Black holes with primary scalar hair

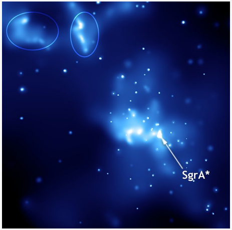
Christos Charmousis

IJCLab-CNRS

ThUG Conference-IPhT Saclay

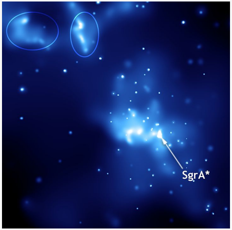
Collaborators : E. Babichev [1312.3204 [gr-qc]]
A. Bakopoulos, N. Lecoeur, P. Kanti, T. Nakas [2310.11919 [gr-qc]]
S. Iteanu, D. Langlois, K Noui [2503.22348 [gr-qc]]

Breakthrough in observational data concerning compact objects

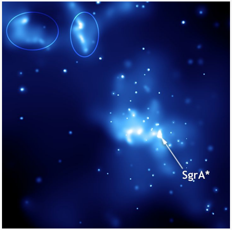


- There is a multitude of observational data for compact objects
- GR with a tiny cosmological constant is an effective theory that is compatible with data

Breakthrough in observational data concerning compact objects




- There is a multitude of observational data for compact objects
- GR with a tiny cosmological constant is an effective theory that is compatible with data
- Can we find alternatives to GR black holes and neutron stars as precise rulers of departure from GR?




- Horndeski theories and beyond... as a measurable departure from GR
- Constructing black holes with primary hair
- Axial perturbations. The effective and background metrics
- Concluding remarks

Key notions :Primary and secondary hair, disformal transformations, local and global symmetries, effective and background metric

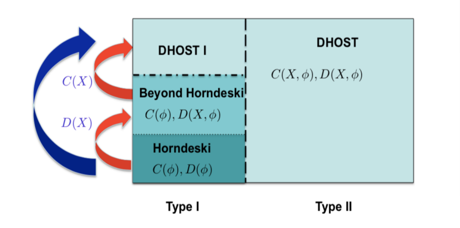
Scalar tensor theories : a robust measurable departure from GR

Simplest modified gravity theory with a single scalar degree of freedom

BD theory,..., Horndeski (or generalised Galileon [Deffayet, Deser, Esposito-Farèse,...]),..., beyond Horndeski,..., DHOST theories [Achour, Crisostomi, Koyama, Langlois, Noui, Piazza, Vernizzi, et.al.]

- *Nothing fundamental* about ST theories, they are just measurable departures from GR which are robust with a single additional degree of freedom
- They are limits of more complex fundamental theories (massive gravity, braneworld models, EFT from string theory, Lovelock theory etc.)
- Horndeski is parametrized by 4 functions of scalar and its kinetic energy, $G_i = G_i(\phi, X)$ with $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$.
- Beyond Horndeski or DHOST are parametrised by two more functions corresponding to conformal and dysformal transformations of Horndeski

$$g_{\mu\nu} \longrightarrow \tilde{g}_{\mu\nu} = C(\phi, X) g_{\mu\nu} + D(\phi, X) \nabla_\mu \phi \nabla_\nu \phi$$



[Langlois, 2018]

We now seek solutions within this vast theory

Stealth solutions with secondary black hole hair [Babichev, CC, 2013]

Scalar tensor theories

Example in Horndeski with $G_2 = \Lambda + \eta X$, $G_4 = 1 + 2\beta X$

$$S = \int d^4x \sqrt{-g} \left[R - 2\Lambda_b - X + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right],$$

- Kinetic term is $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ $(= -\frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu)$.
- The theory has global shift and parity symmetry. Conserved current $\nabla_\mu J^\mu = 0$ for the scalar field equation
- One simple (stealth) solution reads

$$f = h = 1 - \frac{2M}{r} - \frac{1}{6\beta} r^2$$

,

$$\phi = qt + \int dr \, \frac{q}{f} \sqrt{1 - f}$$

with secondary hair $q^2 = \frac{1 - 2\Lambda_b \beta}{\beta}$ related to the action couplings.

Stealth solutions with secondary black hole hair [Babichev, CC, 2013]

Scalar tensor theories

Example in Horndeski with $G_2 = \Lambda + \eta X$, $G_4 = 1 + 2\beta X$

$$S = \int d^4x \sqrt{-g} \left[R - 2\Lambda_b - X + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right],$$

- Kinetic term is $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ $(= -\frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu)$.
- The theory has global shift and parity symmetry. Conserved current $\nabla_\mu J^\mu = 0$ for the scalar field equation
- One simple (stealth) solution reads

$$f = h = 1 - \frac{2M}{r} - \frac{1}{6\beta} r^2$$

,

$$\phi = qt + \int dr \, \frac{q}{f} \sqrt{1 - f}$$

with secondary hair $q^2 = \frac{1 - 2\Lambda_b \beta}{\beta}$ related to the action couplings.

Stealth solutions with secondary black hole hair [Babichev, CC, 2013]

Scalar tensor theories

Example in Horndeski with $G_2 = \Lambda + \eta X$, $G_4 = 1 + 2\beta X$

$$S = \int d^4x \sqrt{-g} \left[R - 2\Lambda_b - X + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right] ,$$

- Kinetic term is $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ $(= -\frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu)$.
- The theory has global shift and parity symmetry. Conserved current $\nabla_\mu J^\mu = 0$ for the scalar field equation
- One simple (stealth) solution reads

$$f = h = 1 - \frac{2M}{r} - \frac{1}{6\beta} r^2$$

,

$$\phi = qt + \int dr \, \frac{q}{f} \sqrt{1 - f}$$

with secondary hair $q^2 = \frac{1 - 2\Lambda_b \beta}{\beta}$ related to the action couplings.

Shift symmetry allows for linear time dependence.

The associated energy-momentum tensor for the scalar must have the same symmetries as the metric,... not the scalar field itself

Stealth solutions with secondary black hole hair [Babichev, CC, 2013]

Scalar tensor theories

Example in Horndeski with $G_2 = \Lambda + \eta X$, $G_4 = 1 + 2\beta X$

$$S = \int d^4x \sqrt{-g} \left[R - 2\Lambda_b - X + \beta G^{\mu\nu} \partial_\mu \phi \partial_\nu \phi \right],$$

- Kinetic term is $X = -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi$ $(= -\frac{1}{2} g^{\mu\nu} \phi_\mu \phi_\nu)$.
- The theory has global shift and parity symmetry. Conserved current $\nabla_\mu J^\mu = 0$ for the scalar field equation
- One simple (stealth) solution reads

$$f = h = 1 - \frac{2M}{r} - \frac{1}{6\beta} r^2$$

,

$$\phi = qt + \int dr \, \frac{q}{f} \sqrt{1 - f}$$

with secondary hair $q^2 = \frac{1 - 2\Lambda_b \beta}{\beta}$ related to the action couplings.

- Properties :
  $X = g^{\mu\nu} \phi_\mu \phi_\nu = -q^2$ is constant and stealth solutions are generic [Kobayashi, Tanahashi]
- Scalar field is regular at the (future) event horizon for all q. $\phi = qv - q \int \frac{dr}{1 + \sqrt{1-f}}$, in advanced EF coordinates where $dv = dt + \frac{dr}{f}$

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S[g_{\mu\nu},\phi] = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\Big\{G_2(X) + G_4(X)R + G_{4X}\left[(\Box\phi)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + F_4(X)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_\mu\phi_\eta\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$ds^2 = -h(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2, \qquad \phi = qt + \psi(r),$$

- Setting, $Z(X) = 2XG_{4X} - G_4 + 4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S\left[g_{\mu\nu},\phi\right] = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big\{G_2\left(X\right) + G_4\left(X\right)R + G_{4X}\left[\left(\Box\phi\right)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + F_4\left(X\right)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_{\mu}\phi_{\eta}\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$\mathrm{d}s^2 = -h\left(r\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f\left(r\right)} + r^2\mathrm{d}\Omega^2, \qquad \phi = qt + \psi\left(r\right),$$

- Setting, $Z\left(X\right) = 2XG_{4X} - G_4 + 4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S\left[g_{\mu\nu},\phi\right]=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big\{G_2\left(X\right) + G_4\left(X\right)R+G_{4X}\left[\left(\Box\phi\right)^2-\phi_{\mu\nu}\phi^{\mu\nu}\right]+ \\ + F_4\left(X\right)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_{\mu}\phi_{\eta}\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$\mathrm{d}s^2=-h\left(r\right)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f\left(r\right)}+r^2\mathrm{d}\Omega^2,\qquad \phi=qt+\psi\left(r\right),$$

- Setting, $Z\left(X\right)=2XG_{4X}-G_4+4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S\left[g_{\mu\nu},\phi\right]=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big\{G_2\left(X\right)+G_4\left(X\right)R+G_{4X}\left[\left(\Box\phi\right)^2-\phi_{\mu\nu}\phi^{\mu\nu}\right]+ \\ +F_4\left(X\right)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_\mu\phi_\eta\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$\mathrm{d}s^2=-h\left(r\right)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f\left(r\right)}+r^2\mathrm{d}\Omega^2,\qquad \phi=qt+\psi\left(r\right),$$

- Setting, $Z\left(X\right)=2XG_{4X}-G_4+4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S\left[g_{\mu\nu},\phi\right]=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big\{G_2\left(X\right)\quad+\quad G_4\left(X\right)R+G_{4X}\left[\left(\Box\phi\right)^2-\phi_{\mu\nu}\phi^{\mu\nu}\right]+$$
$$+\quad F_4\left(X\right)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_\mu\phi_\eta\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$\mathrm{d}s^2=-h\left(r\right)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f\left(r\right)}+r^2\mathrm{d}\Omega^2,\qquad \phi=qt+\psi\left(r\right),$$

- Setting, $Z\left(X\right)=2XG_{4X}-G_4+4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

$$2X'Z_X=Z\left(\frac{h'}{h}-\frac{f'}{f}\right)$$

$$r^2(G_2Z)_X+2(G_4Z)_X\left(1-\frac{q^2\gamma^2}{2Z^2X}\right)=0$$

$$2\gamma^2\left(hr-\frac{q^2r}{2X}\right)'=-r^2G_2Z-2G_4Z\left(1-\frac{q^2\gamma^2}{2Z^2X}\right)+\frac{q^2\gamma^2X'r}{ZX^2}\left(2XG_{4X}-G_4\right)$$

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S\left[g_{\mu\nu},\phi\right]=\frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big\{G_2\left(X\right) + G_4\left(X\right)R+G_{4X}\left[\left(\Box\phi\right)^2-\phi_{\mu\nu}\phi^{\mu\nu}\right]+ + F_4\left(X\right)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_\mu\phi_\eta\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$\mathrm{d}s^2=-h\left(r\right)\mathrm{d}t^2+\frac{\mathrm{d}r^2}{f\left(r\right)}+r^2\mathrm{d}\Omega^2,\qquad \phi=qt+\psi\left(r\right),$$

- Setting, $Z\left(X\right)=2XG_{4X}-G_4+4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

$$\frac{f}{h}=\frac{\gamma^2}{Z^2}$$

$$r^2(G_2Z)_X+2(G_4Z)_X\left(1-\frac{q^2\gamma^2}{2Z^2X}\right)=0$$

$$2\gamma^2\left(hr-\frac{q^2r}{2X}\right)'=-r^2G_2Z-2G_4Z\left(1-\frac{q^2\gamma^2}{2Z^2X}\right)+\frac{q^2\gamma^2X'r}{ZX^2}\left(2XG_{4X}-G_4\right)$$

Obtaining primary hair black holes [Bakopoulos, CC, Kanti, Lecoeur, Nakas]

- Consider shift and parity symmetric beyond Horndeski theory parametrised $G_2(X)$, $G_4(X)$, $F_4(X)$ [Gleyzes, Langlois, Piazza, Vernizzi]
- The theory reads,

$$S\left[g_{\mu\nu},\phi\right] = \frac{1}{2\kappa}\int \mathrm{d}^4x\sqrt{-g}\Big\{G_2\left(X\right) \quad + \quad G_4\left(X\right)R + G_{4X}\left[\left(\Box\phi\right)^2 - \phi_{\mu\nu}\phi^{\mu\nu}\right] + \\ + \quad F_4\left(X\right)\epsilon^{\mu\nu\rho\sigma}\epsilon^{\eta\beta\gamma}{}_{\sigma}\phi_\mu\phi_\eta\phi_{\nu\beta}\phi_{\rho\gamma}\Big\}.$$

- We seek spherically symmetric and static solutions

$$\mathrm{d}s^2 = -h\left(r\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f\left(r\right)} + r^2\mathrm{d}\Omega^2, \qquad \phi = qt + \psi\left(r\right),$$

- Setting, $Z\left(X\right) = 2XG_{4X} - G_4 + 4X^2F_4$, we find
- Field Equations are integrable for given BH theory:

$$f = h, Z = \gamma$$

$$r^2G_{2X} + 2G_{4X}\left(1 - \frac{q^2}{2X}\right) = 0$$

$$2\gamma\left(hr - \frac{q^2r}{2X}\right)' = -r^2G_2 - 2G_4\left(1 - \frac{q^2}{2X}\right) + \frac{q^2X'\gamma r}{X^2}\left(2XG_{4X} - G_4\right)$$

Primary hair black holes: Solving the system

[A. Bakopoulos, C.C, P. Kanti, N. Lecoeur., T. Nakas, 2023]

[Baake, Cisterna, Hassaine, Hernandes-Vera 2024], [Bakopoulos, Chatzifotis, Nakas 2024]

- For $Z = \gamma$ theories therefore we have $f = h$ (homogeneous solutions)
- The class of theories

$$G_2(X) = -\frac{2\eta}{\lambda^2} X^p, \qquad G_4(X) = 1 - \eta X^p, \qquad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

  where η and λ are the coupling constants and p is integer of half-integer.
- the scalar equation,

$$r^2 (G_2)_X + 2(G_4)_X \left(1 - \frac{q^2}{2X}\right) = 0$$

- gives X which is not constant but regular everywhere and the scalar $\phi = qt + \psi(r)$:

$$X = \frac{q^2/2}{1+(r/\lambda)^2} \qquad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right],$$

  Finally we solve for the metric,
- $$f(r) = 1 - \frac{2M}{r} - \frac{2\lambda\xi_p}{r}\int du\, \frac{u^2/\lambda^2}{(1+(u/\lambda)^2)^p}$$

  We have two independent integration constants M and $\xi_p = \eta(2p-1)(q^2/2)^p$
- Noether charge : $Q = \int_\Sigma d^3x\sqrt{\gamma}n_\mu J^\mu \propto q^{2p-1}\left[\frac{\eta}{2^p}\int_0^\infty \frac{r^2}{(1+(r/\lambda)^2)^p}dr\right]$ associated to Noether current $\nabla_\mu J^\mu = 0$.

Primary hair black holes: Solving the system

[A. Bakopoulos, C.C, P. Kanti, N. Lecoeur., T. Nakas, 2023]

[Baake, Cisterna, Hassaine, Hernandes-Vera 2024], [Bakopoulos, Chatzifotis, Nakas 2024]

- For $Z=\gamma$ theories therefore we have $f=h$ (homogeneous solutions)
- The class of theories

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \quad G_4(X) = 1-\eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

where η and λ are the coupling constants and p is integer of half-integer.

- the scalar equation,

$$r^2(G_2)_X + 2(G_4)_X\left(1-\frac{q^2}{2X}\right) = 0$$

- gives X which is not constant but regular everywhere and the scalar $\phi = qt + \psi(r)$:

$$X = \frac{q^2/2}{1+(r/\lambda)^2} \quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1-\frac{f(r)}{1+(r/\lambda)^2}\right],$$

Finally we solve for the metric,

-

$$f(r) = 1-\frac{2M}{r}-\frac{2\lambda\xi_p}{r}\int du\,\frac{u^2/\lambda^2}{(1+(u/\lambda)^2)^p}$$

We have two independent integration constants M and $\xi_p = \eta(2p-1)(q^2/2)^p$

- Noether charge : $Q=\int_\Sigma d^3x\sqrt{\gamma}n_\mu J^\mu \propto q^{2p-1}\left[\frac{\eta}{2^p}\int_0^\infty \frac{r^2}{(1+(r/\lambda)^2)^p}dr\right]$ associated to Noether current $\nabla_\mu J^\mu = 0$.

Primary hair black holes: Solving the system

[A. Bakopoulos, C.C, P. Kanti, N. Lecoeur., T. Nakas, 2023]

[Baake, Cisterna, Hassaine, Hernandes-Vera 2024], [Bakopoulos, Chatzifotis, Nakas 2024]

- For $Z = \gamma$ theories therefore we have $f = h$ (homogeneous solutions)
- The class of theories

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \qquad G_4(X) = 1 - \eta X^p, \qquad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

  where η and λ are the coupling constants and p is integer of half-integer.
- the scalar equation,

$$r^2(G_2)_X + 2(G_4)_X\left(1 - \frac{q^2}{2X}\right) = 0$$

- gives X which is not constant but regular everywhere and the scalar $\phi = qt + \psi(r)$:

$$X = \frac{q^2/2}{1+(r/\lambda)^2} \quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right],$$

  Finally we solve for the metric,
- $$f(r) = 1 - \frac{2M}{r} - \frac{2\lambda\xi_p}{r}\int du\,\frac{u^2/\lambda^2}{(1+(u/\lambda)^2)^p}$$

  We have two independent integration constants M and $\xi_p = \eta(2p-1)(q^2/2)^p$
- Noether charge : $Q = \int_\Sigma d^3x\sqrt{\gamma}n_\mu J^\mu \propto q^{2p-1}\left[\frac{\eta}{2^p}\int_0^\infty \frac{r^2}{(1+(r/\lambda)^2)^p}dr\right]$ associated to Noether current $\nabla_\mu J^\mu = 0$.

Primary hair black holes: Solving the system

[A. Bakopoulos, C.C, P. Kanti, N. Lecoeur., T. Nakas, 2023]

[Baake, Cisterna, Hassaine, Hernandes-Vera 2024], [Bakopoulos, Chatzifotis, Nakas 2024]

- For $Z = \gamma$ theories therefore we have $f = h$ (homogeneous solutions)
- The class of theories

$$G_2(X) = -\frac{2\eta}{\lambda^2} X^p, \qquad G_4(X) = 1 - \eta X^p, \qquad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

  where η and λ are the coupling constants and p is integer of half-integer.
- the scalar equation,

$$r^2 (G_2)_X + 2(G_4)_X \left(1 - \frac{q^2}{2X}\right) = 0$$

- gives X which is not constant but regular everywhere and the scalar $\phi = qt + \psi(r)$:

$$X = \frac{q^2/2}{1+(r/\lambda)^2} \quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right],$$

  Finally we solve for the metric,
- $$f(r) = 1 - \frac{2M}{r} - \frac{2\lambda\xi_p}{r}\int du\, \frac{u^2/\lambda^2}{(1+(u/\lambda)^2)^p}$$

  We have two independent integration constants M and $\xi_p = \eta(2p-1)(q^2/2)^p$
- Noether charge : $Q = \int_\Sigma d^3x\sqrt{\gamma}n_\mu J^\mu \propto q^{2p-1}\left[\frac{\eta}{2^p}\int_0^\infty \frac{r^2}{(1+(r/\lambda)^2)^p}dr\right]$ associated to Noether current $\nabla_\mu J^\mu = 0$.

Primary hair black holes: Solving the system

[A. Bakopoulos, C.C, P. Kanti, N. Lecoeur., T. Nakas, 2023]

[Baake, Cisterna, Hassaine, Hernandes-Vera 2024], [Bakopoulos, Chatzifotis, Nakas 2024]

- For $Z = \gamma$ theories therefore we have $f = h$ (homogeneous solutions)
- The class of theories

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \quad G_4(X) = 1 - \eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}.$$

  where η and λ are the coupling constants and p is integer of half-integer.
- the scalar equation,

$$r^2(G_2)_X + 2(G_4)_X\left(1 - \frac{q^2}{2X}\right) = 0$$

- gives X which is not constant but regular everywhere and the scalar $\phi = qt + \psi(r)$:

$$X = \frac{q^2/2}{1+(r/\lambda)^2} \quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right],$$

  Finally we solve for the metric,
-

$$f(r) = 1 - \frac{2M}{r} - \frac{2\lambda\xi_p}{r}\int du\,\frac{u^2/\lambda^2}{(1+(u/\lambda)^2)^p}$$

  We have two independent integration constants M and $\xi_p = \eta(2p-1)(q^2/2)^p$
- Noether charge : $Q = \int_\Sigma d^3x\sqrt{\gamma}n_\mu J^\mu \propto q^{2p-1}\left[\frac{\eta}{2^p}\int_0^\infty \frac{r^2}{(1+(r/\lambda)^2)^p}dr\right]$ associated to Noether current $\nabla_\mu J^\mu = 0$.

Primary hair black holes: Solving the system

[A. Bakopoulos, C.C, P. Kanti, N. Lecoeur., T. Nakas, 2023]

[Baake, Cisterna, Hassaine, Hernandes-Vera 2024], [Bakopoulos, Chatzifotis, Nakas 2024]

- For $Z = \gamma$ theories therefore we have $f = h$ (homogeneous solutions)
- The class of theories

$$G_2(X) = -\frac{2\eta}{\lambda^2} X^p, \qquad G_4(X) = 1 - \eta X^p, \qquad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

where η and λ are the coupling constants and p is integer of half-integer.

- the scalar equation,

$$r^2 (G_2)_X + 2(G_4)_X \left(1 - \frac{q^2}{2X}\right) = 0$$

- gives X which is not constant but regular everywhere and the scalar $\phi = qt + \psi(r)$:

$$X = \frac{q^2/2}{1+(r/\lambda)^2} \quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right],$$

Finally we solve for the metric,

- $$f(r) = 1 - \frac{2M}{r} - \frac{2\lambda\xi_p}{r}\int du\, \frac{u^2/\lambda^2}{(1+(u/\lambda)^2)^p}$$

We have two independent integration constants M and $\xi_p = \eta(2p-1)(q^2/2)^p$

- Noether charge : $Q = \int_\Sigma d^3x \sqrt{\gamma} n_\mu J^\mu \propto q^{2p-1}\left[\frac{\eta}{2^p}\int_0^\infty \frac{r^2}{(1+(r/\lambda)^2)^p}\, dr\right]$ associated to Noether current $\nabla_\mu J^\mu = 0$.

Primary hair black hole examples and properties

For generic theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \quad G_4(X) = 1 - \eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

We have primary hair black holes with :

-Scalar charge $Q(\eta, q, p) < \infty$ finite iff $p > 3/2$

-Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$

Examples parametrised by M and $\xi_p = \eta(2p-1)\frac{q^{2p}}{2^p}$

- We have :

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2,$$

Primary hair black hole examples and properties

For generic theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \qquad G_4(X) = 1 - \eta X^p, \qquad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

We have primary hair black holes with :

-Scalar charge $Q(\eta, q, p) < \infty$ finite iff $p > 3/2$

-Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$

Examples parametrised by M and $\xi_p = \eta(2p-1)\frac{q^{2p}}{2^p}$

- We have :

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2,$$

$p = 1/2$: Stealth solution as $f(r) = 1 - \frac{2M}{r}$. This is the only homogeneous Horndeski case!

Primary hair black hole examples and properties

For generic theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2} X^p, \quad G_4(X) = 1 - \eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}.$$

We have primary hair black holes with :

-Scalar charge $Q(\eta, q, p) < \infty$ finite iff $p > 3/2$

-Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$

Examples parametrised by M and $\xi_p = \eta(2p-1)\frac{q^{2p}}{2^p}$

- We have :

$$ds^2 = -f(r)\,dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2,$$

$p = 1$, Canonical kinetic term : $f(r) = 1 - 2\xi_1 - \frac{2M}{r} - 2\xi_1 \frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda}$.
The solution is only locally asymptotically flat.
Similar to gravitational monopole [Bariola, Villenkin]

Primary hair black hole examples and properties

For generic theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2} X^p, \quad G_4(X) = 1 - \eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}.$$

We have primary hair black holes with :

-Scalar charge $Q(\eta, q, p) < \infty$ finite iff $p > 3/2$

-Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$

Examples parametrised by M and $\xi_p = \eta(2p-1)\frac{q^{2p}}{2^p}$

- We have :

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2 \mathrm{d}\Omega^2,$$

$p = 2$ Asymptotically flat black hole with primary hair,
$f(r) = 1 - \frac{2M}{r} + \xi_2\left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2}\right)$

Primary hair black hole examples and properties

For generic theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \qquad G_4(X) = 1 - \eta X^p, \qquad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

We have primary hair black holes with :

-Scalar charge $Q(\eta, q, p) < \infty$ finite iff $p > 3/2$

-Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$

Examples parametrised by M and $\xi_p = \eta(2p-1)\frac{q^{2p}}{2^p}$

- We have :

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2,$$

$p = 5/2$ etc...

$$f(r) = 1 - \frac{2M}{r} + \frac{2\xi_{5/2}\,\lambda}{3r}\left(1 - \frac{(r/\lambda)^3}{(1+(r/\lambda)^2)^{3/2}}\right)$$

Primary hair black hole examples and properties

For generic theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2} X^p, \quad G_4(X) = 1 - \eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

We have primary hair black holes with :

-Scalar charge $Q(\eta, q, p) < \infty$ finite iff $p > 3/2$

-Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\quad \psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$

Examples parametrised by M and $\xi_p = \eta(2p-1)\frac{q^{2p}}{2^p}$

- We have :

$$\mathrm{d}s^2 = -f(r)\,\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(r)} + r^2\mathrm{d}\Omega^2,$$

What is the asymptotic structure? What are the possible horizons?

Take $p = 2$ and $f(r) = 1 - \frac{2M}{r} + \xi_2\left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2}\right)$

Horizons and asymptotics

Far away the black hole behaves much like RN but with the scalar playing the role of EM charge

$$f(r) = 1 - \frac{2M}{r} + \xi\left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2}\right)$$
$$= 1 - \frac{2M}{r} + 2\lambda^2\frac{\xi}{r^2} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad r \to \infty$$

while close to the origin we get,

$$f(r) = 1 - \frac{2M - \pi\xi\lambda/2}{r} - \frac{2\xi r^2}{3\lambda^2} + \mathcal{O}\left(r^4\right).$$

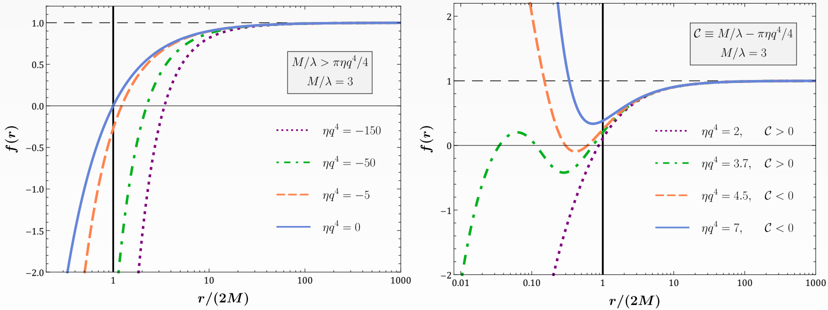


Figure: Left: $\eta < 0$, unique horizon greater than the Schwarzschild radius $r_S = 2M$. Right: $\eta > 0$, one, two, three or zero horizons, horizon smaller than Schwarzschild.

Regular spacetime (black hole or soliton)

For $M = \pi\xi_{reg}\lambda/4$, the central singularity disappears and all curvature invariants become infinitely regular:

$$f(r) = 1 - \frac{4M}{\pi\lambda}\left(\frac{\arctan(r/\lambda)}{r/\lambda} - \frac{1}{1+(r/\lambda)^2}\right)$$

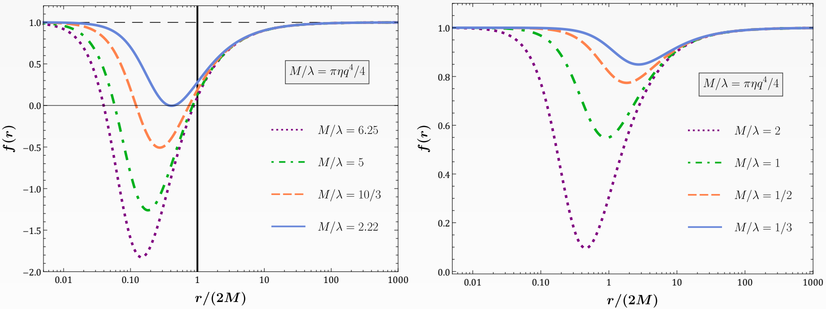


Figure: Left: Regular BH solutions. Right: regular solitonic solutions.

We have a regular black hole for $M > M_{bh} = \frac{3\sqrt{3}}{4}$ or a regular soliton for $M < M_{bh}$

Disformal transformations and non homogeneous black holes [CC, S Iteanou, D Langlois, K Noui], [Bakopoulos, Chatzifotis, Nakas]

For spherical symmetry disformal transformations $\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X)\, \partial_\mu \phi\, \partial_\nu \phi$

take us from homogeneous ($f = h$) to non homogenous black holes ($f = \frac{h}{Z^2}$).

- $Z(X)$ and $D(X)$ are mathematically equivalent. We have a class of equivalence defined modulo Z taking us from homogeneous ($Z = 1$) to non homogeneous solutions ($Z \neq constant$)
- $c_g = c$ frame given by $\tilde{Z} = -\tilde{G}_4 \Rightarrow D(X) = \frac{\eta}{2} X^{p-1}$.
- Horndeski frame given by $\tilde{F}_4 = 0 \Rightarrow D(X) = \eta \frac{2p-1}{2(p-1)X^{p-1}}$

Disformal transformations and non homogeneous black holes [CC, S Iteanou, D Langlois, K Noui], [Bakopoulos, Chatzifotis, Nakas]

For spherical symmetry disformal transformations $\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X)\,\partial_\mu\phi\,\partial_\nu\phi$

take us from homogeneous ($f = h$) to non homogenous black holes ($f = \frac{h}{Z^2}$).

- $c_g = c$ frame given by $\tilde{Z} = -\tilde{G}_4 \Rightarrow D(X) = \frac{\eta}{2}X^{p-1}$.
- Horndeski frame given by $\tilde{F}_4 = 0 \Rightarrow D(X) = \eta\frac{2p-1}{2(p-1)X^{p-1}}$

$p = 2$ homogeneous black hole with $Z = \gamma$

$$f(r) = 1 - \frac{2M}{r} + \xi_2\left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2}\right)$$

Disformal transformations and non homogeneous black holes [CC, S Iteanou, D Langlois, K Noui], [Bakopoulos, Chatzifotis, Nakas]

For spherical symmetry disformal transformations $\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X)\,\partial_\mu\phi\,\partial_\nu\phi$

take us from homogeneous ($f = h$) to non homogenous black holes ($f = \frac{h}{Z^2}$).

- $c_g = c$ frame given by $\tilde{Z} = -\tilde{G}_4 \Rightarrow D(X) = \frac{\eta}{2} X^{p-1}$.
- Horndeski frame given by $\tilde{F}_4 = 0 \Rightarrow D(X) = \eta \frac{2p-1}{2(p-1)X^{p-1}}$

$p = 2$ homogeneous black hole with $Z = \gamma$

$f(r) = 1 - \frac{2M}{r} + \xi_2 \left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2} \right)$

- for $p = 2$ the $c_g = c$ frame gives,

$$d\tilde{s}^2 = -\tilde{f}(r)\,dt_\star^2 + \left(1 - \frac{\xi_2}{3(1+\frac{r^2}{\lambda^2})^2}\right)\frac{dr^2}{\tilde{f}(r)} + r^2 d\Omega^2$$

with $\tilde{f}(r) = 1 - \frac{2M}{r} + \xi_2 \left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} \right) + \frac{2\xi_2}{3} \frac{1}{1+(r/\lambda)^2}$

For spherical symmetry disformal transformations $\tilde{g}_{\mu\nu} = g_{\mu\nu} + D(X)\,\partial_\mu\phi\,\partial_\nu\phi$

take us from homogeneous ($f = h$) to non homogenous black holes ($f = \frac{h}{Z^2}$).

- $c_g = c$ frame given by $\tilde{Z} = -\tilde{G}_4 \Rightarrow D(X) = \frac{\eta}{2} X^{p-1}$.
- Horndeski frame given by $\tilde{F}_4 = 0 \Rightarrow D(X) = \eta \frac{2p-1}{2(p-1)X^{p-1}}$

$p = 2$ homogeneous black hole with $Z = \gamma$

$f(r) = 1 - \frac{2M}{r} + \xi_2 \left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} + \frac{1}{1+(r/\lambda)^2} \right)$

- for $p = 2$ the Horndeski frame gives

$$d\tilde{s}^2 = -\tilde{f}(r)\,dt_\star^2 + \left(1 - \frac{\xi_2}{(1+(r/\lambda)^2)^2}\right)\frac{dr^2}{\tilde{f}(r)} + r^2 d\Omega^2$$

$\tilde{f}(r) = 1 - \frac{2M}{r} + \xi_2 \left(\frac{\pi/2 - \arctan(r/\lambda)}{r/\lambda} \right)$

- In order for disformal transformation to be invertible, we have the constraint $\xi_p < 2p - 1$

In summary

For theories :

$$G_2(X) = -\frac{2\eta}{\lambda^2}X^p, \quad G_4(X) = 1 - \eta X^p, \quad F_4(X) = \frac{\eta}{4}(2p-1)X^{p-2}\,.$$

We have primary hair black holes with :

- Homogeneous and non homogeneous solutions modulo disformal transformations controlled by $Z = -G_4 + 2XG_{4X} + 4X^2F_4$
- Two independent charges, mass M and scalar charge ξ
- Noether charge $Q(\eta, q, p) < \infty$ finite for $p > 3/2$. We then have good asymptotics, horizons for all solutions
- Regular black hole at the maximal charge $\xi_p^{reg} \sim M_p^{reg}$ parametrised by ADM mass
- Regular scalar for all q at all future horizons $X = \frac{q^2/2}{1+(r/\lambda)^2}$ $\psi'(r)^2 = \frac{q^2}{f^2(r)}\left[1 - \frac{f(r)}{1+(r/\lambda)^2}\right]$
- One can sum sources and mix solutions with different values of p

Black hole perturbations of spherically symmetric spacetimes

GW astronomy provides a window to test gravity

-Ringdown phase of a BH merger is described by linear BH perturbation theory

- We go to the frequency domain $f(t,r) = f(r)e^{-i\omega t}$ and expand in spherical harmonics.
- We have Axial and Polar modes which to linear order are independent for given boundary conditions
- In GR we get two master equations in Schrodinger form. The effective metric where gravitons propagate and the background metric are identical.
- In scalar tensor theories we have an additional polar mode. We have one axial mode but, the effective and background metrics are now different!

Axial perturbations [Regge, Wheeler]

For GR Schwarzschild,

- Given spherical symmetry we can expand metric perturbations in spherical harmonics (ℓ, m) and (odd-even) parity
- Given staticity we can separate modes in each given frequency ω in the Regge-Wheeler gauge

$$h_{t\theta} = \frac{1}{\sin\theta}\sum_{\ell,m} h_0^{\ell m}(t,r)\partial_\varphi Y_{\ell m}(\theta,\varphi), \qquad h_{t\varphi} = -\sin\theta\sum_{\ell,m} h_0^{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\varphi),$$
$$h_{r\theta} = \frac{1}{\sin\theta}\sum_{\ell,m} h_1^{\ell m}(t,r)\partial_\varphi Y_{\ell m}(\theta,\varphi), \qquad h_{r\varphi} = -\sin\theta\sum_{\ell,m} h_1^{\ell m}(t,r)\partial_\theta Y_{\ell m}(\theta,\varphi), \quad (1)$$

  Essentially we seek $h_0 = h_0(r)$, $h_1 = h_1(r)$ labelled each by eigenvalues ℓ, m, ω.
- The analysis boils down to a single second order master equation for one radially dependent function $\mathcal{Y}(r)$ for each label, l, m

$$-\frac{d^2\mathcal{Y}}{dr_*^2} + V_\ell\,\mathcal{Y} = \omega^2\,\mathcal{Y}$$

,

- with outgoing and ingoing boundary conditions for $dr_* = dr/f_{GR}$ the tortoise coordinate.
- We have $f_{GR} = 1 - \frac{2M}{r}$ with a positive definite potential $V_\ell(r) = \frac{\ell^2+\ell-2}{r^2} f_{GR} - \frac{1}{r} f_{GR} f'_{GR} + \frac{2}{r^2} f^2_{GR}$,

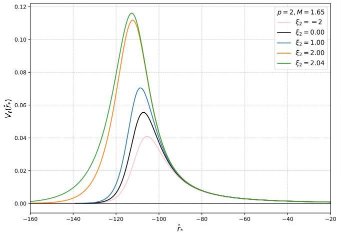
Axial (odd parity) perturbations [Langlois, Noui, Roussille], [Kobayashi, Motohashi Suyama]

- Using Regge Wheeler gauge we get a modified Schrodinger potential for all p and η parametrising the theories :

$$V_\ell(r) = \frac{\ell^2+\ell-2}{r^2}\Phi - \frac{\kappa_1(p,\eta,q)}{r}\Phi\Phi' + \frac{2\kappa_2(p,\eta,q)}{r^2}\Phi^2$$

with $\kappa_{1,2}$ depending on the theory at hand for example $\kappa_1 = \frac{1+\eta(\frac{p}{2}-1)X^p - \frac{\eta p}{q^2}X^{p+1}}{(1-\eta X^p)^2}$

- and effective metric $\Phi(r) = f(r) - \frac{\eta q^2}{2}X^{p-1}$
- Effective metric where axial gravitons propagate is not the background metric [Langlois, Noui and Roussille]. It is the $c_g = c$ frame with a conformal factor $C = 1$!
- Axial gravitons see a different metric than light or matter close to the black hole horizon. Axial gravitons propagate in the $c_g = c$ frame.
- Axial perturbations are stable in their domain of definition due to the well defined disformal geometries as long as $\xi_p < 2p-1$ and $p > 3/2$.



Phase diagram for the background and effective metrics

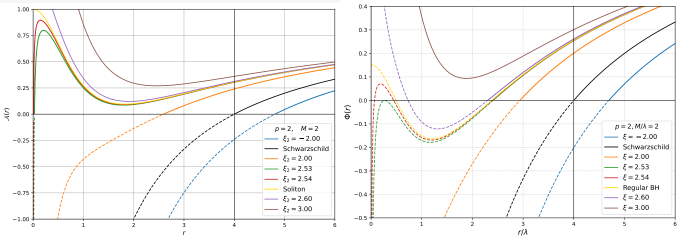
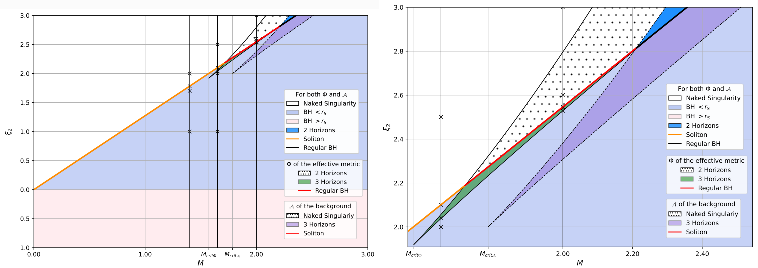


Figure: Plot of the background and effective metrics for $p = 2$ and mass $M = 2$ and different values of ξ_2



Concluding remarks

- We have constructed static black holes with primary hair. We need global symmetry and time dependent scalar
- Solutions become regular at the origin for maximal scalar charge without any fine tuning of theory parameters.
- Homogeneous and non homogeneous black holes belong to the same equivalence class related via the theory function $Z = Z(X)$. Axial perturbations of all members of this class have identical axial perturbations.
- Only in the $c = c_g$ frame the effective and background metrics are identical.
- Axial stability of the solutions is generic modulo within well defined effective metric and essentially the presence of a finite scalar charge.
- Polar perturbations are the crucial step in understanding the stability. Is there a notion of effective metric in this case and what is this effective metric?
- Can we find rotating counterparts to these solutions? Is there some geometrical interpretation for the precise form of X bifurcating in between timelike and null geodesics