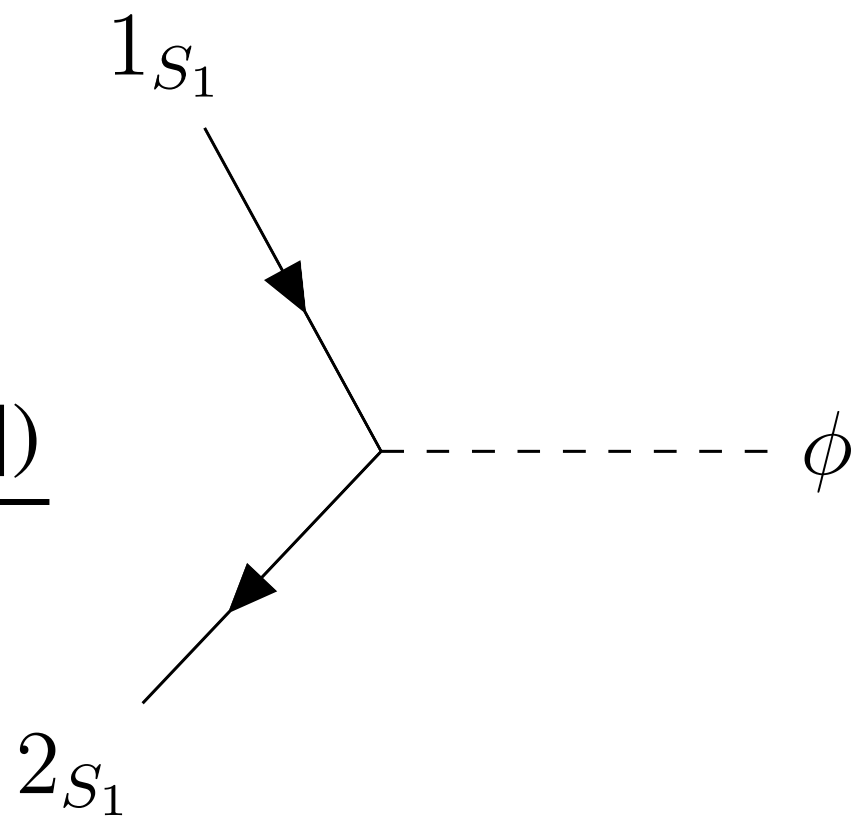
$$\tilde{g}_{\mu\nu} = \exp\left[C(\frac{\phi}{M_{Pl}})\right] g_{\mu\nu}$$

$$\boxed{\exp\left[C(\frac{\phi}{M_{Pl}})\right] \approx 1 + c\frac{\phi}{M_{Pl}}}$$

3-point amplitudes for arbitrary spinning black holes:

$$\mathscr{M}_{3,bos.}[1_{\Phi_n},2_{\bar{\Phi}_n}3_{\phi}] = -\frac{c_n}{M_{Pl}}\frac{\langle \mathbf{21}\rangle^{S_n}[\mathbf{21}]^{S_n}}{m_n^{2S_n-2}}$$

$$\mathscr{M}_{3,ferm.}[1_{\Psi_n},2_{\bar{\Psi}_n}3_{\phi}] = -\frac{c_n}{M_{Pl}}\frac{\langle \mathbf{21}\rangle^{S_n-1/2}[\mathbf{21}]^{S_n-1/2}(\langle \mathbf{21}\rangle + [\mathbf{21}])}{m_n^{2S-2}}$$



In the classical limit conformal coupling is spin-independent!

$$\mathscr{M}^{\text{cl}}[1_{\Phi_n},2_{\bar{\Phi}_n}3_{\phi}] = -\frac{c_n m_n^2}{M_{Pl}}$$

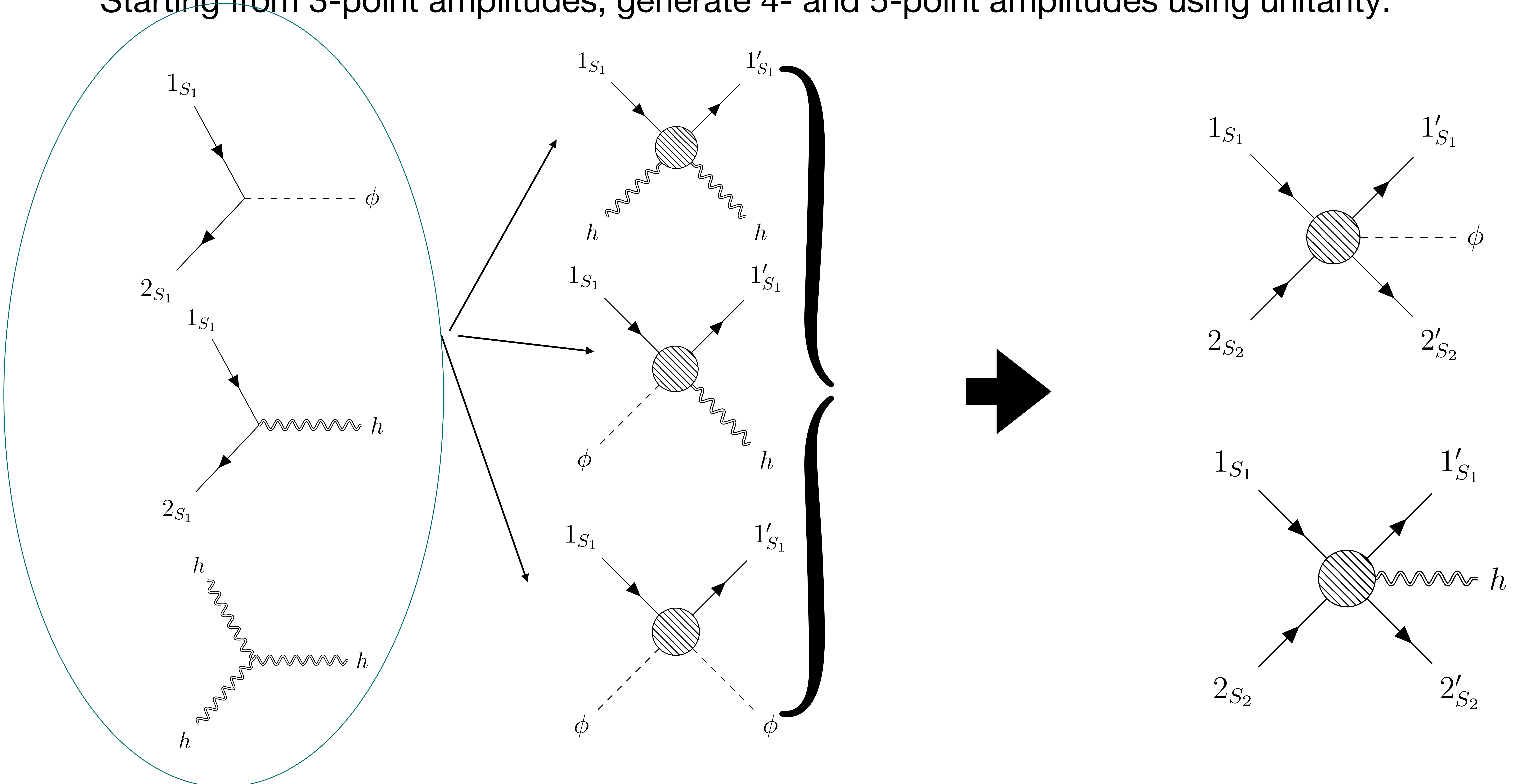
Classical conformal coupling
maps to monopole charge
in scalar tensor theory

At the lagrangian level for any spin coupling can be obtained by mass redefinition:

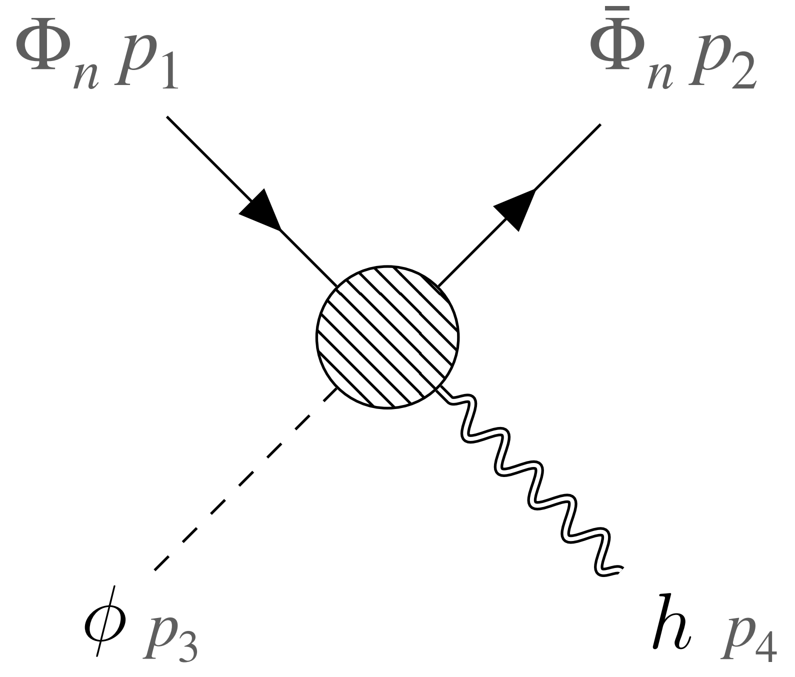
$$m \to e^{C/2} m$$

Scalar-Tensor Theories: amplitudes

Starting from 3-point amplitudes, generate 4- and 5-point amplitudes using unitarity:

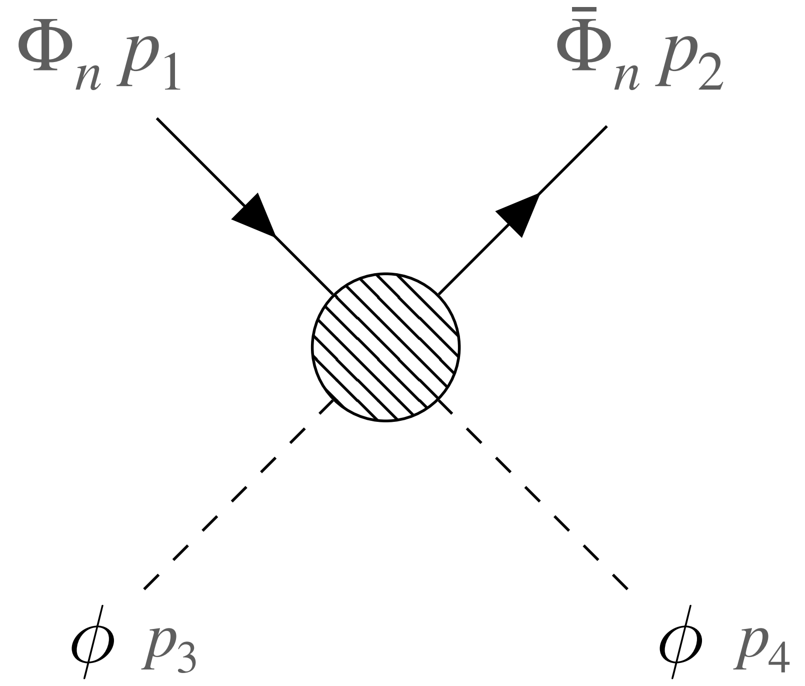


Scalar-Tensor Theories: amplitudes



$$\mathscr{M}_U^{\text{cl}}\left[1_{\Phi_n} 2_{\bar{\Phi}_n} 3_\phi 4_h^-\right] = -\frac{c_n m_n^2}{M_{\text{Pl}}^2} \frac{\langle 4 | p_3 p_1 | 4 \rangle^2}{s(t-m_n^2)^2}$$

Contact terms start at quadratic order in spin



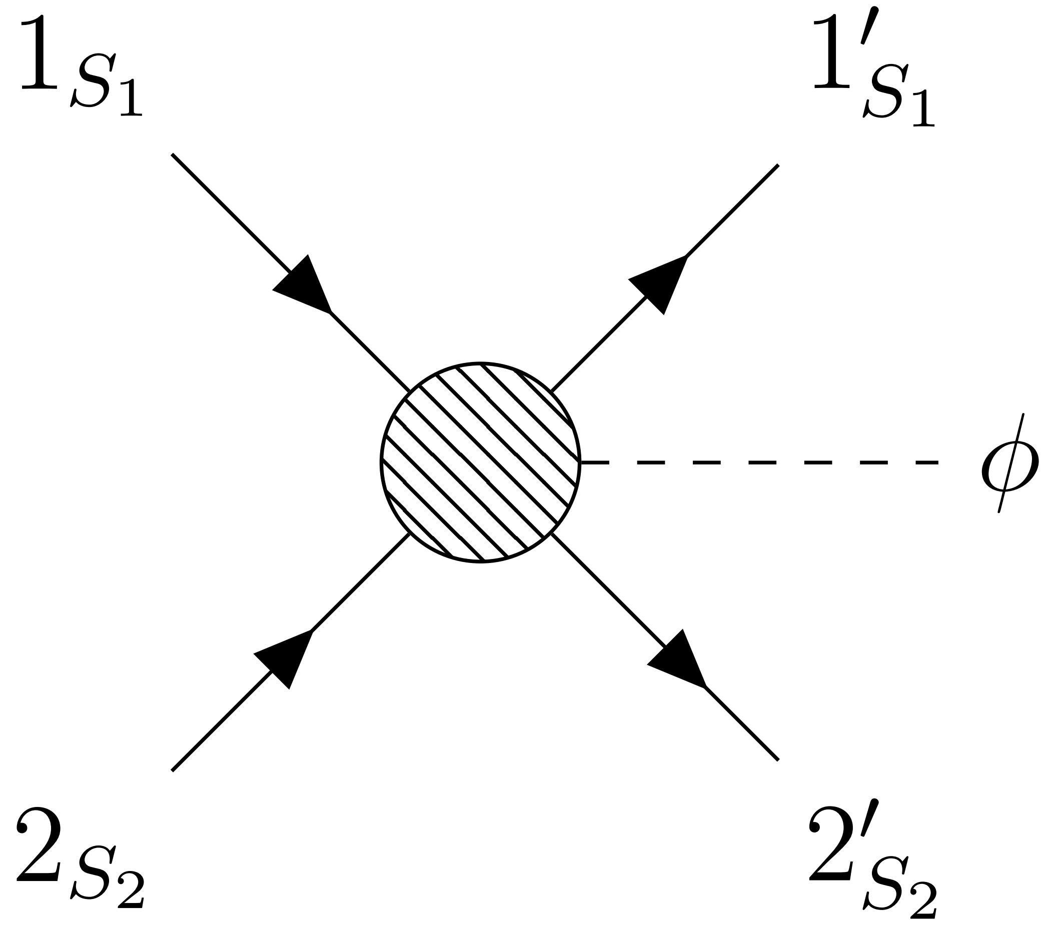
$$\mathscr{M}_U^{\text{cl}}\left[1_{\Phi_n} 2_{\bar{\Phi}_n} 3_\phi 4_\phi\right] = \frac{t-m_n^2}{M_{\text{Pl}}^2 s} \left\{ -(t-m_n^2)\cosh(q\cdot a_n) + 2i\varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_3^\nu p_4^\rho a_n^\sigma \frac{\sinh(q\cdot a_n)}{q\cdot a_n} \right\} + \mathscr{O}(c_n^2)$$

Contact terms start at zeroth-order in spin

$$\mathscr{M}_C^{\text{cl},(0)}\left[1_{\Phi_n} 2_{\bar{\Phi}_n} 3_\phi 4_\phi\right] = \frac{C_n^{(0)} m_n^2}{M_{\text{Pl}}^2}$$

$$\mathscr{M}_C^{\text{cl},(1)}\left[1_{\Phi_n} 2_{\bar{\Phi}_n} 3_\phi 4_\phi\right] = -i\frac{C_n^{(1)} m_n^2}{M_{\text{Pl}}^2}(qa_n)$$

Scalar-Tensor Theories: amplitudes



$$R_X = -\sum_{i=h,\phi} \mathscr{M}^{\rm cl}_X[(p_1+w_1)_{\Phi_1}(-p_1)_{\bar{\Phi}_1}(-k)_\phi(w_2)_i]\mathscr{M}^{\rm cl}[(p_2+w_2)_{\Phi_2}(-p_2)_{\bar{\Phi}_2}(-w_2)_i]$$

One finds the pole part of the residue to all orders in spin, plus contact terms in systematic expansion in spin vector

Part originating from pole terms in 4-point amplitude

$$R_U = \frac{2}{M_{\rm Pl}^3}\Bigg\{ \frac{(kw_2)[(p_1p_2)^2 - \frac{1}{2}m_1^2m_2^2] + m_2^2(p_1k)^2 - 2(p_1p_2)(p_1k)(p_2k)}{(p_1k)^2} c_1m_1^2\cosh(w_2a_2)$$
$$+i\frac{c_1m_1^2(p_1p_2)}{(p_1k)^2}p_1^\mu p_2^\nu k^\rho w_2^\sigma \varepsilon_{\mu\nu\rho\sigma}\sinh(w_2a_2) - 2\frac{c_2m_2^2(p_1k)^2\cosh(w_1a_1) + c_1m_1^2(p_2k)^2\cosh(w_2a_2)}{w_1^2}$$
$$+\frac{2i\varepsilon_{\mu\nu\rho\sigma}p_1^\mu k^\nu w_2^\rho}{w_1^2}\left[a_1^\sigma(p_1k)\frac{c_2m_2^2\sinh(w_1a_1)}{w_1a_1} + p_2^\sigma\frac{c_1m_1^2(p_2k)}{p_1k}\sinh(w_2a_2)\right]\Bigg\}$$

Part originating from contact terms in 4-point amplitude

$$R_C^{(0)} = \frac{C_1^{(0)}c_2m_1^2m_2^2}{M_{\rm Pl}^3} \qquad R_C^{(1)} = -i\frac{C_1^{(1)}c_2m_1^2m_2^2}{M_{\rm Pl}^3}(w_1a_1) \qquad R_C^{(2)} = \ldots$$

Scalar-Tensor Theories: scalar waveforms

Scalar radiation observable $\mathscr{R}_{\phi} \equiv {}_{\text{out}}\langle \psi | \phi(x) | \psi \rangle_{\text{out}}$

Scalar waveform $$\mathscr{R}_X(x) = \frac{W_X(t)}{|\boldsymbol{x}|} \qquad |\boldsymbol{x}| \to \infty \qquad t \equiv x^0 - |\boldsymbol{x}|$$

Using KMOC one can relate the scalar waveshape to the residues of the 5-point scalar emission amplitude

$$\mathrm{R} \equiv \mathrm{Res}_{w_2^2 \to 0} \mathscr{M}^{\text{cl}}_{\text{tree}}[p_1 + w_1, p_2 + w_2 \to p_1, p_2, k_\phi]$$

$$f_\phi(\omega) = -\frac{1}{64\pi^2 m_1 m_2 \sqrt{\gamma^2 - 1}} \int_{-\infty}^{\infty} \frac{dz e^{ib_1 k + iz(\hat{u}_1 k) b}}{\sqrt{z^2+1}} \frac{1}{2} \Big\{ R\big(w_2 \to (\hat{u}_1 k)\big[\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} + i\sqrt{z^2+1}\,\tilde{\mathrm{v}}\big]\big) + R\big(w_2 \to (\hat{u}_1 k)\big[\gamma \hat{u}_2 - \hat{u}_1 + z\tilde{b} - i\sqrt{z^2+1}\,\tilde{\mathrm{v}}\big]\big) \Big\} + (1 \leftrightarrow 2) \qquad k = \omega n$$

From this, waveform is calculated via inverse Fourier transform

$$W_\phi(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} e^{-i\omega t} f_\phi(\omega)$$

Scalar-Tensor Theories: scalar waveforms

$$W_\phi = W_\phi^{(0)} + W_\phi^{(1)} + \dots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b} \qquad \hat{u}_i = \frac{u_i}{\sqrt{\gamma^2 - 1}} \qquad \gamma = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$

LO Scalar Waveforms-Spinless part:

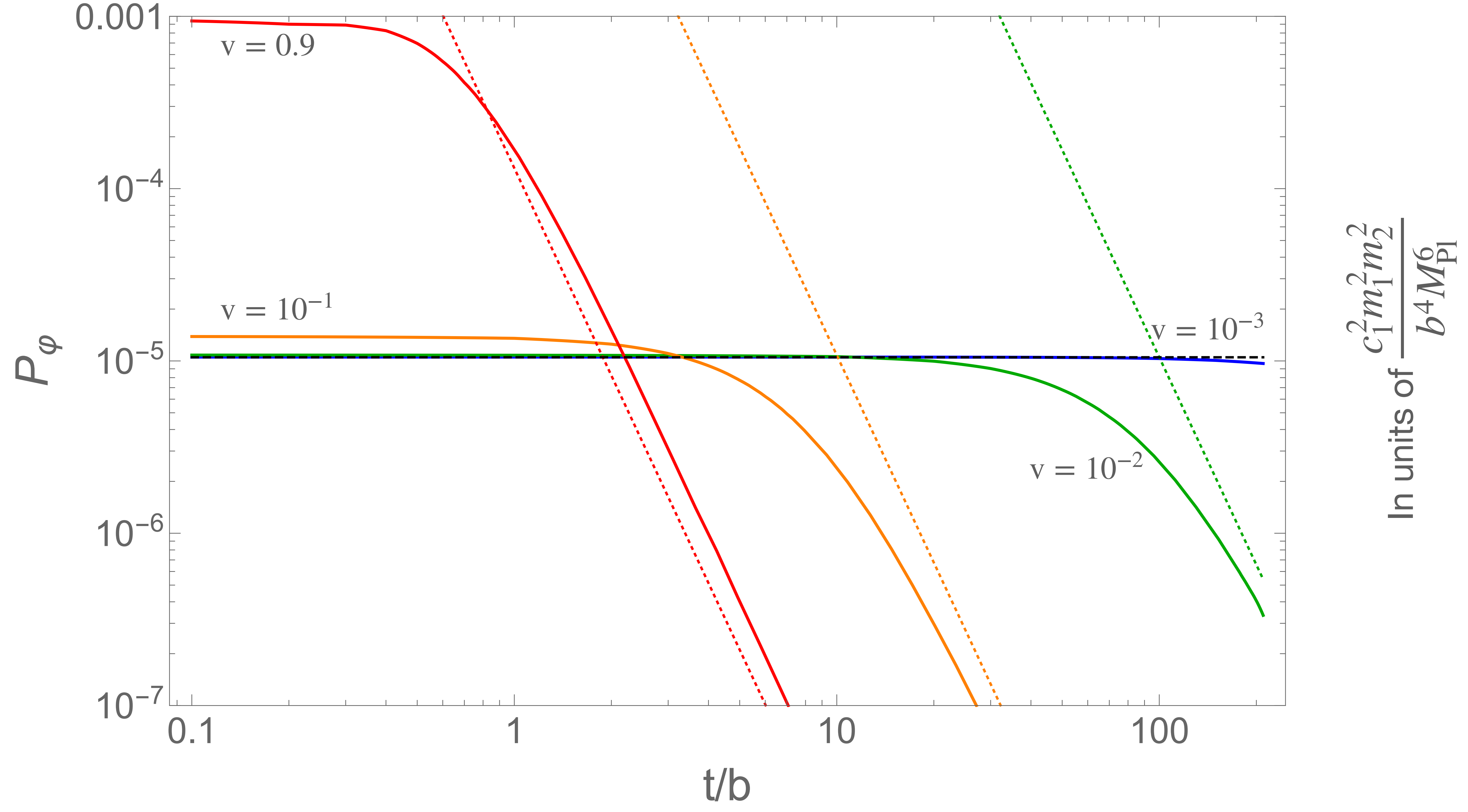
Contact term in 4-point matter-scalar scattering amplitude

$$W_\phi^{(0)} = -\frac{m_1 m_2}{32\pi^2 M_{Pl}^3 \sqrt{\gamma^2 - 1}(\hat{u}_1 n)^2 b} \frac{1}{\sqrt{1 + T_1^2}} \left\{ \frac{(\gamma^2 - 1)[c_1(\hat{u}_2 n)^2 + c_2(\hat{u}_1 n)^2][\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b} n)T_1]}{[-(\hat{u}_1 n) + \gamma(\hat{u}_2 n) + (\tilde{b} n)T_1]^2 + (\tilde{v} n)^2(1 + T_1^2)} \right.$$

$$\left. -\frac{c_1}{2} \frac{(\hat{u}_1 n) + (2\gamma^2 - 3)\gamma(\hat{u}_2 n) - (2\gamma^2 - 1)(\tilde{b} n)T_1}{\gamma^2 - 1} + \frac{C_1^{(0)}}{2} c_2(\hat{u}_1 n) \right\} + (1 \leftrightarrow 2)$$

Emitted power

$$\frac{dP_\phi}{d\Omega} = (\partial_t W_\phi)^2$$



Scalar-Tensor Theories: scalar waveforms

$$W_\phi = W_\phi^{(0)} + W_\phi^{(1)} + \dots$$

$$T_i \equiv \frac{t - b_i n}{(\hat{u}_i n) b} \qquad \hat{u}_i = \frac{u_i}{\sqrt{\gamma^2 - 1}} \qquad \gamma = u_1 u_2 = \frac{1}{\sqrt{1 - \mathrm{v}^2}}$$

LO Scalar Waveforms-Spinless part

$$W_\phi^{(0)} = -\frac{m_1 m_2}{32\pi^2 M_{Pl}^3 \sqrt{\gamma^2 - 1}(\hat{u}_1 n)^2 b} \frac{1}{\sqrt{1 + T_1^2}} \left\{ \frac{(\gamma^2 - 1)[c_1(\hat{u}_2 n)^2 + c_2(\hat{u}_1 n)^2][\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b}n)T_1]}{[-(\hat{u}_1 n) + \gamma(\hat{u}_2 n) + (\tilde{b}n)T_1]^2 + (\tilde{v}n)^2(1 + T_1^2)} \right.$$

$$\left. - \frac{c_1}{2} \frac{(\hat{u}_1 n) + (2\gamma^2 - 3)\gamma(\hat{u}_2 n) - (2\gamma^2 - 1)(\tilde{b}n)T_1}{\gamma^2 - 1} + \frac{C_1^{(0)}}{2} c_2(\hat{u}_1 n) \right\} + (1 \leftrightarrow 2)$$

Center-of-mass frame, non-relativistic limit of small relative velocity v.
PN expansion of the waveform independent of contact terms at this order

$$W_\phi^{(0)} = \frac{m_1 m_2 (c_1 - c_2)}{64\pi^2 M_{\mathrm{Pl}}^3 b} \left\{ -\frac{(\hat{\mathbf{v}}\hat{\boldsymbol{n}})}{\mathrm{v}} + (\hat{\boldsymbol{b}}\hat{\boldsymbol{n}})\frac{t}{b} \right\} + O(\mathrm{v})$$

Phys.Rev.D 85 (2012) 064022, Phys.Rev.D 93 (2016) 2, 029902 [Yagi, Stein, Yunes, Tanaka]
Class.Quant.Grav. 39 (2022) 3, 035002 [Shiralilou, Hinderer, Nissanke, Ortiz, Witek]

Emitted power in the limit of of small velocity

$$P_\phi^{(0)} = \frac{m_1^2 m_2^2}{3072\pi^3 M_{\mathrm{Pl}}^6 b^4}(c_1 - c_2)^2 + O(\mathrm{v}) \xrightarrow{\hspace{2cm}} P_\phi^{(0)} = \frac{4\pi m_1^2 m_2^2 M_{\mathrm{Pl}}^2 (c_1 - c_2)^2}{3(m_1 + m_2)^4}\mathrm{v}^8 + O(\mathrm{v}^9)$$

Kepler's law

$$b^{-1} \to \frac{8\pi M_{\mathrm{Pl}}^2 \mathrm{v}^2}{m_1 + m_2}$$

Agreement with existing classical results for SGB!

In particular, $\frac{P_\phi^{(0)}}{P_h(0)} \sim \frac{1}{\mathrm{v}^2}$

Scalar-Tensor Theories: scalar waveforms

Comparison beyond leading PN order for quasi-circular orders

Amplitudes

$$\begin{aligned} P_\phi^{(0)} = {} & \frac{m_1^2 m_2^2}{3072\pi^3 M_{\rm Pl}^6 b^4}(c_1 - c_2)^2 \\ & + \frac{m_1^2 m_2^2}{\pi^3 M_{\rm Pl}^6 b^4}\left[\frac{(c_1+c_2)^2}{3840} + \frac{(c_1+c_2)(c_1-c_2)}{3840}\frac{m_1-m_2}{m_1+m_2} + (6-\eta)\frac{(c_1-c_2)^2}{7680} \right. \\ & + \frac{(c_1-c_2)\big(m_2 c_2 C_1^{(0)} - m_1 c_1 C_2^{(0)}\big)}{1536(m_1+m_2)} \\ & \left. - \frac{(c_1-c_2)^2}{1536}\frac{t^2}{b^2}\right]\mathrm{v}^2, \end{aligned}$$

Classical

Class.Quant.Grav. 39 (2022) 3, 035002 [Shiralilou, Hinderer, Nissanke, Ortiz, Witek]

$$\begin{aligned} \dot{E}_S = {} & \frac{\eta^2}{G\bar{\alpha}c^3}\left(\frac{G\bar{\alpha}m}{r}\right)^4 \left\{\frac{4}{3}\mathcal{S}_-^2 + \frac{8}{15\,c^2}\left(\frac{G\bar{\alpha}m}{r}\left[\left(-23+\eta-10\bar{\gamma}-10\beta_+ + 10\frac{\Delta m}{m}\beta_-\right)\mathcal{S}_-^2 \right.\right.\right. \\ & \left. -2\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_-\right] + v^2\left[2\mathcal{S}_+^2 + 2\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_- + (6-\eta+5\bar{\gamma})\mathcal{S}_-^2 - \frac{10}{\bar{\gamma}}\frac{\Delta m}{m}\mathcal{S}_-\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right)\right. \\ & \left. + \frac{10}{\bar{\gamma}}\mathcal{S}_-\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right)\right] + \dot{r}^2\left[\frac{23}{2}\mathcal{S}_+^2 - 8\frac{\Delta m}{m}\mathcal{S}_+\mathcal{S}_- + \left(9\eta - \frac{37}{2} - 10\bar{\gamma}\right)\mathcal{S}_-^2 - \frac{80}{\bar{\gamma}}\mathcal{S}_+\left(\mathcal{S}_+\beta_+ \right.\right. \\ & \left.\left.\left. + \mathcal{S}_-\beta_-\right) + \frac{30}{\bar{\gamma}}\frac{\Delta m}{m}\mathcal{S}_-\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right) - \frac{10}{\bar{\gamma}}\mathcal{S}_-\left(\mathcal{S}_-\beta_+ + \mathcal{S}_+\beta_-\right) + \frac{120}{\bar{\gamma}^2}\left(\mathcal{S}_+\beta_+ + \mathcal{S}_-\beta_-\right)^2\right]\right) \\ & \left. - \frac{\Delta m}{m}\frac{\eta}{6\,c^2}\left(\frac{\alpha f'(\phi_0)\mathcal{S}_-\mathcal{S}_+}{\sqrt{\bar{\alpha}}r^2}\right)\left(\mathcal{S}_+ + \frac{\Delta m}{m}\mathcal{S}_-\right)\left[-9\dot{r}^2 + 3v^2 - \frac{2G\bar{\alpha}m}{r}\right] + \mathcal{O}(c^{-3})\right\}. \end{aligned}$$

higher PM

irrelevant for quasi-circular orbits

suppressed by more powers of distance

Scalar-Tensor Theories: scalar waveforms

Using amplitudes it is straightforward to continue beyond linear order in spin

$$W_\phi = W_\phi^{(0)} + W_\phi^{(1)} + \ldots$$

$$a_i \equiv \frac{S_i}{m_i}$$

one more power of impact parameter

$$W_\phi^{(1)} = \frac{m_1 m_2}{32\pi^2 M_{\rm Pl}^3 b^2 (\hat{u}_1 n)^2} \frac{\partial}{\partial z}\Bigg(\frac{1}{\sqrt{z^2+1}} \mathrm{Re}\Big\{ c_1 \big[z(\tilde{v}n) - i\sqrt{z^2+1}(\tilde{b}n)\big]\big[-(\hat{u}_1 a_2) + z(\tilde{b}a_2) + i\sqrt{z^2+1}(\tilde{v}a_2)\big]$$

$$\times \left[\frac{\gamma}{\gamma^2-1} - \frac{(\hat{u}_2 n)}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b}n) + i\sqrt{z^2+1}(\tilde{v}n)}\right] - \frac{c_2(\hat{u}_1 n)}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + z(\tilde{b}n) + i\sqrt{z^2+1}(\tilde{v}n)}$$

$$\times \left[\big[i\sqrt{z^2+1}(\tilde{b}n) - z(\tilde{v}n)\big](\hat{u}_2 a_1) + \big[\gamma(\tilde{v}n) + i\sqrt{z^2+1}\big(\gamma(\hat{u}_1 n) - \hat{u}_2 n\big)\big](\tilde{b}a_1) + \big[z\big(\hat{u}_2 n - \gamma(\hat{u}_1 n)\big) - \gamma(\tilde{b}n)\big](\tilde{v}a_1)\right]$$

$$+ \frac{C_1^{(1)} c_2}{2\sqrt{\gamma^2-1}}\left[(\hat{u}_1 n)\big[\gamma(\hat{u}_2 a_1) + z(\tilde{b}a_1)\big] - (a_1 n)\right]\Big\}\Bigg)\Big|_{z=T_1} + (1 \leftrightarrow 2)$$

New contact term

$$W_\phi^{(1)} = \frac{m_1 m_2}{32\pi^2 M_{\rm Pl}^3 b^2}\Big\{ \left[c_2 \boldsymbol{a_1} - c_1 \boldsymbol{a_2}\right] \cdot \hat{\mathbf{v}}\big(\hat{\mathbf{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}\big) + \frac{1}{2}\left[c_1 C_2^{(1)} \boldsymbol{a}_2 - c_2 C_1^{(1)} \boldsymbol{a}_1\right] \cdot \hat{\boldsymbol{b}}$$

$$+ \mathrm{v}\left[\left[c_1 \boldsymbol{a_2} - c_2 \boldsymbol{a_1}\right] \cdot \left[2\hat{\boldsymbol{b}}\big(\hat{\mathbf{v}} \times \hat{\boldsymbol{b}} \cdot \hat{\boldsymbol{n}}\big) + \hat{\mathbf{v}} \times \hat{\boldsymbol{b}}\big(\hat{\boldsymbol{b}}\hat{\boldsymbol{n}}\big)\right] + \frac{1}{2}\left[c_1 C_2^{(1)} \boldsymbol{a}_2 - c_2 C_1^{(1)} \boldsymbol{a}_1\right] \cdot \hat{\mathbf{v}}\right]\frac{t}{b}\Big\}$$

$$+ O(\mathrm{v}^2)\,.$$

Linear-in-spin correction to emitted power

$$P_\phi^{(1)} = \frac{m_1^2 m_2^2 (c_1 - c_2)}{768\pi^3 M_{\rm Pl}^6 b^5}(\hat{\mathbf{v}} \times \hat{\boldsymbol{b}}) \cdot \left[c_1 \boldsymbol{a_2} - c_2 \boldsymbol{a_1}\right]\mathrm{v}$$

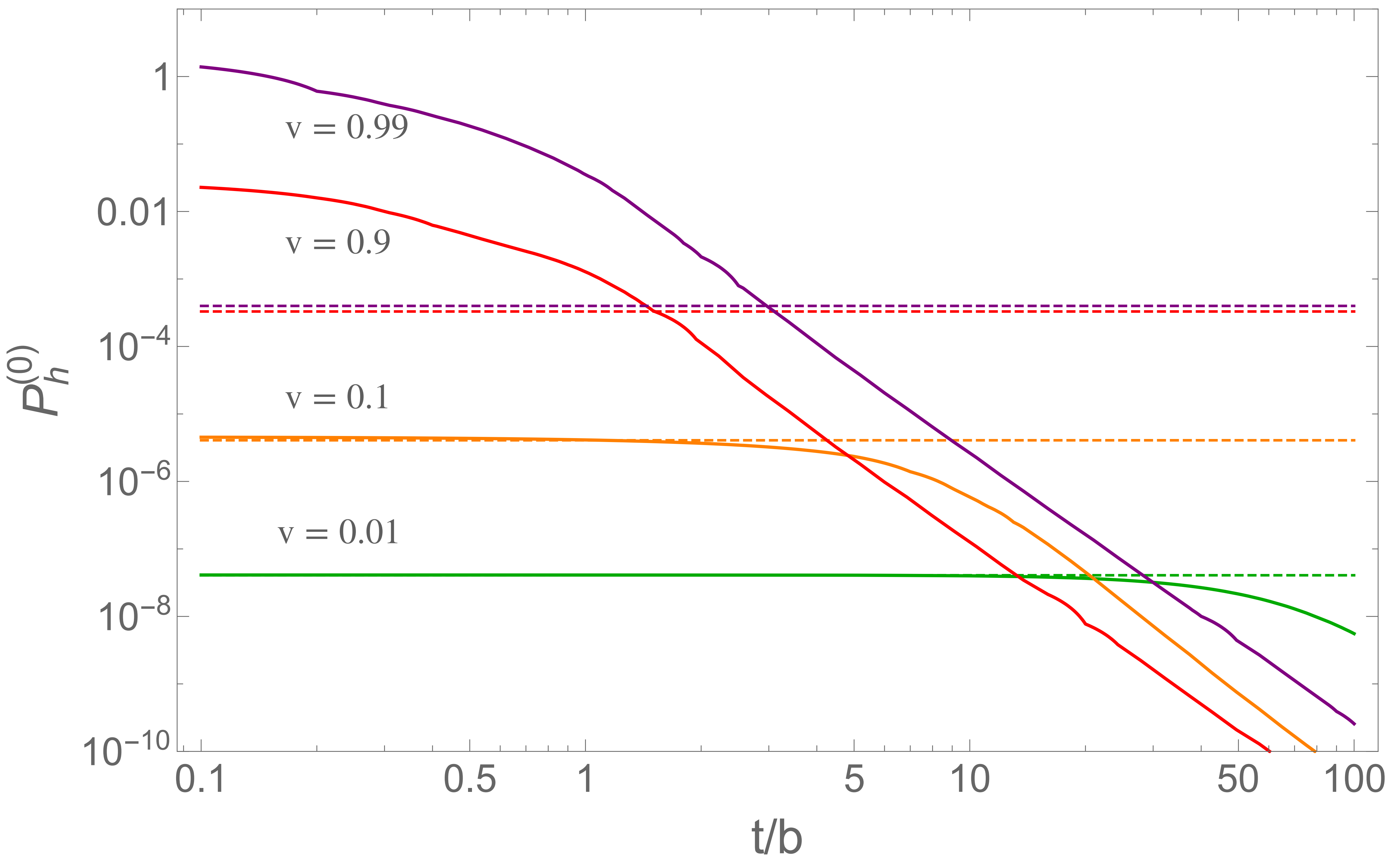
Scalar-Tensor Theories: gravitational waveforms

$$W_h = W_h^{(0)} + W_h^{(1)} + \dots$$

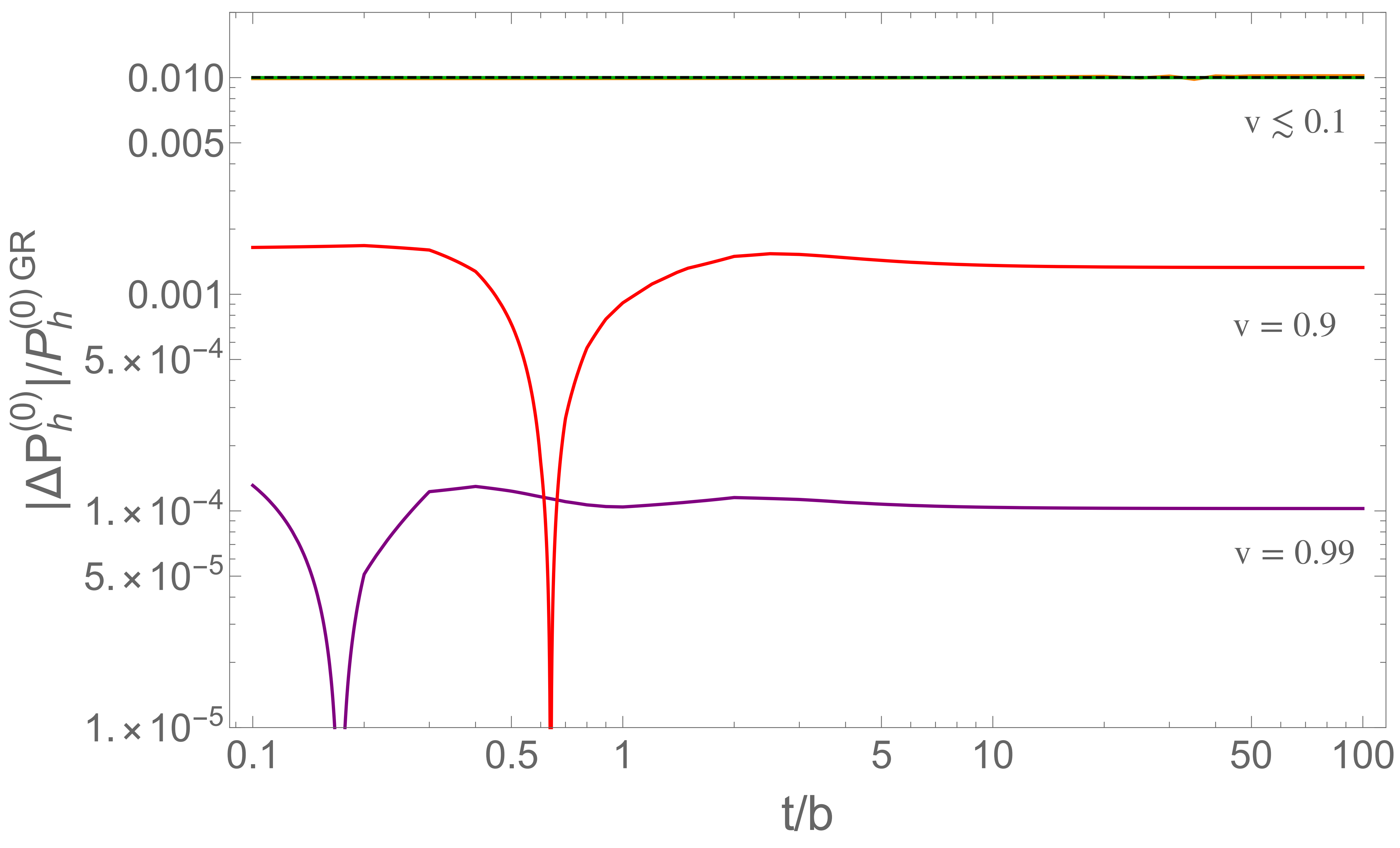
LO Gravitational Waveforms-Spinless part:

$$\Delta W_h^{(0)} = -\frac{c_1 c_2 m_1 m_2}{512\pi^2 M_{Pl}^3 \sqrt{\gamma^2-1}(\hat{u}_1 n)^2 b}\frac{1}{\sqrt{1+T_1^2}}\mathrm{Re}\left\{\frac{(\lambda_n[\gamma\hat{u}_2\sigma + T_1\tilde{b}\sigma + i\sqrt{T_1^2+1}\tilde{v}\sigma]\hat{u}_1\bar{\sigma}\lambda_n)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + T_1(\tilde{b}n) + i\sqrt{T_1^2+1}(\tilde{v}n)}\right\} + (1 \leftrightarrow 2)\,.$$

Emitted power in gravitational waves $\frac{dP_h}{d\Omega} = 2\,|\,\partial_t W_h|^2$



Correction of emitted power due to scalar exchange



Scalar-Tensor Theories: gravitational waveforms

$$W_h = W_h^{(0)} + W_h^{(1)} + \dots$$

LO Gravitational Waveforms-Spinless part:

$$\Delta W_h^{(0)} = -\frac{c_1 c_2 m_1 m_2}{512\pi^2 M_{Pl}^3 \sqrt{\gamma^2-1}(\hat{u}_1 n)^2 b}\frac{1}{\sqrt{1+T_1^2}}\text{Re}\left\{\frac{(\lambda_n[\gamma\hat{u}_2\sigma + T_1\tilde{b}\sigma + i\sqrt{T_1^2+1}\tilde{v}\sigma]\hat{u}_1\bar{\sigma}\lambda_n)^2}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + T_1(\tilde{b}n) + i\sqrt{T_1^2+1}(\tilde{v}n)}\right\} + (1 \leftrightarrow 2)\,.$$

$$\left.\frac{dP_h}{d\Omega}\right|_{\mathcal{O}(a^0)} \sim (1+c_1c_2)\frac{\mathrm{v}^2}{b^4} \xrightarrow{\text{For closed orbits}} \left.\frac{dP_h}{d\Omega}\right|_{\mathcal{O}(a^0)} \sim (1+c_1c_2)\mathrm{v}^{10} \longrightarrow$$

Suppression compared to scalar radiation in agreement with classical literature

Phys.Rev.D 85 (2012) 064022, Phys.Rev.D 93 (2016) 2, 029902 (erratum) [Yagi, Stein, Yunes, Tanaka]

Class.Quant.Grav. 39 (2022) 3, 035002 [Shiralilou, Hinderer, Nissanke, Ortiz, Witek]

In fact, at leading PN order emitted power rescaled compared to GR by factor $1 + c_1 c_2$

Summary

- Classical observables often can be efficiently calculated using quantum amplitudes in the framework of the KMOC formalism
- One very fruitful application of this formalism is to calculate corrections to the gravitational potential and waveforms from systems of compact objects in general relativity
- The formalism can be readily extended to scalar-tensor theories of gravity, where both gravitational and scalar radiation is present
- Many results for emitted power in gravitational and scalar waves found in the classical literature can reproduced in the KMOC approach
- We also derive novel results at higher PN and spin orders

Backup

Quantum Amplitudes and Radiation Observables

Expansion in QFT vs in classical GR for 2-to-2 scattering

	0PN	1PN	2PN	3PN	4PN
1PM	$\frac{1}{M_{\rm Pl}^2}$	$\frac{{\rm v}^2}{M_{\rm Pl}^2}$	$\frac{{\rm v}^4}{M_{\rm Pl}^2}$	$\frac{{\rm v}^6}{M_{\rm Pl}^2}$	$\frac{{\rm v}^8}{M_{\rm Pl}^2}$
2PM		$\frac{1}{M_{\rm Pl}^4}$	$\frac{{\rm v}^2}{M_{\rm Pl}^4}$	$\frac{{\rm v}^4}{M_{\rm Pl}^4}$	$\frac{{\rm v}^6}{M_{\rm Pl}^4}$
3PM			$\frac{1}{M_{\rm Pl}^6}$	$\frac{{\rm v}^2}{M_{\rm Pl}^6}$	$\frac{{\rm v}^4}{M_{\rm Pl}^6}$
4PM				$\frac{1}{M_{\rm Pl}^8}$	$\frac{{\rm v}^2}{M_{\rm Pl}^8}$
5PM					$\frac{1}{M_{\rm Pl}^{10}}$

Loop expansion

Velocity expansion

tree level $\sim \frac{1}{bM_{\rm Pl}^2}$

1-loop $\sim \frac{1}{b^2M_{\rm Pl}^4} \sim \frac{R_s}{b}\frac{1}{bM_{\rm Pl}^2}$

2-loop $\sim \frac{1}{b^3M_{\rm Pl}^6} \sim \left(\frac{R_s}{b}\right)^2\frac{1}{bM_{\rm Pl}^2}$

3-loop

4-loop

Kepler's law

$${\rm v}^2 \sim \frac{1}{bM_{\rm Pl}^2}$$

Schwarzschild radius

$$R_s \sim \frac{m}{M_{\rm Pl}^2}$$

Scalar-Tensor Theories: contact term domination

For previous results, take the limit of large contact terms $C_i \to \infty,\qquad c_i \to 0,\qquad C_i c_i$ finite

$$\partial_t W_\phi^{(0)} = \mathcal{O}(\mathrm{v}^2)$$

$$\partial_t W_\phi^{(1)} = \frac{m_1 m_2}{64\pi^2 M_{\rm Pl}^3 b^3}\left(\left[c_1 C_2^{(1)} \boldsymbol{a}_2 - c_2 C_1^{(1)} \boldsymbol{a}_1\right]\cdot \hat{\mathbf{v}}\right)\mathrm{v} + \mathcal{O}(\mathrm{v}^2)\,.$$

Emitted power in scalar radiation

$$P_\phi = \frac{m_1^2 m_2^2}{1024\pi^3 M_{\rm Pl}^6 b^6}\left(\left[c_1 C_2^{(1)} \boldsymbol{a}_2 - c_2 C_1^{(1)} \boldsymbol{a}_1\right]\cdot \hat{\mathbf{v}}\right)^2 \mathrm{v}^2 + \mathcal{O}(\mathrm{v}^3)$$

This resembles the formula valid for DCS gravity $(\alpha = 0,\quad \tilde{\alpha} \neq 0)$: $P_\phi \sim (\boldsymbol{a}\mathbf{v})^2 + \#\boldsymbol{a}^2\mathrm{v}^2$

But we don't know how to produce the second term above

Tanaka, Yagi, Yunes
arXiv:1208.5102

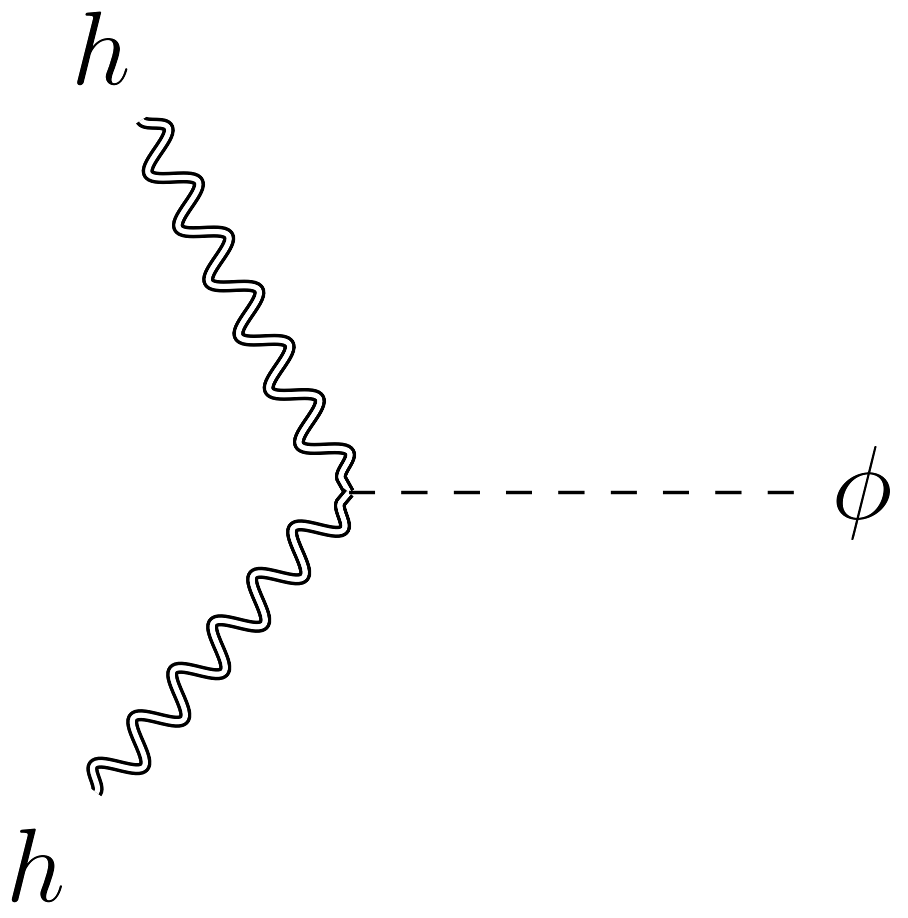
Waveforms for scattering of compact objects without scalar hair

Scalar-Tensor Theories

- $S_{SGB,DCS} = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}^2}{\Lambda^2}\left(f(\phi)\mathscr{G} + \tilde{f}(\phi)R\tilde{R}\right) + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi\right)\right]$

$$f(\phi) = \text{const} + \alpha\frac{\phi}{M_{\rm Pl}} + O(\phi^2) \qquad \tilde{f}(\phi) = \text{const} + \tilde{\alpha}\frac{\phi}{M_{\rm Pl}} + O(\phi^2)$$

The linear term induces a shift-symmetric cubic interaction between scalar and gravity corresponding to 3-point amplitudes



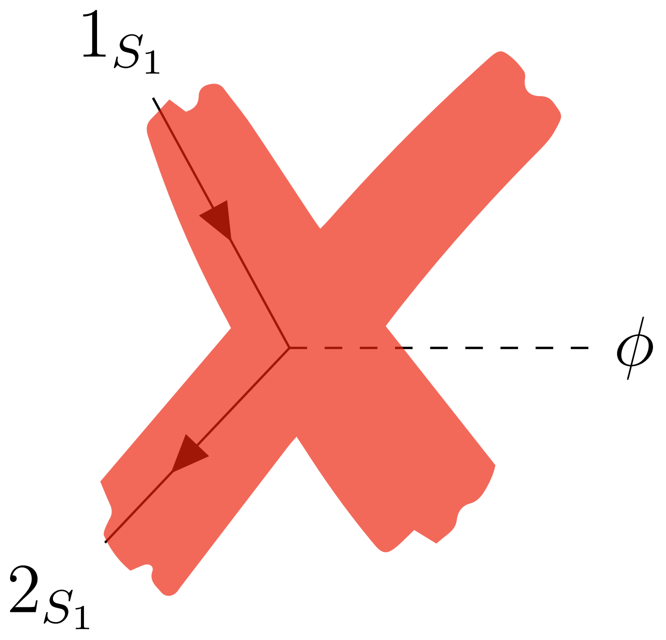
$$\mathscr{M}\left[1_h^- 2_h^- 3_\phi\right] = \frac{2\hat{\alpha}}{\Lambda^2 M_{\rm pl}}\langle 12\rangle^4$$

$$\mathscr{M}\left[1_h^+ 2_h^+ 3_\phi\right] = \frac{2\hat{\alpha}^*}{\Lambda^2 M_{\rm pl}}[12]^4$$

$$\mathscr{M}\left[1_h^- 2_h^+ 3_\phi\right] = 0$$

$\hat{\alpha} \equiv \alpha + i\tilde{\alpha}$

Here we assume no scalar hair



Scalar-Tensor Theories

- $S_{SGB,DCS} = \int d^4x\sqrt{-g}\left[\frac{M_{Pl}}{\Lambda^2}\left(f(\phi)\mathscr{G} + \tilde{f}(\phi)R\tilde{R}\right) + \frac{1}{2}\left(\partial^\mu\phi\partial_\mu\phi\right)\right]$

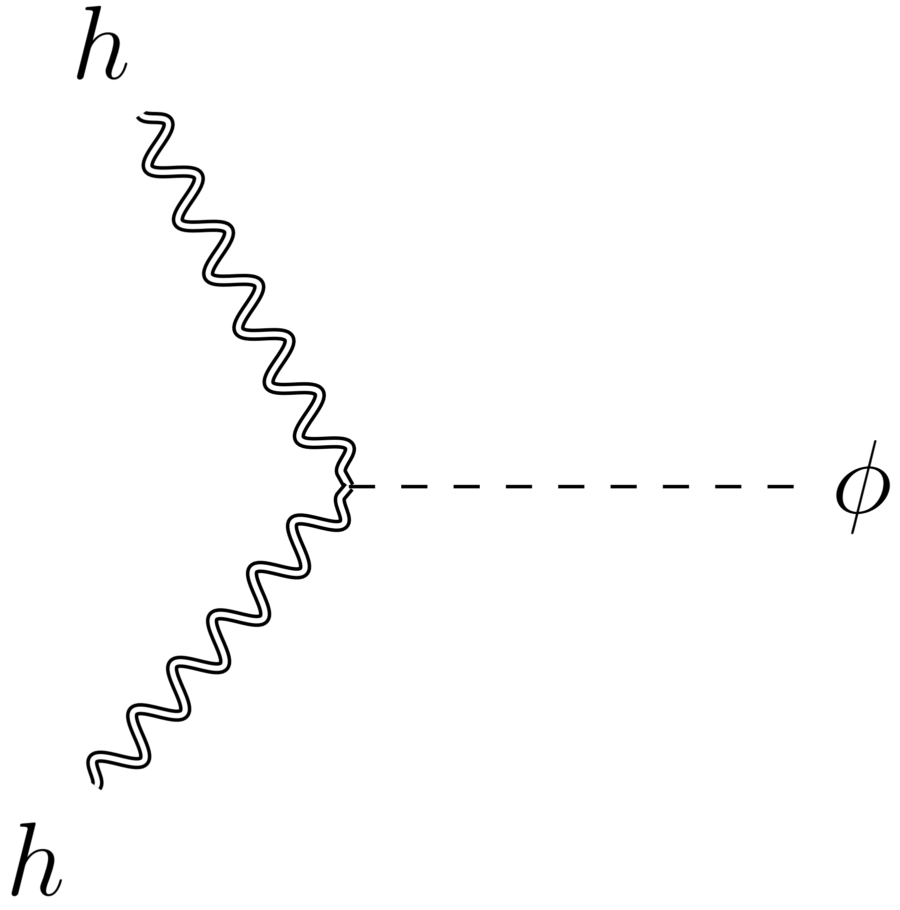
We assume scalar couplings to gravity are shift-symmetric:

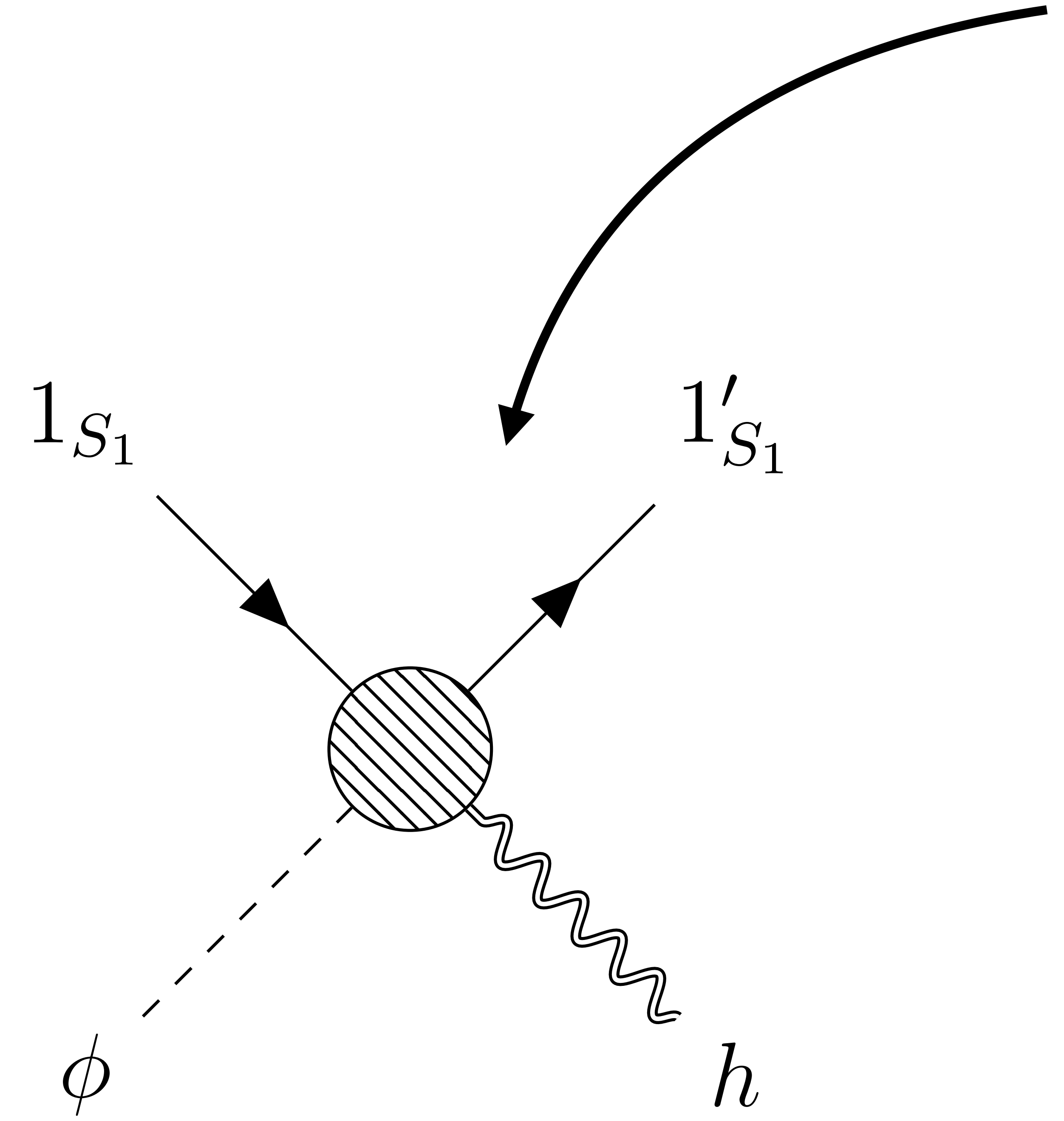
$f(\phi) = \mathrm{const} + \alpha\phi \qquad \tilde{f}(\phi) = \mathrm{const} + \alpha\tilde{\phi}$

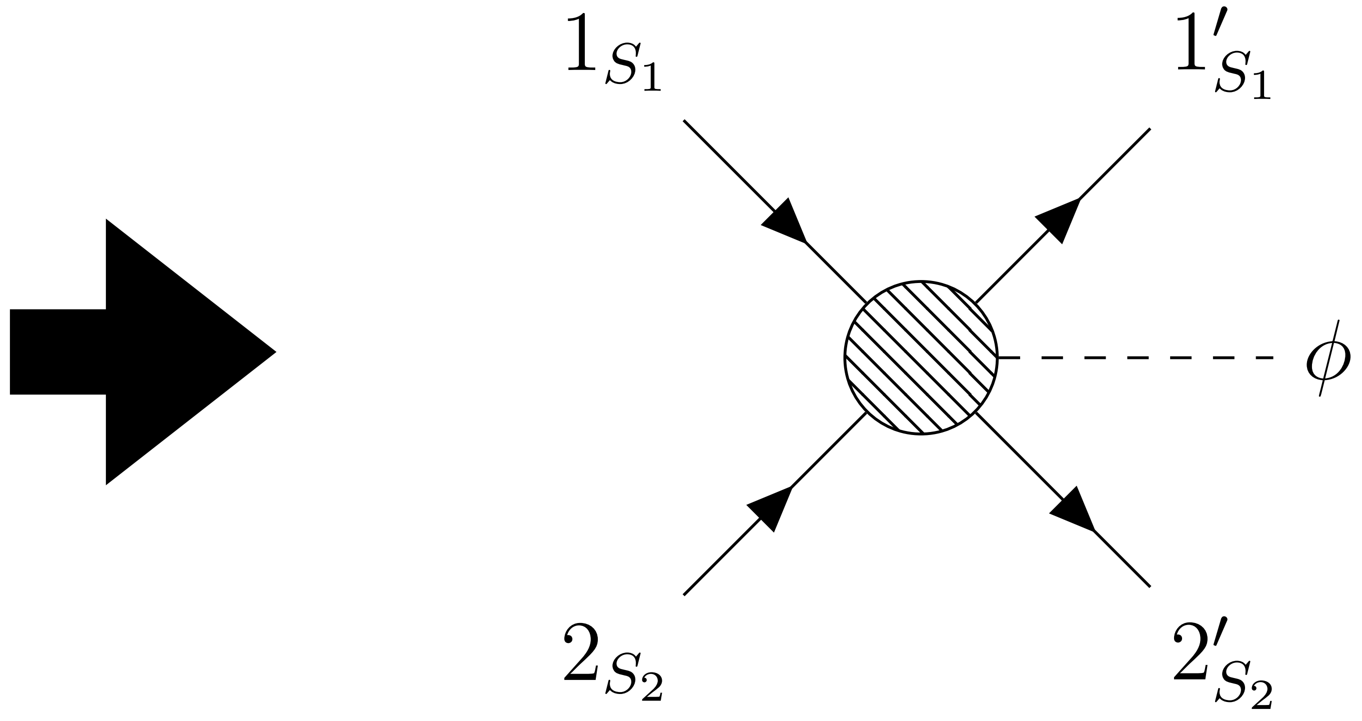
$$\mathscr{M}\left[1_h^- 2_h^- 3_\phi\right] = \frac{2\hat{\alpha}}{\Lambda^2 M_{\mathrm{pl}}}\langle 12\rangle^4 \qquad \hat{\alpha} \equiv \alpha + i\tilde{\alpha}$$

$$\mathscr{M}\left[1_h^+ 2_h^+ 3_\phi\right] = \frac{2\hat{\alpha}^*}{\Lambda^2 M_{\mathrm{pl}}}[12]^4$$

$$\mathscr{M}\left[1_h^- 2_h^+ 3_\phi\right] = 0$$







Scalar-Tensor Theories

In this case, the relevant 4-point amplitude

$q = p_1 + p_2 = - p_3 - p_4$

$$\mathscr{M}_U^{\mathrm{cl}}\left[1_{\Phi_i} 2_{\bar{\Phi}_i} 3_h^- 4_\phi\right] = \frac{2\hat{\alpha}\langle 3 | p_1 q | 3\rangle^2}{M_{\mathrm{Pl}}^2 \Lambda^2 q^2} e^{q a_1}$$

scales as $\mathscr{M}_U^{\mathrm{cl}} \to \mathscr{M}_U^{\mathrm{cl}} \hbar^2$ under classical scaling,
as opposed to $\mathscr{M}_U^{\mathrm{cl}} \to \mathscr{M}_U^{\mathrm{cl}} \hbar^0$ for Compton amplitudes in GR or QED
of for scalar analogous amplitude in the presence of scalar hair

$$\mathscr{M}_U^{\mathrm{cl}}\left[1_{\Phi_n} 2_{\bar{\Phi}_n} 3_\phi 4_h^-\right] = - \frac{c_n m_n^2}{M_{\mathrm{Pl}}^2} \frac{\langle 4 | p_3 p_1 | 4\rangle^2}{s(t - m_n^2)^2}$$

Additional suppression does not mean however that the effect is necessarily quantum,
but places more stringent constraints on resolvability of radiation

Bellazzini, Isabella, Riva
arXiv:2211.00085

We find the effect is resolvable when

$$R_s \ll b \ll b_{\max}, \qquad b_{\max} = R_s \frac{M_{\mathrm{Pl}}}{m^{1/3} \Lambda^{2/3}}$$

Cf. resolvability on nPM corrections in GR

$$\frac{m^2}{M_{\mathrm{Pl}}^2}\left(\frac{R_s}{b}\right)^{n-1} \gg 1$$

Scalar-Tensor Theories

LO scalar waveform in CP-even limit :

$$
\begin{aligned}
W_\phi = {} & \frac{m_1 m_2}{8\pi^2 b^3 M_{Pl}^3 \Lambda^2 (\hat{u}_1 n)^2 \sqrt{\gamma^2-1}} \Bigg(\alpha \Bigg\{ -\frac{d^2}{dz^2} \Bigg[\frac{1}{\sqrt{z^2+1}} \frac{2(\gamma^2-1)^2 (\hat{u}_2 n)^2 \big[\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b} n) z\big]}{\big[\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b} n) z\big]^2 + (\tilde{v} n)^2 (z^2+1)} \Bigg] + \big[(\hat{u}_1 n) + \gamma\big(2\gamma^2 - 3\big)(\hat{u}_2 n)\big] \frac{2z^2-1}{(z^2+1)^{5/2}} + \big(2\gamma^2-1\big)(\tilde{b} n) \frac{3z}{(z^2+1)^{5/2}} \Bigg\} \\
& + \frac{2\alpha}{b} \frac{d^3}{dz^3} \mathrm{Re} \Bigg\{ \frac{1}{\sqrt{z^2+1}} \frac{(\hat{u}_2 n) - \gamma(\hat{u}_1 n) + \gamma(\tilde{b} n) z + i\gamma(\tilde{v} n)\sqrt{z^2+1}}{\gamma(\hat{u}_2 n) - (\hat{u}_1 n) + (\tilde{b} n) z + i(\tilde{v} n)\sqrt{z^2+1}} \Bigg[z(\tilde{v} n) - i\sqrt{z^2+1}(\tilde{b} n) \Bigg] \times \Bigg[\frac{(a_1 n)}{(\hat{u}_1 n)} - \gamma(a_1 \hat{u}_2) - (a_2 \hat{u}_1) - (a_1^A - a_2^A)(z\tilde{b}^A + i\sqrt{z^2+1}\,\tilde{v}^A) \Bigg] \Bigg\} \\
& - \sqrt{\gamma^2-1} \frac{C_1}{b} \frac{d^3}{dz^3} \Bigg\{ \frac{1}{\sqrt{z^2+1}} \mathrm{Re} \Bigg[\Bigg(z\big[(\tilde{v} n) a_1^A + (a_1 \tilde{v}) n^A\big] - i\sqrt{z^2+1}\big[(\tilde{b} n) a_1^A + (a_1 \tilde{b}) n^A\big] \Bigg) \times \Bigg(\hat{u}_2^A - \gamma \hat{u}_1^A + \gamma\big[z\tilde{b}^A + i\sqrt{z^2+1}\,\tilde{v}^A\big] \Bigg) \Bigg] \Bigg\} \Bigg) \Bigg|_{z=T_1} + (1 \leftrightarrow 2) + \mathcal{O}(a^2)\,.
\end{aligned}
$$

Scalar-Tensor Theories

LO scalar waveform for scalar Gauss-Bonnet:

$$
\begin{aligned}
W_\phi = \frac{m_1 m_2}{8\pi^2 b^3 M_{Pl}^3 \Lambda^2 (\hat{u}_1 n)^2 \sqrt{\gamma^2-1}} \Bigg(\alpha \Bigg\{ -\frac{d^2}{dz^2}\Bigg[\frac{1}{\sqrt{z^2+1}} \frac{2(\gamma^2-1)^2(\hat{u}_2 n)^2\big[\gamma(\hat{u}_2 n)-(\hat{u}_1 n)+(\tilde{b}n)z\big]}{\big[\gamma(\hat{u}_2 n)-(\hat{u}_1 n)+(\tilde{b}n)z\big]^2+(\tilde{v}n)^2(z^2+1)}\Bigg] + \big[(\hat{u}_1 n)+\gamma\big(2\gamma^2-3\big)(\hat{u}_2 n)\big]\frac{2z^2-1}{(z^2+1)^{5/2}} + \big(2\gamma^2-1\big)(\tilde{b}n)\frac{3z}{(z^2+1)^{5/2}} \Bigg\} \\
+\frac{2\alpha}{b}\frac{d^3}{dz^3}\mathrm{Re}\Bigg\{\frac{1}{\sqrt{z^2+1}}\frac{(\hat{u}_2 n)-\gamma(\hat{u}_1 n)+\gamma(\tilde{b}n)z+i\gamma(\tilde{v}n)\sqrt{z^2+1}}{\gamma(\hat{u}_2 n)-(\hat{u}_1 n)+(\tilde{b}n)z+i(\tilde{v}n)\sqrt{z^2+1}}\Big[z(\tilde{v}n)-i\sqrt{z^2+1}(\tilde{b}n)\Big]\times\Bigg[\frac{(a_1 n)}{(\hat{u}_1 n)}-\gamma(a_1\hat{u}_2)-(a_2\hat{u}_1)-(a_1^A-a_2^A)(z\tilde{b}^A+i\sqrt{z^2+1}\tilde{v}^A)\Bigg]\Bigg\} \\
-\sqrt{\gamma^2-1}\frac{C_1}{b}\frac{d^3}{dz^3}\Bigg\{\frac{1}{\sqrt{z^2+1}}\mathrm{Re}\Bigg[\Bigg(z\big[(\tilde{v}n)a_1^A+(a_1\tilde{v})n^A\big]-i\sqrt{z^2+1}\big[(\tilde{b}n)a_1^A+(a_1\tilde{b})n^A\big]\Bigg)\times\Bigg(\hat{u}_2^A-\gamma\hat{u}_1^A+\gamma\big[z\tilde{b}^A+i\sqrt{z^2+1}\tilde{v}^A\big]\Bigg)\Bigg]\Bigg\}\Bigg)\Bigg|_{z=T_1} + (1\leftrightarrow 2)+\mathcal{O}(a^2)\,.
\end{aligned}
$$

$$\left.\frac{dP_\phi}{d\Omega}\right|_{\mathcal{O}(a^0)} \sim \frac{\mathrm{v}^6}{b^8} \xrightarrow{\boxed{\text{For closed orbits}}} \left.\frac{dP_\phi}{d\Omega}\right|_{\mathcal{O}(a^0)} \sim \mathrm{v}^{22}$$

$$\left.\frac{dP_\phi}{d\Omega}\right|_{\mathcal{O}(a^1)} \sim \frac{\mathrm{v}^4}{b^{10}} \xrightarrow{\boxed{\text{For closed orbits}}} \left.\frac{dP_\phi}{d\Omega}\right|_{\mathcal{O}(a^1)} \sim \mathrm{v}^{24}$$

Much **bigger suppression** compared to $P_\phi \sim \mathrm{v}^8$ in the presence of scalar hair

Connect to observables: Power emitted in scalar radiation

$$\frac{dP_\phi}{d\Omega} = (\partial_t W_\phi)^2$$