Simulating structure formation in modified gravity

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Modified gravity models (see also the review from Llinares, 2021)

Type of equation	Model	Method
$L(\phi, \dot{\phi}, \ddot{\phi}) = f(\rho, \phi)$ L, f linear	Dynamical DE	Modified expansion
	Coupled DE	Modified expansion + G_{eff}
	Non-local	Modified expansion + G_{eff}
	Vector DE	Modified expansion, power spectrum, and growth history
$L(\phi, \partial_x\phi, \partial_x^2\phi) = f(\rho, \phi)$ L linear in $\partial_x^2\phi$	$f(R)$	Multigrid (Jordan / Einstein frame)
	Generalized chameleon	Multigrid
	Symmetron	Multigrid
	Dilaton	Multigrid
$L(\phi, \partial_x\phi, \partial_x^2\phi) = f(\rho, \phi)$ L non-linear in $\partial_x^2\phi$	DGP	Multigrid or FFT-relaxation with smoothing / splitting
	Cubic Galileon	Multigrid with operator splitting
	Quartic Galileon	Multigrid with operator splitting
$L(\ddot{\phi}, \dot{\phi}, \partial_x\phi, \partial_x^2\phi) = f(\rho, \dot{\phi}, \phi)$	Symmetron	Leap-frog
	$f(R)$	Implicit
	Disformal	Leap-frog

"GR" simulations

GR simulations ?

Cosmological action (GR + ΛCDM)

$$S = \int \left[\frac{1}{2\kappa}(R - 2\Lambda) + \mathcal{L}_m\right]\sqrt{-g}\,d^4x \tag{1}$$

Weakly-perturbed FLRW metric in Newtonian gauge

$$ds^2 = a^2(\eta)\left[-(1+2\psi)\,d\eta^2 + (1-2\phi)\,d\boldsymbol{x}^2\right] \tag{2}$$

In GR, $\phi = \psi$

$$\nabla^2\phi - 3\mathcal{H}\left(\phi' + \mathcal{H}\phi\right) = 4\pi G a^2 \delta\bar{\rho} \tag{3}$$

GR simulations ?

$$ds^2 = a^2(\eta)\left[-(1+2\psi)\,d\eta^2 + \underbrace{(1-2\phi)}_{\text{Problem !}}\,d\boldsymbol{x}^2\right] \tag{4}$$

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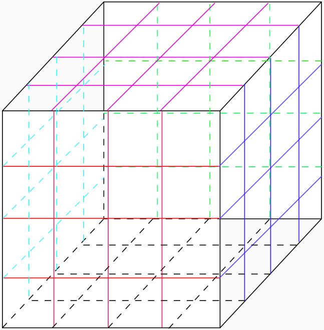

Not consistent with N-body simulations (fixed grid geometry) !

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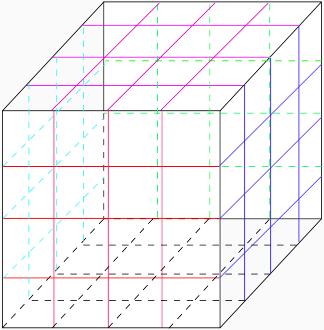

Not consistent with N-body simulations (fixed grid geometry) !

Solution

- Use the so-called N-body gauge
- Newtonian gauge + corrections in post-processing

Newtonian simulations

From GR...

$$\nabla^2\phi - 3\mathcal{H}\left(\phi' + \mathcal{H}\phi\right) = 4\pi G a^2 \delta\bar{\rho} \tag{5}$$

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... to Newton!

$$\boxed{\nabla^2\phi = 4\pi G a^2 \delta\bar{\rho}}$$

Can be solved using particle–tree or particle–mesh methods.

Solving PDEs in N-body simulations

Particle methods

- Particle-Particle, Tree codes (Barnes & Hut), Fast Multipole Methods (FMM)
- Solves $1/r^2$ interactions
- Pros : very fast, accurate on small scales
- Con : only works for Newtonian simulations

Particle-Mesh (PM) methods

- FFT, Multigrid, Conjugate gradient
- Solves $\nabla^2\phi = 4\pi G a^2 \delta\bar{\rho}$
- Pros : extremely fast, solves non-linear elliptic/parabolic PDEs
- Con : inaccurate on small scales

PM with Adaptive-Mesh Refinement (PM-AMR) methods

- Multigrid, Conjugate gradient
- Pros : accurate on small scales, solves non-linear elliptic/parabolic PDEs
- Cons : Not as fast as previous methods, anisotropic

PM (+ AMR) solver

- Project the particle mass (density) onto a grid
- Compute the gravitational potential (solver !)
- Compute the force
- Interpolate back to particles
- Update velocities, then positions

First possibility : FFT

$$\phi(k) = -k^{-2} W(k) \times 4\pi G a^2 \delta\bar{\rho} \tag{6}$$

- "Exact" case $W(k) = 1$ is inaccurate on small scales
- Several possibilities for $W(k)$
- Also possible to directly compute the force !

Multigrid

Discretize the Laplacian operator : $\nabla^2 \phi_i = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{h^2}$

Iterative method (smoothing) : $\phi_i = \frac{1}{2}\left(\phi_{i+1} + \phi_{i-1} - h^2 4\pi G a^2 \delta\bar{\rho}\right)$

Principles of Multigrid

- Smooth on fine grid (pre-smoothing)
- Compute residual $r_h = f_h(\delta) - \nabla_h^2 \phi_h$ $\rightarrow$ Interpolate to coarser grid
- Compute the correction term $\nabla_H^2 \tilde{\phi}_H = -\mathcal{R}_H(r_h)$
- Apply to correction on the fine grid $\phi_h = \phi_h + \mathcal{P}_h(\tilde{\phi}_H)$
- Smooth on fine grid (post-smoothing)

Can solve non-linear equations using the Newton-Raphson method

$$\phi^{\text{new}} = \phi^{\text{old}} - \frac{\mathcal{L}(\phi^{\text{old}})}{\partial\mathcal{L}/\partial\phi^{\text{old}}} \tag{7}$$

Modified gravity simulations

Minimally coupled Quintessence

Action

$$S = \int \sqrt{-g} d^4x \left[\frac{1}{2\kappa} R - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) + \mathcal{L}_m \right] \quad (8)$$

Dark-energy equation of state :

$$w = \frac{\dot{\phi}^2/2 - V(\phi)}{\dot{\phi}^2/2 + V(\phi)} \quad (9)$$

Popular models : Ratra-Peebles, SUGRA...

How to simulate ?

- Newtonian dynamical solver (no change)
- Modification of cosmological tables (background expansion)

Parametrized gravity

Set of equations

$$\nabla^2 \psi = 4\pi G \mu(a) a^2 \delta\bar{\rho} \tag{10}$$

$$\nabla^2 (\psi + \phi) = 8\pi G \Sigma(a) a^2 \delta\bar{\rho} \tag{11}$$

How to simulate ?

- Trivial change to the dynamical solver (one multiplication)
- Depending on applications, the second equation is not mandatory
- Background expansion is the same, but D_+ is not (initial rescaling)

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$$\ddot{D} + \mathcal{H}\dot{D} - \beta D = 0 \qquad \text{first-order growth factor} \tag{12}$$

$$\ddot{E} + \mathcal{H}\dot{E} - \beta\left[E - D^2\right] = 0 \qquad \text{second-order growth factor} \tag{13}$$

with $\beta = \frac{3}{2}\mu(a)\Omega_m(a)$

Modified Newtonian Dynamics (MOND)

Modified Poisson equation (AQUAL)

$$\nabla \left[\mu \left(\frac{|\nabla \phi|}{g_0} \right) \nabla \phi \right] = 4\pi G \delta \bar{\rho} \qquad (14)$$

Modified Newtonian Dynamics (MOND)

Modified Poisson equation (AQUAL)

$$\nabla\left[\mu\left(\frac{|\nabla\phi|}{g_0}\right)\nabla\phi\right] = 4\pi G\delta\bar{\rho} \qquad (14)$$

Quasi-linear MOND (QUMOND)

$$\nabla^2\phi^N = 4\pi G\rho \qquad (15)$$

$$\nabla^2\phi = \nabla\left[\nu\left(\left|\nabla\phi^N\right|/g_0\right)\nabla\phi^N\right] \qquad (16)$$

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- **Simple :**

$$\nu(y)=\frac{1}{2}+\frac{\sqrt{1+4/y}}{2} \tag{17}$$

- **The n-family :**

$$\nu(y)=\left[\frac{1}{2}+\frac{\sqrt{1+4/y^n}}{2}\right]^{1/n} \tag{18}$$

- **The β-family :**

$$\nu(y)=\left(1-e^{-y}\right)^{-1/2}+\beta e^{-y} \tag{19}$$

- **The γ-family :**

$$\nu(y)=\left(1-e^{-y^{\gamma/2}}\right)^{-1/\gamma} \tag{20}$$

$$+\left(1-\gamma^{-1}\right)e^{-y^{\gamma/2}} \tag{21}$$

- **The δ-family :**

$$\nu(y)=\left(1-e^{-y^{\delta/2}}\right)^{-1/\delta} \tag{22}$$

Modified Newtonian Dynamics (MOND)

QUMOND equations

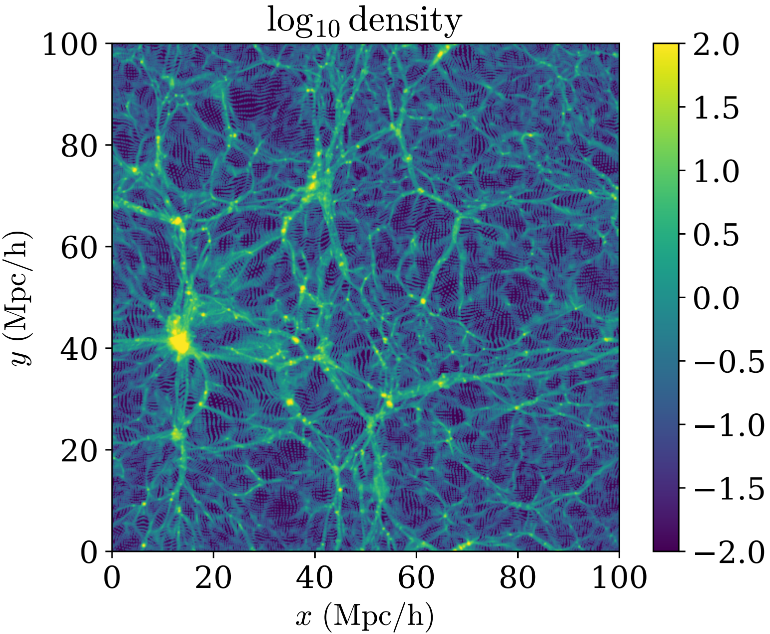
$$\nabla^2 \phi^N = 4\pi G \rho \tag{23}$$

$$\nabla^2 \phi = \nabla \left[\nu \left(\left| \nabla \phi^N \right| / g_0 \right) \nabla \phi^N \right] \tag{24}$$

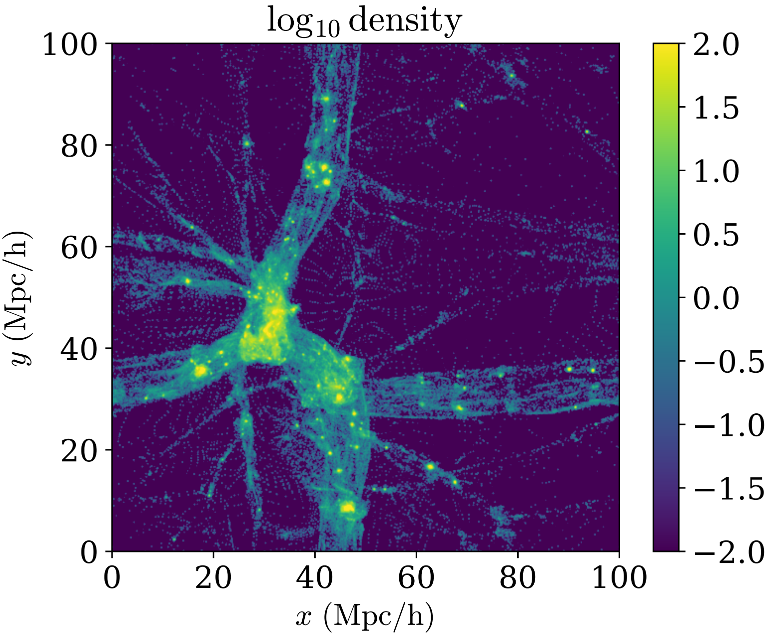
How to simulate ?

- Solve for a Newtonian potential (no change)
- Solve for a MOND potential (compute new source term)
- No modification of cosmological tables
- Arbitrary initial conditions rescaling

Newtonian simulation at $z = 0$



MOND simulation at $z = 0$



$f(R)$ gravity

$f(R)$ action

$$S = \int \left[\frac{1}{2\kappa} \left(R + f(R) \right) + \mathcal{L}_m \right] \sqrt{-g} d^4 x \tag{25}$$

Hu & Sawicki (2007) model

$$f(R) = -m^2 \frac{c_1 (R/m^2)^n}{c_2 (R/m^2)^n + 1} \tag{26}$$

Only two free parameters (one is constrained by observations)

$$\frac{c_1}{c_2} = 6 \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \qquad \frac{c_1}{c_2^2} = -\frac{1}{n} \left[3 + 12 \frac{\Omega_{\Lambda,0}}{\Omega_{m,0}} \right] f_{R0} \tag{27}$$

$f(R)$ gravity

Modified set of equations

$$\nabla^2 \phi = \underbrace{4\pi G a^2 \delta\bar{\rho}}_{Newton} - \frac{c^2}{2}\nabla^2 f_R \tag{28}$$

$$\nabla^2 f_R = A\delta\bar{\rho} + B + C f_R^{-\frac{1}{n+1}} \tag{29}$$

How to simulate ?

- Solve the non-linear equation for the f_R field with Multigrid
- Then, solve a linear equation with modified source term
- In practice, we add the contribution directly in the Force

$f(R)$ gravity

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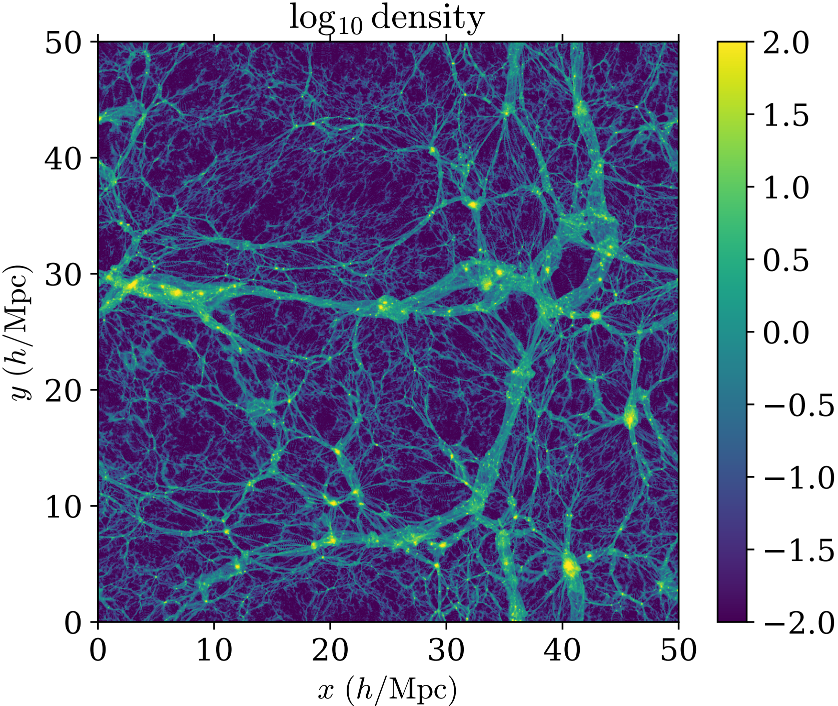
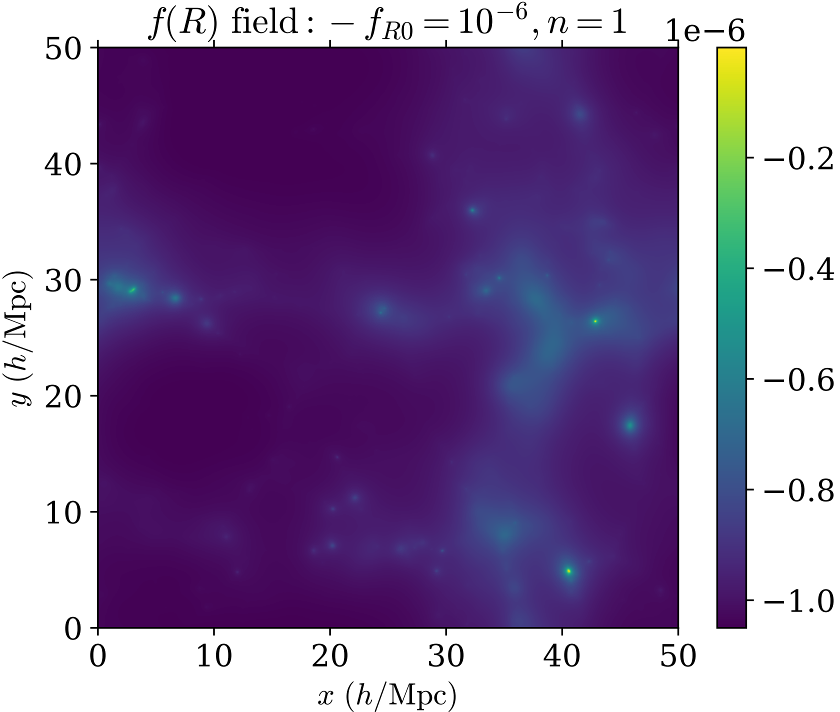
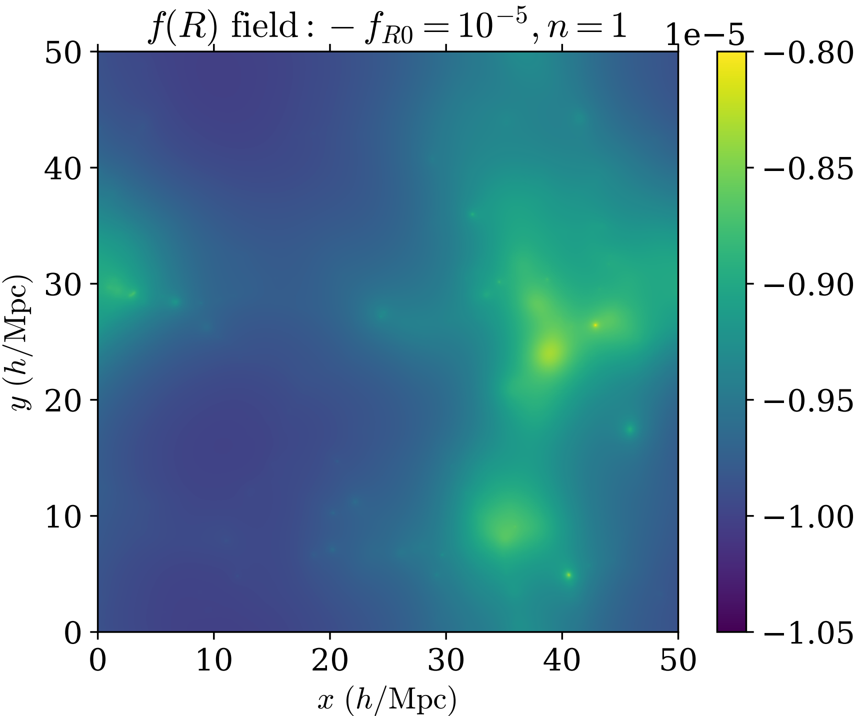
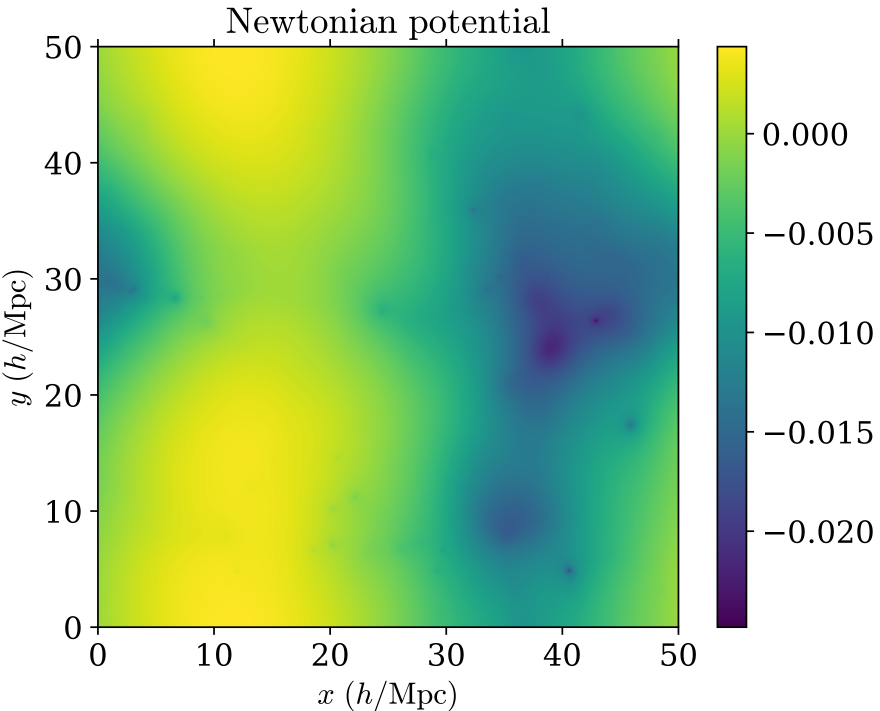
Approximation

$$-k^2 \phi = 4\pi G_{eff}(a,k) a^2 \delta\bar{\rho} \tag{30}$$


$\log_{10}$ density
y (h/Mpc)
x (h/Mpc)
0
10
20
30
40
50
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−2.0

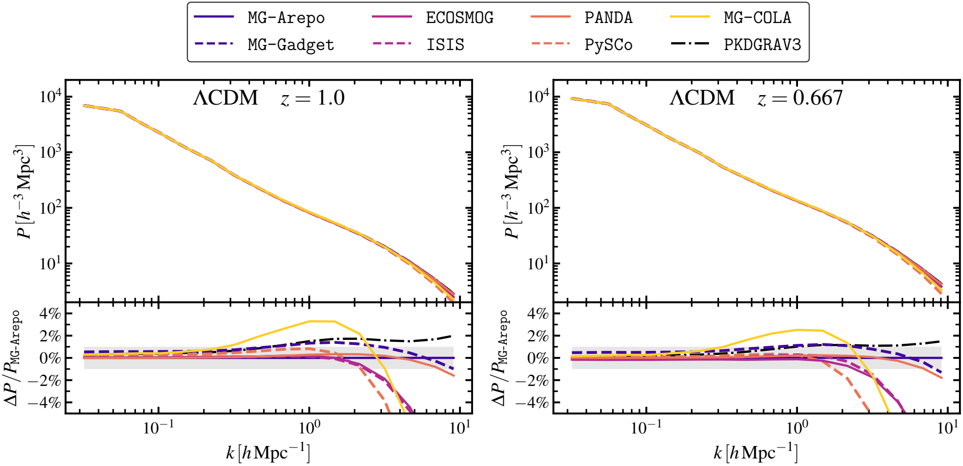

$f(R)$ field $: -f_{R0} = 10^{-6}, n = 1$
1e−6
−0.2
−0.4
−0.6
−0.8
−1.0
50
40
30
20
10
0
y (h/Mpc)
0
10
20
30
40
50
x (h/Mpc)


$f(R)$ field : $-f_{R0} = 10^{-5}, n = 1$
1e−5
−0.80
−0.85
−0.90
−0.95
−1.00
−1.05
50
40
30
20
10
0
0
10
20
30
40
50
y (h/Mpc)
x (h/Mpc)


Newtonian potential
50
40
30
20
10
0
0
10
20
30
40
50
y (h/Mpc)
x (h/Mpc)
0.000
−0.005
−0.010
−0.015
−0.020

Some code-comparison challenge

ΛCDM Power spectrum (Euclid Collaboration : Adamek et al., 2025)



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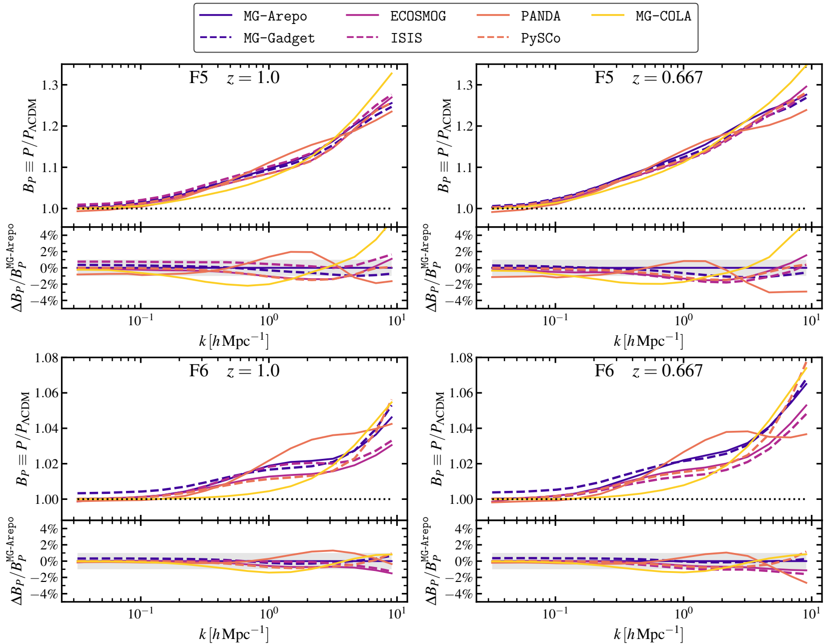
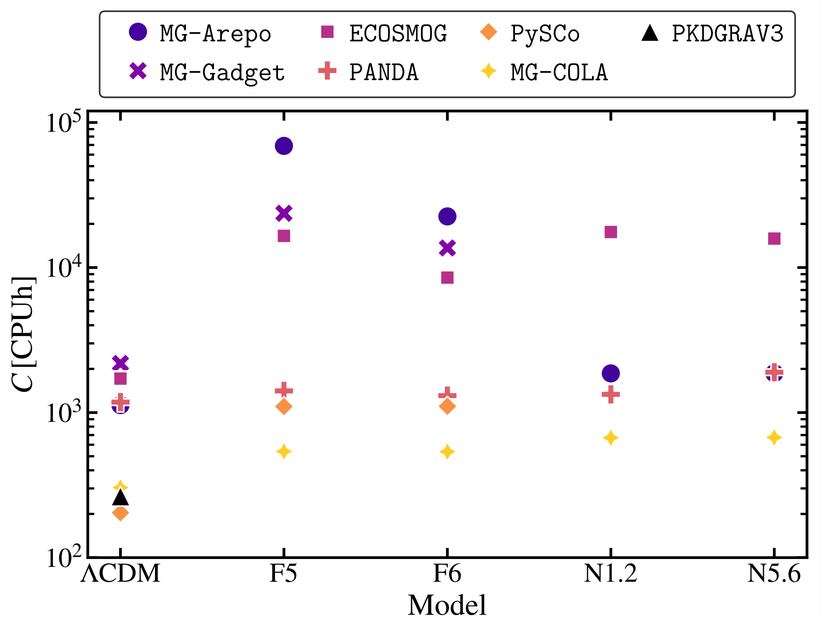


Runtime comparison (Euclid Collaboration : Adamek et al., 2025)



Summary

Summary

Current status

- Different theories need appropriate numerical tools
- Fair consensus among different numerical implementation
- Increase the range of models we can simulate
- Ongoing work to implement EFTofDE in `RAMSES` and `PySCo` by Himanish Ganjoo at LUX

Limitations

- MG simulations are generally more expensive than Newtonians
- MG simulations usually restricted to snapshots
- Need to produce lightcones !
- Current MG N-body codes do not run on GPUs

Run MG simulations with PySCo github.com/mianbreton/pysco

Installation as package (also installs dependencies for you !)

```
python -m pip install pysco-nbody
```

Run from Python script

```python
import pysco

param = {
    "nthreads": 1,
    "theory": "newton", # mond, fr...
    "H0": 72,
    "z_out": "[10, 5, 2, 1, 0.5, 0]"
    # Add other parameters
}

pysco.run(param)
```

Appendix

Preliminary work : EFTofDE (by Himanish Ganjoo)

The Effective-field theory of dark energy framework

General action (second order)

$$S = \int d^4x\sqrt{-g}\left[\frac{M_*^2}{2}f(t)R - \Lambda(t) - c(t)g^{00} + \frac{M_2^4(t)}{2}\left(\delta g^{00}\right)^2 - \frac{\bar{M}_1^3(t)}{2}\delta g^{00}\delta K - \frac{\bar{M}_2^2(t)}{2}(\delta K)^2 - \frac{\bar{M}_3^2(t)}{2}\delta K_{ij}\delta K^{ij} + \frac{\hat{M}^2(t)}{2}\delta g^{00}\delta R^{(3)} + \dots\right] \tag{31}$$

Rewritten with 4 time-varying α-functions, encoding deviations from GR/ΛCDM

Cubic screening (simplest non-linear)

$$\nabla^2\psi = 4\pi G a^2\delta\bar{\rho} + (\alpha_B - \alpha_M)\nabla^2\chi \tag{32}$$

$$p\nabla^2\chi + q4\pi G a^2\delta\bar{\rho} + r\left[\left(\nabla^2\chi\right)^2 - \partial_i\partial_j\chi\partial^i\partial^j\chi\right] = 0 \tag{33}$$

EFTofDE results (PRELIMINARY) Courtesy of H. Ganjoo