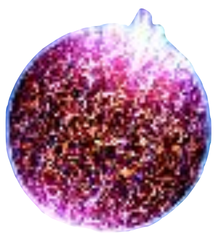
Static Quadratic Love Numbers of Neutron Stars



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In preparation with Paolo Pani, Massimiliano Maria Riva, Luca Santoni and Nikola Savic

Previous work: 2312.05065, 2410.03542 on Schwarzschild Black Holes

15 October 2025

IPhT, TUG - Théorie, Univers et Gravitation

Newtonian case

- Poisson equation: $\nabla^2 \Phi = 4\pi G \rho$

tidal field

$$\Phi = G \int d^3x' \frac{\rho(\vec{x})}{|\vec{x} - \vec{x}'|} + \sum_{\ell \geq 2, m} \mathcal{E}_{\ell m} r^{\ell} Y_{\ell m}$$

Newtonian case

- Poisson equation: $\nabla^2\Phi = 4\pi G\rho$

tidal field

$$\Phi = \underbrace{G\int d^3x' \frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}}_{-\frac{M}{r} + \sum_{\ell\geq 2,m} \frac{Q_{\ell m}}{r^{\ell+1}} Y_{\ell m}} + \sum_{\ell\geq 2,m} \mathcal{E}_{\ell m} r^{\ell} Y_{\ell m}$$

induced mass multipoles $\qquad Q_\ell \sim MR^\ell$

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$$-\frac{M}{r} + \sum_{\ell\geq 2,m}\frac{Q_{\ell m}}{r^{\ell+1}}Y_{\ell m}$$

induced mass multipoles $Q_\ell \sim MR^\ell$

- Linear response $Q_{\ell m} = \lambda_\ell\mathcal{E}_{\ell m} = k_\ell\frac{R^{2\ell+1}}{G}\mathcal{E}_{\ell m}$ linear tidal Love number

$$\Phi = -\frac{GM}{r} + \sum_{\ell\geq 2,m} r^{\ell}\mathcal{E}_{\ell m}Y_{\ell m}\left[1 + k_\ell\left(\frac{R}{r}\right)^{2\ell+1}\right]$$

Newtonian case

- Poisson equation: $\nabla^2\Phi = 4\pi G\rho$

tidal field

$$\Phi = G\underbrace{\int d^3x'\,\frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}}_{-\frac{M}{r} + \sum_{\ell\geq 2,m}\frac{Q_{\ell m}}{r^{\ell+1}}Y_{\ell m}} + \sum_{\ell\geq 2,m}\mathcal{E}_{\ell m} r^{\ell} Y_{\ell m}$$

induced mass quadrupole $\qquad Q_2 \sim MR^2$

- Linear response $\qquad Q_2 = \lambda_2\mathcal{E}_2 = k_2\frac{R^5}{G}\mathcal{E}_2$ $\qquad$ linear tidal Love number

$$\Phi = -\frac{GM}{r} + r^2\mathcal{E}_2 P_2\left[1 + k_2\left(\frac{R}{r}\right)^5\right]$$

Newtonian case

- Poisson equation: $$\nabla^2\Phi = 4\pi G\rho$$

tidal field

$$\Phi = G\underbrace{\int d^3x' \frac{\rho(\vec{x})}{|\vec{x}-\vec{x}'|}} + \sum_{\ell\geq 2,m} \mathcal{E}_{\ell m} r^{\ell} Y_{\ell m}$$

$$-\frac{M}{r} + \sum_{\ell\geq 2,m} \frac{Q_{\ell m}}{r^{\ell+1}} Y_{\ell m}$$

induced mass quadrupole $\quad Q_\ell \sim M R^\ell$

- Quadratic response $$Q_2 = \lambda_2 \mathcal{E}_2 + \lambda_{222}\mathcal{E}_2\mathcal{E}_2 \qquad \lambda_{222} = p_2 \frac{R^8}{G^2 M}$$

$$\Phi = -\frac{GM}{r} + r^2 P_2\left[\mathcal{E}_2 + \mathcal{E}_2 k_2 \left(\frac{R}{r}\right)^5 + \mathcal{E}_2\mathcal{E}_2 \frac{r^3}{GM} p_2 \left(\frac{R}{r}\right)^8\right]$$

Effect on the waveform

$$\Phi = -\frac{GM}{r} + r^2 P_2 \left[\mathcal{E}_2 + \mathcal{E}_2 k_2 \left(\frac{R}{r} \right)^5 + \mathcal{E}_2 \mathcal{E}_2 \frac{r^3}{GM} p_2 \left(\frac{R}{r} \right)^8 \right]$$

Effect on the waveform

$$\Phi = -\frac{GM}{r} + r^2 P_2 \left[\mathcal{E}_2 + \underbrace{\mathcal{E}_2 k_2 \left(\frac{R}{r}\right)^5}_{\frac{k_2}{C^5} \cdot \left(\frac{r_s}{r}\right)^5} + \mathcal{E}_2 \mathcal{E}_2 \frac{r^3}{GM} \underbrace{p_2 \left(\frac{R}{r}\right)^8}_{\frac{p_2}{C^8} \cdot \left(\frac{r_s}{r}\right)^8} \right]$$

$$r_s = 2GM \ , \qquad C \equiv \frac{GM}{R} \qquad \Rightarrow \qquad \frac{k_2}{C^5} \cdot \left(\frac{r_s}{r}\right)^5 \qquad \frac{p_2}{C^8} \cdot \left(\frac{r_s}{r}\right)^8$$

compactness

5PN effect

8PN effect

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$r_s = 2GM$, $C \equiv \frac{GM}{R}$ $\Rightarrow$

compactness

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- Waveform: $h(f) = \mathcal{A} f^{-7/6} e^{i\Psi(f)}$ $\quad v(f) = (\pi G M f)^{1/3}$ $\quad \eta = \frac{M_1 M_2}{(M_1+M_2)^2}$

$$\Psi(f) = 2\pi f t_c + \frac{3}{128\eta v^5}\left[\underbrace{1 + \# v^2 + \ldots \mathcal{O}(v^7)}_{\text{point particle}} - 24\Lambda_L v^{10} + \frac{90}{11}\Lambda_Q v^{16}\right]$$

$$\Lambda_L = \frac{(M_1 + 12 M_2) M_1^4 [k_2/C^5]_1 + (1 \leftrightarrow 2)}{(M_1+M_2)^5}$$

Flanagan & Hinderer 2007

$$\Lambda_Q = \frac{(M_1 + 35 M_2) M_1^5 [p_2/C^8]_1 + (1 \leftrightarrow 2)}{(M_1+M_2)^6}$$

Pani et al. in prep.

Effect on the waveform

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$$\frac{\text{quadratic TLN}}{\text{linear TLN}} \sim \frac{\Lambda_Q}{\Lambda_L} v^6 \sim \frac{p_2/C^3}{k_2}\left(\frac{GM}{r_{\rm orb}}\right)^3 \sim \frac{p_2}{k_2}\left(\frac{R}{r_{\rm orb}}\right)^3$$

Effect on the waveform

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compactness — 5PN effect — 8PN effect

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⇒ Improved waveform model

⇒ Additional information on the NS internal structure

⇒ Same order as first <u>dynamical Love numbers</u>: NS frequency, resonances, etc.

Quadratic tidal Love number must be included for the next generation of GW detectors

Relativistic case

- In general relativity we have Einstein equations:

$$\nabla^2\Phi = 4\pi G\rho \qquad \Rightarrow \qquad G_{\mu\nu}[g_{\mu\nu}] = R_{\mu\nu}[g_{\mu\nu}] - \frac{1}{2}g_{\mu\nu}R[g_{\mu\nu}] = 8\pi G T_{\mu\nu}$$

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- Polar (parity even) perturbations:

PM non-linearities

$$\delta g_{tt} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} \underbrace{r^\ell \mathcal{E}_{\ell m}^+ Y_{\ell m} \Bigg[\left(1 + a_1^+ \frac{r_s}{r} + \dots \right)}_{\delta \bar{g}_{tt}} + k_\ell{}^+ \left(\frac{R}{r}\right)^{2\ell+1} \left(1 + b_1^+ \frac{r_s}{r} + \dots\right) \Bigg]$$

tidal field $\delta \bar{g}_{tt}$ solution of $\bar{G}_{\mu\nu}[\bar{g}_{\mu\nu}] = 0$ regular at the origin

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tidal field $\quad \delta\bar{g}_{tt}\quad$ solution of $\quad \bar{G}_{\mu\nu}[\bar{g}_{\mu\nu}] = 0 \quad$ regular at the origin

- Axial (parity odd) perturbations:

$$\delta g_{tA} = -\sum_{\ell\geq 2,m} r^\ell \mathcal{E}^-_{\ell m} \nabla_A Y_{\ell m} \left[\left(1 + a_1^- \frac{r_s}{r} + \dots\right) + k_\ell{}^- \left(\frac{R}{r}\right)^{2\ell+1} \left(1 + b_1^- \frac{r_s}{r} + \dots\right)\right]$$

- Expressions above are not-gauge invariant. Ambiguity in definition of Love numbers. [Gralla '17]
- Things get worse when including the nonlinear response to the tidal field.

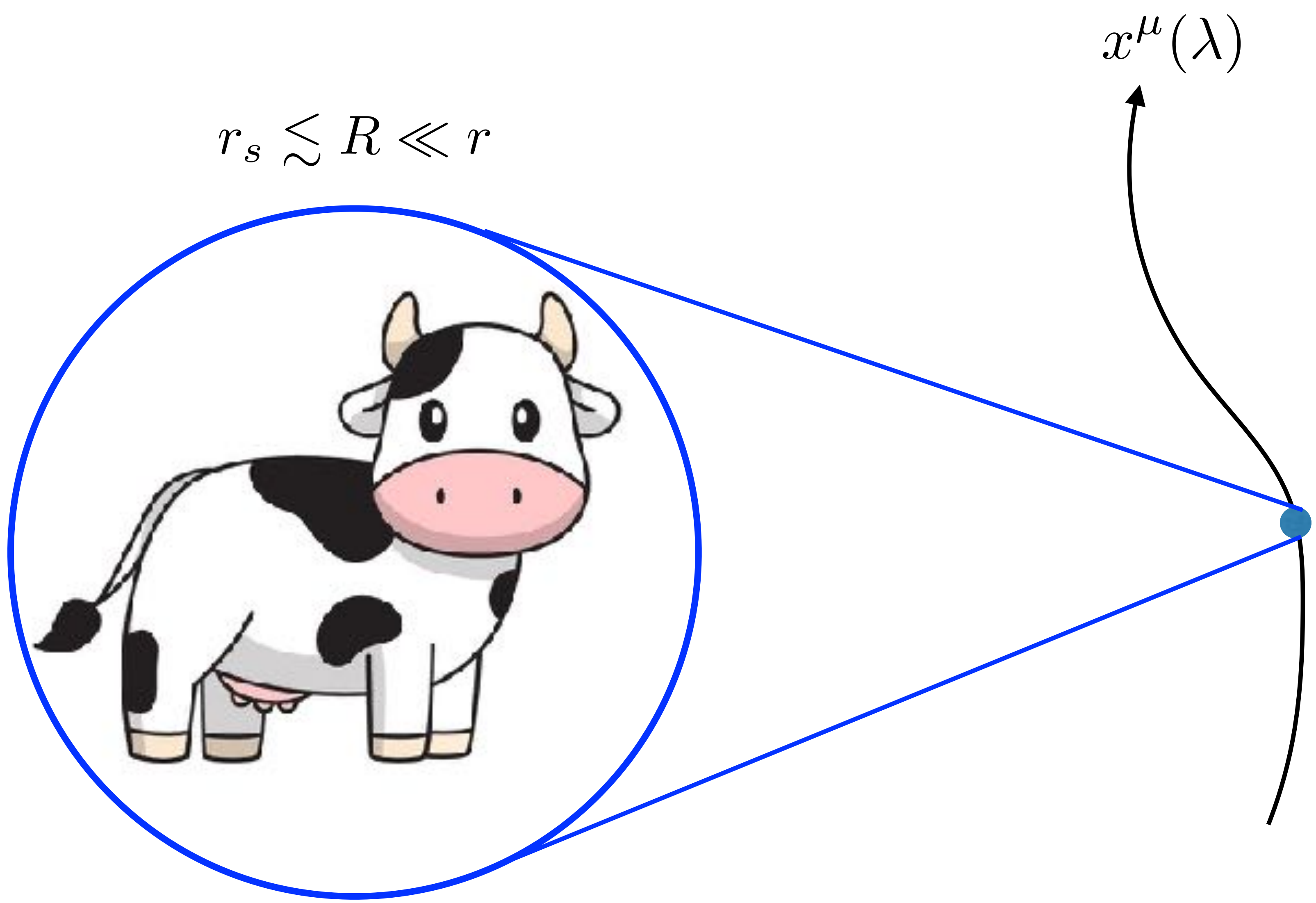
Worldline EFT

[Goldberger & Rothstein ‘04, Porto, others]

- EFT of self-interacting gravitons coupled to classical worldline sources

$$S = S_{\rm EH}[g_{\mu\nu}] + S_{\rm pp}[g_{\mu\nu}, x^\mu(\lambda)]$$

- From afar, everything looks point-like and localized on a center-of-mass worldline $x^\mu(\lambda)$



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- Micro-structure integrated out. Modelled as series of multipole moments

quadrupole

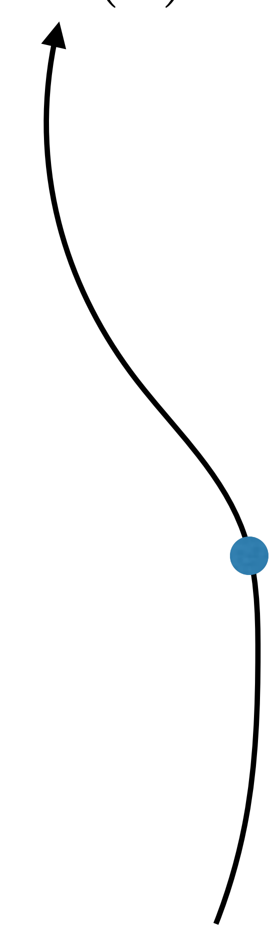
$$S_{\rm pp} = \int d\tau \left[M + Q_{ij}E^{ij} + Q_{ijk}\nabla^{\langle i}E^{jk\rangle} + \text{magnetic} + \cdots\right]$$

mass

octupole

$$E_{ij} = C_{i0j0}\ , \qquad B_{ij} = \frac{1}{2}\epsilon_{i0kl}C^{kl}{}_{j0} \qquad C_{\mu\nu\rho\sigma}$$ Weyl tensor

$x^\mu(\lambda)$



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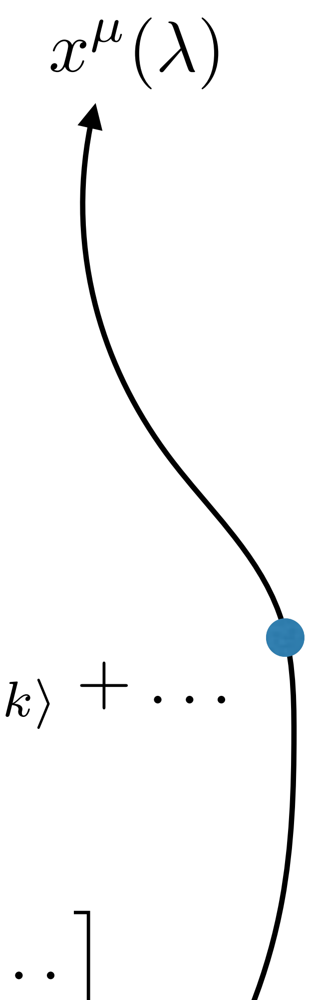
- Integrating out multipoles $\quad Q_{ij} = \lambda_2 E_{ij} + \ldots\ , \qquad Q_{ijk} = \lambda_3 \nabla_{\langle i}E_{jk\rangle} + \ldots$

linear response + nonlinearities

$$S_{\rm pp} = \int d\tau \left[M + \lambda_2 E_{ij}E^{ij} + \lambda_3 \nabla_{\langle i}E_{jk\rangle}\nabla^{\langle i}E^{jk\rangle} + {\rm magnetic} + \cdots\right]$$



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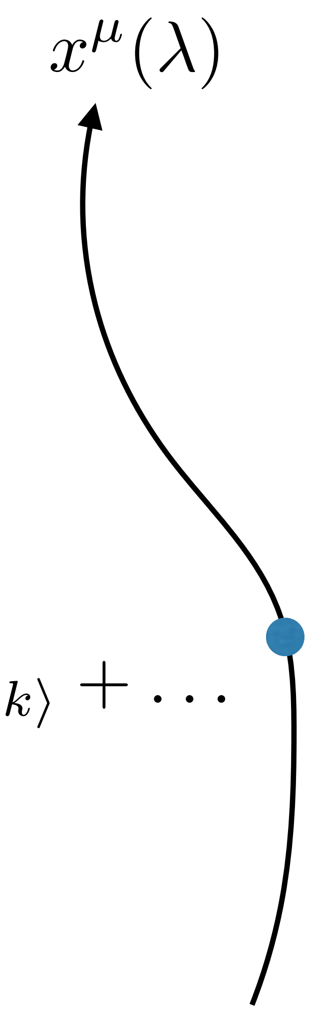

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quadrupole

$$S_{\rm pp} = \int d\tau \left[M + Q_{ij} E^{ij} + Q_{ijk} \nabla^{\langle i} E^{jk\rangle} + \text{magnetic} + \cdots \right]$$

mass octupole

- Integrating out multipoles $\quad Q_{ij} = \lambda_2 E_{ij} + \ldots \,, \qquad Q_{ijk} = \lambda_3 \nabla_{\langle i} E_{jk\rangle} + \ldots$

linear response + nonlinearities

$$S_{\rm pp} = \int d\tau \left[M + \sum_{L=2} \lambda_\ell E_L E^L + \text{magnetic} + \cdots \right]$$

Tidal effects in the WEFT

- Action $$S = S_{\rm EH}[g_{\mu\nu}] + S_{\rm pp}[g_{\mu\nu}, x^\mu(\lambda)]$$

Linear Love number couplings

$$S_{\rm pp} = \int d\tau \Big\{ M + \sum_L \left(\lambda_\ell^E E_L E^L + \lambda_\ell^B B_L B^L\right)$$

(partity invariant operators)

$$+ \sum_{L_1 L_2 L_3} \left[\lambda^{E^3}_{\ell_1\ell_2\ell_3} \left(E_{L_1} E_{L_2} E_{L_3}\right) + \lambda^{EB^2}_{\ell_1\ell_2\ell_3} \left(E_{L_1} B_{L_2} B_{L_3}\right)\right] + \mathcal{O}(E^4, \ldots)\Big\}$$

Quadratic Love number couplings

- WEFT provides coordinate-independent definition of Love numbers and systematically accounts for non-linearities

- Controlled expansion parameters: $$\mathcal{E}^{1/\ell} \ll \frac{r_s}{r} \ll \frac{r_s}{R} \leq 1$$

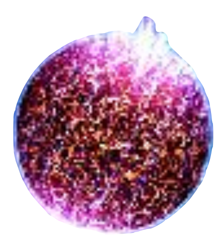
- See Poisson (e.g. 2012.10184) and Poisson & Pitre (e.g. 2506.08722) for a different approach

Matching

- Action $S = S_{\rm EH}[g_{\mu\nu}] + S_{\rm pp}[g_{\mu\nu}, x^\mu(\lambda)]$

$$S_{\rm pp} = \int d\tau \Big\{ M + \sum_L \left(\lambda_\ell^E E_L E^L + \lambda_\ell^B B_L B^L\right)$$
$$+ \sum_{L_1 L_2 L_3} \left[\lambda^{E^3}_{\ell_1\ell_2\ell_3} \left(E_{L_1} E_{L_2} E_{L_3}\right) + \lambda^{EB^2}_{\ell_1\ell_2\ell_3} \left(E_{L_1} B_{L_2} B_{L_3}\right)\right] + \mathcal{O}(E^4, \ldots)\Big\}$$

- Compute $\lambda_i's$ by matching black hole or stellar perturbation theory in GR to the WEFT

NSPT  $\overset{r \gg R}{=}$  WEFT

$\delta g_{\mu\nu}$ $\quad\quad$ $\langle \delta g_{\mu\nu}(\lambda_i)\rangle_{\rm EFT}$

$$\Downarrow$$

$$\lambda_i$$

λ is gauge independent, but $\delta g_{\mu\nu}$ must be matched in the same gauge

WEFT calculation

- Action $S = S_{\mathrm{EH}}[g_{\mu\nu}] + S_{\mathrm{pp}}[g_{\mu\nu}, x^\mu(\lambda)]$

$$S_{\mathrm{pp}} = \int d\tau \Big\{ M + \sum_L \left(\lambda_\ell^E E_L E^L + \lambda_\ell^B B_L B^L \right)$$

$$+ \sum_{L_1 L_2 L_3} \left[\lambda^{E^3}_{\ell_1 \ell_2 \ell_3} \left(E_{L_1} E_{L_2} E_{L_3} \right) + \lambda^{EB^2}_{\ell_1 \ell_2 \ell_3} \left(E_{L_1} B_{L_2} B_{L_3} \right) \right] + \mathcal{O}(E^4, \ldots) \Big\}$$

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- Background-field method

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu} = \eta_{\mu\nu} + H_{\mu\nu} + h_{\mu\nu}$$

Background tidal field
(satisfies vacuum Einstein eqs.)

$$\bar{G}_{\mu\nu}[\bar{g}_{\mu\nu}] = 0$$

WEFT calculation

- Action $$S = S_{\rm EH}[H_{\mu\nu}, h_{\mu\nu}] + S_{\rm GF}[H_{\mu\nu}, h_{\mu\nu}] + S_{\rm pp}[H_{\mu\nu}, h_{\mu\nu}, x^\mu]$$

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- Gauge-fixing term (background de Donder)

$$S_{\rm GF} = -\int d^4x \sqrt{-\bar{g}}\, \bar{g}^{\mu\nu} \bar{\Gamma}_\mu \bar{\Gamma}_\nu \,, \qquad \bar{\Gamma}_\nu = \bar{g}^{\rho\sigma} \left(\bar{\nabla}_\rho h_{\sigma\nu} - \frac{1}{2}\bar{\nabla}_\nu h_{\rho\sigma}\right)$$

WEFT calculation

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- One-point expectation value $$\langle h_{\mu\nu}(x) \rangle = \int \mathcal{D}[h]\, h_{\mu\nu}\, e^{iS[H_{\alpha\beta}, h_{\alpha\beta}, x^\rho]}$$

= all Feynman diagrams with one external leg of $h_{\mu\nu}$

WEFT calculation

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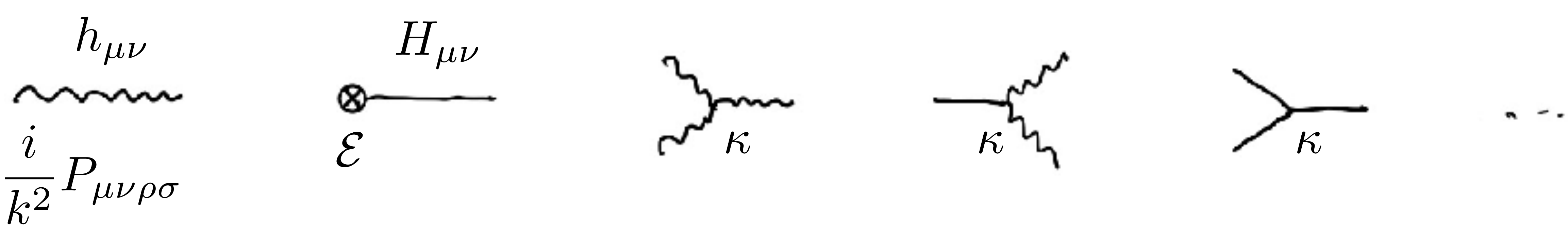

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M

λ_ℓ

$\lambda_\ell \;, \lambda_{\ell\ell_1\ell_2}$

...

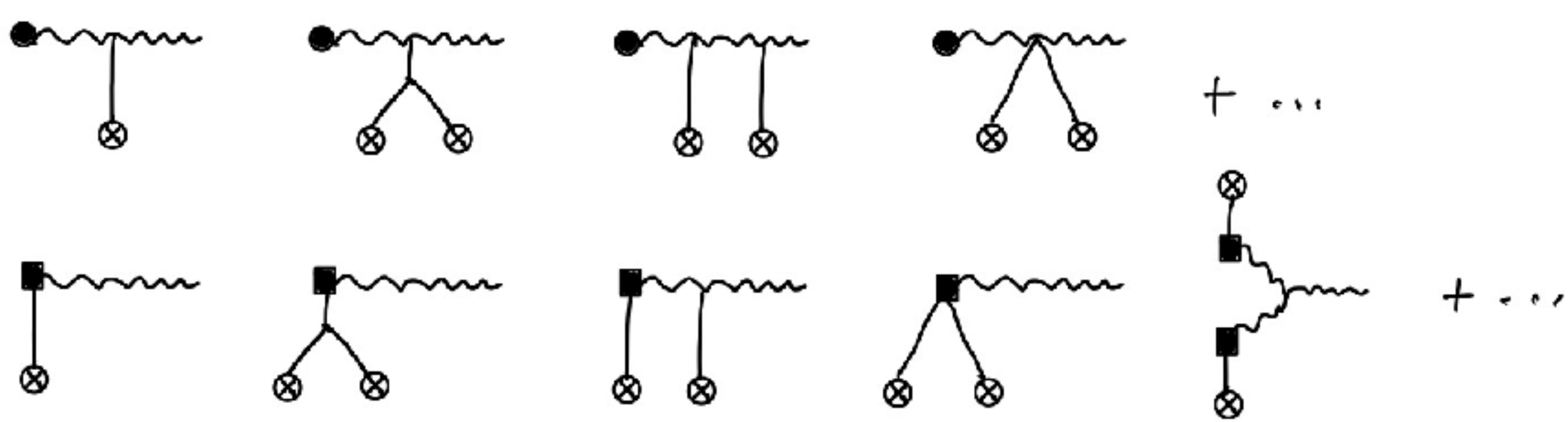
WEFT calculation

- Action $S = S_{\mathrm{EH}}[H_{\mu\nu}, h_{\mu\nu}] + S_{\mathrm{GF}}[H_{\mu\nu}, h_{\mu\nu}] + S_{\mathrm{pp}}[H_{\mu\nu}, h_{\mu\nu}, x^{\mu}]$

$$S_{\mathrm{pp}} = \int d\tau \Big\{ M + \sum_{L} \left(\lambda_\ell^E E_L E^L + \lambda_\ell^B B_L B^L\right)$$
$$+ \sum_{L_1 L_2 L_3} \left[\lambda^{E^3}_{\ell_1\ell_2\ell_3} \left(E_{L_1} E_{L_2} E_{L_3}\right) + \lambda^{EB^2}_{\ell_1\ell_2\ell_3} \left(E_{L_1} B_{L_2} B_{L_3}\right)\right] + \mathcal{O}(E^4, \ldots) \Big\}$$

- One-point expectation value $\langle h_{\mu\nu}(x)\rangle = \int \mathcal{D}[h]\, h_{\mu\nu}\, e^{iS[H_{\alpha\beta}, h_{\alpha\beta}, x^\rho]}$

= all Feynman diagrams with one external leg of $h_{\mu\nu}$



Full GR calculation (NSPT)

- Tolman–Oppenheimer–Volkoff equations inside the star and vacuum Einstein equations outside

$$G_{\mu\nu} = T_{\mu\nu} = 8\pi G \left[(\rho + p) u_\mu u_\nu + p g_{\mu\nu} \right] \qquad p = p(\rho) = C\rho^{1+1/n}$$

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- Expand metric and fluid variables around background <u>in the static limit</u> (no time dependence)

$$g_{\mu\nu} = g^{\rm bkgd}_{\mu\nu}(r) + \delta g_{\mu\nu}(r,\theta,\phi) = g^{\rm bkgd}_{\mu\nu}(r) + {}^{(1)}\delta g_{\mu\nu}(r,\theta,\phi) + {}^{(2)}\delta g_{\mu\nu}(r,\theta,\phi) + \ldots$$

$$ds^2_{\rm bkgd} = -e^{\nu(r)}dt^2 + e^{\Lambda(r)}dr^2 + r^2 d\Omega^2$$

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- Regge-Wheeler approach

$$\delta g_{\mu\nu} = \delta g^+_{\mu\nu} + \delta g^-_{\mu\nu}$$

[Regge & Wheeler '57]

$$\delta g^+_{\mu\nu} = \begin{pmatrix} f(r)H_0 & H_1 & \nabla_A \cancel{\mathcal{H}_0} \\ * & f^{-1}(r)H_2 & \nabla_A \cancel{\mathcal{H}_1} \\ * & * & r^2 K\gamma_{AB} + r^2(\nabla_A\nabla_B - \frac{1}{2}\gamma_{AB})\cancel{G} \end{pmatrix}$$

$$\delta g^-_{\mu\nu} = \begin{pmatrix} 0 & 0 & -\epsilon_A{}^C\nabla_C h_0 \\ * & 0 & -\epsilon_A{}^C\nabla_C h_1 \\ * & * & -\frac{1}{2}(\epsilon_A{}^C\nabla_C\nabla_B + \epsilon_B{}^C\nabla_C\nabla_A)\cancel{h_2} \end{pmatrix}.$$

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$$G_{\mu\nu} = T_{\mu\nu} = 8\pi G\left[(\rho + p)u_\mu u_\nu + p g_{\mu\nu}\right] \qquad p = p(\rho) = C\rho^{1+1/n}$$

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$$ds^2_{\rm bkgd} = -e^{\nu(r)}dt^2 + e^{\Lambda(r)}dr^2 + r^2 d\Omega^2$$

- Solve Einstein equations perturbatively at linear and quadratic order, imposing regularity at the star center and continuity at the star surface

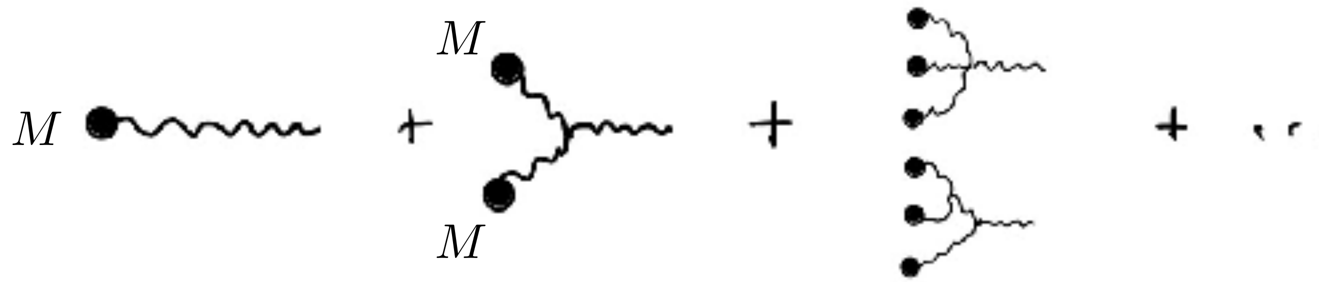
$$G_{\mu\nu} = \bar{G}_{\mu\nu} + {}^{(1)}G_{\mu\nu} + {}^{(2)}G_{\mu\nu} \qquad T_{\mu\nu} = \bar{T}_{\mu\nu} + {}^{(1)}T_{\mu\nu} + {}^{(2)}T_{\mu\nu}$$

$$\mathcal{D}[\delta g^{(1)}_{\mu\nu}] = 0$$ Linear

$$\mathcal{D}[\delta g^{(2)}_{\mu\nu}] = \mathcal{S}[\delta g^{(1)}_{\mu\nu} \star \delta g^{(1)}_{\mu\nu}]$$ Quadratic

Matching at $\mathcal{O}(\mathcal{E}^0)$

$\mathcal{O}(r_s/r)$ $\mathcal{O}(r_s^2/r^2)$ $\mathcal{O}(r_s^3/r^3)$



$$\delta g_{tt}^{\mathrm{EFT}} = -\frac{r_s}{r}$$

Schwarzschild reconstructed

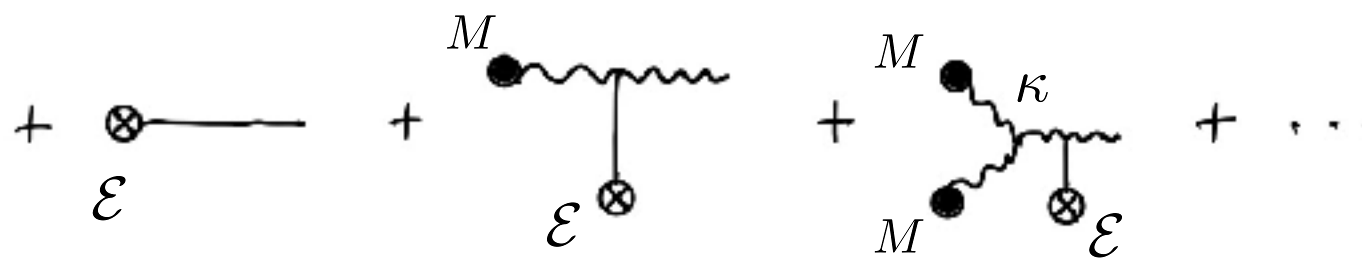
[Mougiakakos & Vanhove '24]

=

$$\delta g_{tt}^{\mathrm{NSPT}} = -\frac{r_s}{r}$$

Matching at $\mathscr{O}(\mathscr{E}^1)$

$\mathcal{O}(r_s^0/r^0)$ $\mathcal{O}(r_s/r)$ $\mathcal{O}(r_s^2/r^2)$ $\mathcal{O}(r_s^3/r^3)$



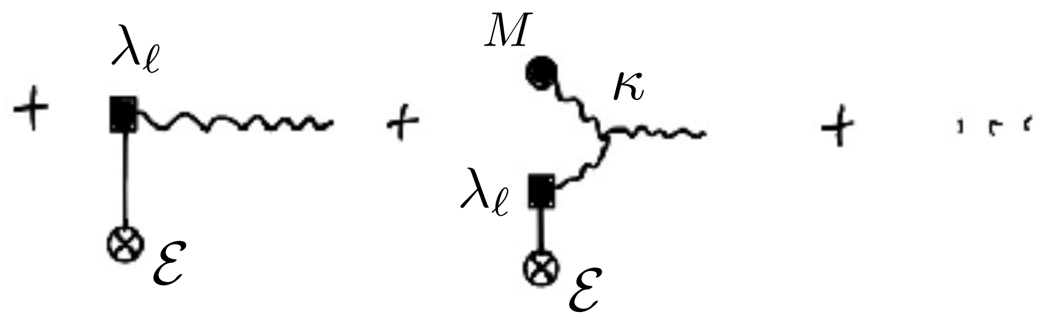
$$\delta g_{tt}^{\mathrm{EFT}} = -\frac{r_s}{r} - \sum_{\ell\geq 2,m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m} \left[\left(1 + a_1 \frac{r_s}{r} + \ldots \right) \right.$$

=

$$\delta g_{tt}^{\mathrm{NSPT}} = -\frac{r_s}{r} - \sum_{\ell\geq 2,m} \mathcal{E}_{\ell m} \left(1 - \frac{r_s}{r}\right) \left[P_\ell^2(2r/r_s - 1) + b\, Q_\ell^2(2r/r_s - 1) \right] Y_{\ell m}$$

Matching at $\mathscr{O}(\mathscr{E}^1)$

$\mathcal{O}(r_s^0/r^0)$ $\mathcal{O}(r_s/r)$ $\mathcal{O}(r_s^2/r^2)$ $\mathcal{O}(r_s^3/r^3)$



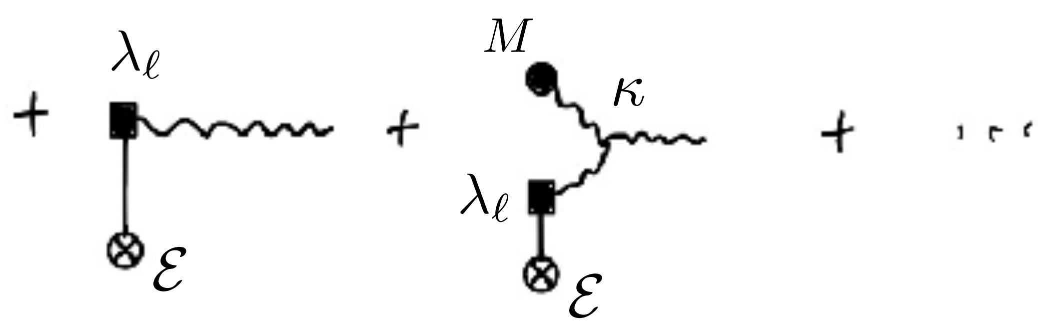
$$\delta g_{tt}^{\mathrm{EFT}} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m} \left[\left(1 + a_1 \frac{r_s}{r} + \ldots \right) + k_\ell \left(\frac{R}{r}\right)^{2\ell+1} \left(1 + b_1 \frac{r_s}{r} + \ldots \right) \right]$$

$$=$$

$$\delta g_{tt}^{\mathrm{NSPT}} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} \mathcal{E}_{\ell m} \left(1 - \frac{r_s}{r}\right) \left[P_\ell^2(2r/r_s - 1) + b\, Q_\ell^2(2r/r_s - 1) \right] Y_{\ell m}$$

Matching at $\mathcal{O}(\mathscr{E}^1)$

$\mathcal{O}(r_s^0/r^0)$ $\quad$ $\mathcal{O}(r_s/r)$ $\quad$ $\mathcal{O}(r_s^2/r^2)$ $\quad$ $\mathcal{O}(r_s^3/r^3)$



$$\delta g_{tt}^{\mathrm{EFT}} = -\frac{r_s}{r} - \sum_{\ell\geq 2,m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m}\left[\left(1 + a_1\frac{r_s}{r} + \ldots\right) + k_\ell\left(\frac{R}{r}\right)^{2\ell+1}\left(1 + b_1\frac{r_s}{r} + \ldots\right)\right]$$

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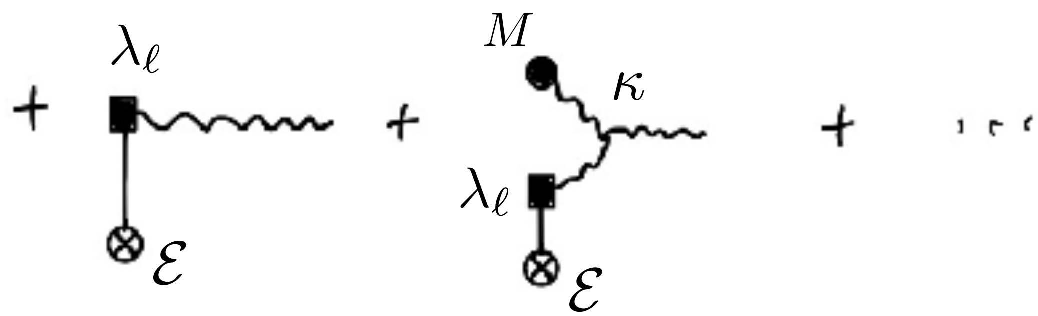
- Matching b at the surface $\quad \Rightarrow \quad k_\ell = k_\ell(C, y)$ $\qquad y = \left.\frac{R\, {}^{(1)}H_0'}{{}^{(1)}H_0}\right|_{r=R}$ [Hinderer '06]

Matching at $\mathcal{O}(\mathscr{E}^1)$

$\mathcal{O}(r_s^0/r^0)$ $\quad$ $\mathcal{O}(r_s/r)$ $\quad$ $\mathcal{O}(r_s^2/r^2)$ $\quad$ $\mathcal{O}(r_s^3/r^3)$



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$$=$$

$$\delta g_{tt}^{\mathrm{NSPT}} = -\frac{r_s}{r} - \sum_{\ell\geq 2,m} \mathcal{E}_{\ell m}\left(1 - \frac{r_s}{r}\right)\left[P_\ell^2(2r/r_s - 1) + \underbrace{b\, Q_\ell^2(2r/r_s - 1)}\right] Y_{\ell m}$$

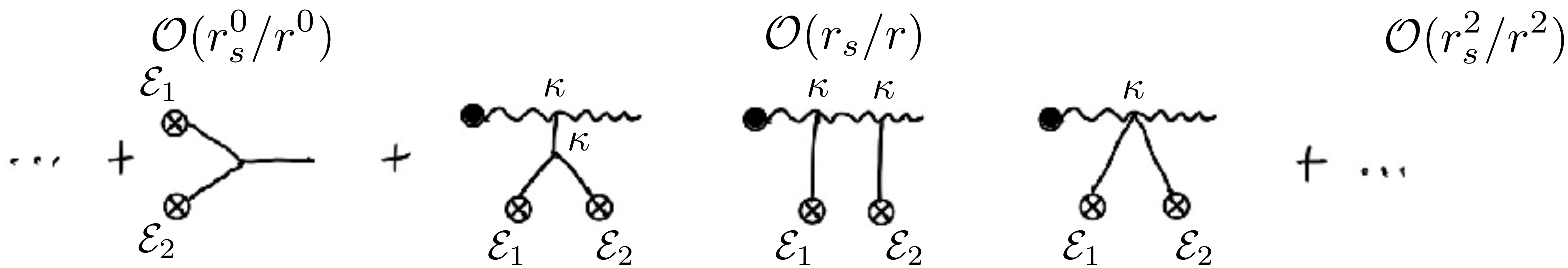
absent for Schwartzschild BH $b = 0$

$\Rightarrow \quad \lambda_\ell = 0$ $\quad$ Vanishing LN of Schwarzschild BHs

[Fang & Lovelace ’05; Binnington & Poisson ’09; Damour & Nagar ’09]

Matching at $\mathcal{O}(\mathcal{E}^2)$

[Riva, Santoni, Savic, FV '23;
Iteanu, Riva, Santoni, Savic, FV '24;]



$$\delta g_{tt}^{\mathrm{EFT}} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m} \left[\left(1 + a_1 \frac{r_s}{r} + \dots \right) + k_\ell \left(\frac{R}{r}\right)^{2\ell+1} \left(1 + b_1 \frac{r_s}{r} + \dots \right) \right]$$

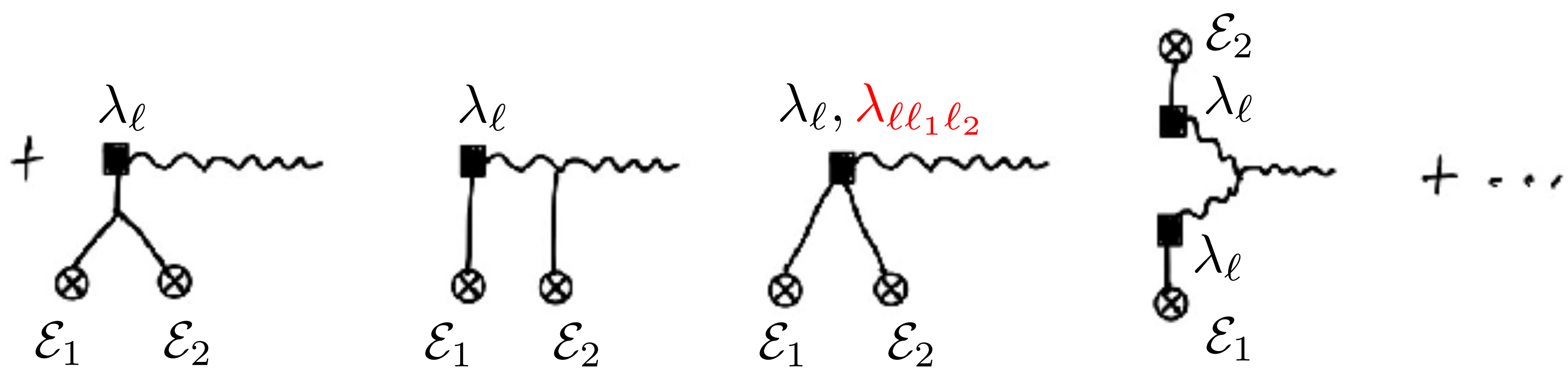
$$+ \sum_{\ell m \ell_1 m_1 \ell_2 m_2} \mathcal{I}_{\ell \ell_1 \ell_2}^{m m_1 m_2} \, \mathcal{E}_{\ell_1 m_1} \mathcal{E}_{\ell_2 m_2} r^{\ell_1 + \ell_2} \Big[(1 + \cdots)$$

=

$$\delta g_{tt}^{\mathrm{NSPT}} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} \mathcal{E}_{\ell m} \left(1 - \frac{r_s}{r}\right) \left[P_\ell^2(2r/r_s - 1) + b\, Q_\ell^2(2r/r_s - 1)\right] Y_{\ell m} + \dots$$

Matching at $\mathcal{O}(\mathcal{E}^2)$

[Riva, Santoni, Savic, FV, Pani, in prep.]



$$\delta g_{tt}^{\rm EFT} = -\frac{r_s}{r} - \sum_{\ell\geq 2,m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m}\left[\left(1 + a_1\frac{r_s}{r} + \dots\right) + k_\ell\left(\frac{R}{r}\right)^{2\ell+1}\left(1 + b_1\frac{r_s}{r} + \dots\right)\right]$$

$$+ \sum_{\ell m \ell_1 m_1 \ell_2 m_2} \mathcal{I}^{m m_1 m_2}_{\ell\ell_1\ell_2}\, \mathcal{E}_{\ell_1 m_1}\mathcal{E}_{\ell_2 m_2} r^{\ell_1+\ell_2}\left[(1+\cdots) + k_\ell\left(\frac{R}{r}\right)^{2\ell+1}(1+\cdots)\right.$$

$$\left. + k_{\ell_1}k_{\ell_2}\left(\frac{R}{r}\right)^{2(\ell_1+\ell_2+1)}(1+\cdots) + \frac{R}{GM}p_{\ell\ell_1\ell_2}\left(\frac{R}{r}\right)^{\ell+\ell_1+\ell_2+1}(1+\cdots)\right]$$

$$=$$

$$\delta g_{tt}^{\rm NSPT} = -\frac{r_s}{r} - \sum_{\ell\geq 2,m}\mathcal{E}_{\ell m}\left(1 - \frac{r_s}{r}\right)\left[P_\ell^2(2r/r_s - 1) + b\, Q_\ell^2(2r/r_s - 1)\right] Y_{\ell m} + \dots$$

Matching at $\mathcal{O}(\mathscr{E}^2)$

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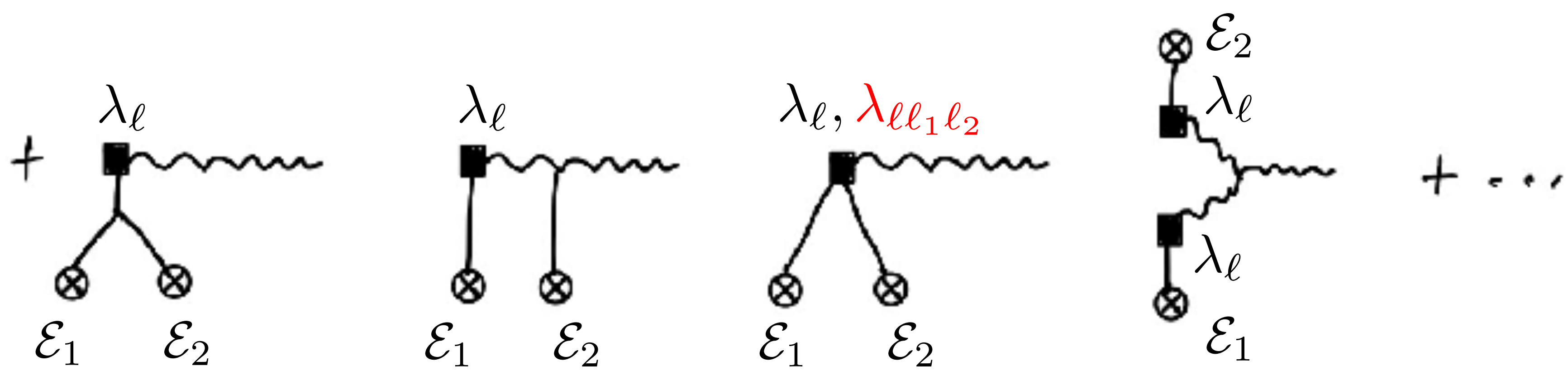

$$\delta g_{tt}^{\rm EFT} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} r^\ell \mathcal{E}_{\ell m} Y_{\ell m} \left[\left(1 + a_1 \frac{r_s}{r} + \ldots\right) + k_\ell \left(\frac{R}{r}\right)^{2\ell+1} \left(1 + b_1 \frac{r_s}{r} + \ldots\right)\right]$$

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$$+ k_{\ell_1} k_{\ell_2} \left(\frac{R}{r}\right)^{2(\ell_1+\ell_2+1)} (1 + \cdots) + \frac{R}{GM} p_{\ell\ell_1\ell_2} \left(\frac{R}{r}\right)^{\ell+\ell_1+\ell_2+1} (1 + \cdots) \Bigg]$$

$$=$$

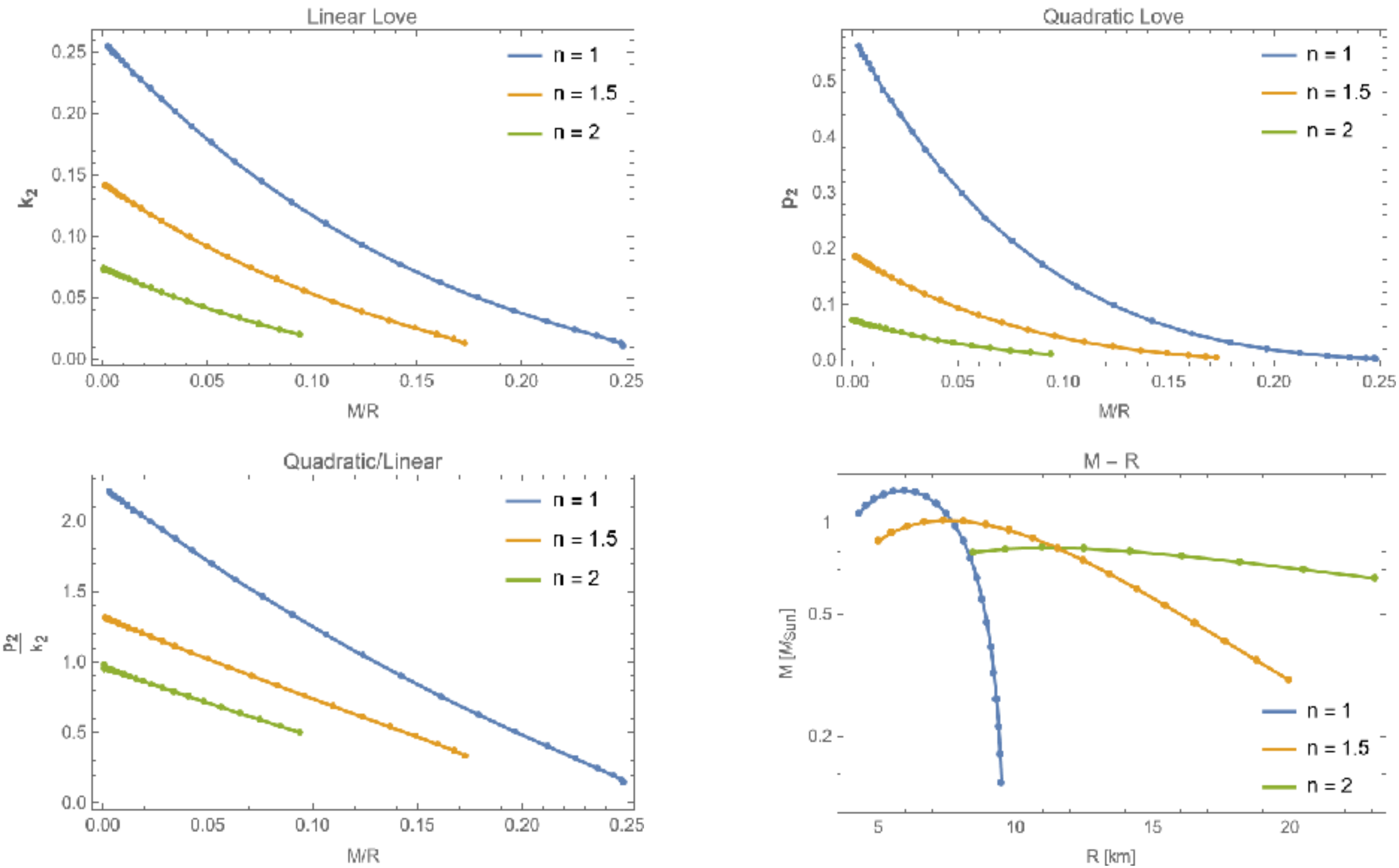
$$\delta g_{tt}^{\rm NSPT} = -\frac{r_s}{r} - \sum_{\ell \geq 2, m} \mathcal{E}_{\ell m} \left(1 - \frac{r_s}{r}\right) \left[P_\ell^2(2r/r_s - 1) + b\, Q_\ell^2(2r/r_s - 1)\right] Y_{\ell m} + \ldots$$

$$\Rightarrow \quad p_{\ell\ell_1\ell_2} = p_{\ell\ell_1\ell_2}(C, y, z, x) \qquad y = \left.\frac{R\, {}^{(1)}H_0'}{{}^{(1)}H_0}\right|_{r=R} \quad z = \left.\frac{R\, {}^{(2)}H_0'}{{}^{(2)}H_0}\right|_{r=R}, \quad x = \left.\frac{{}^{(1)}H_0^2}{{}^{(2)}H_0}\right|_{r=R}$$

Results

[Riva, Santoni, Savic, FV, Pani, in prep.
Agreement with Pitre & Poisson '25 when overlap]

- Polytropic equation of state: $p = C\rho^{1+1/n}$



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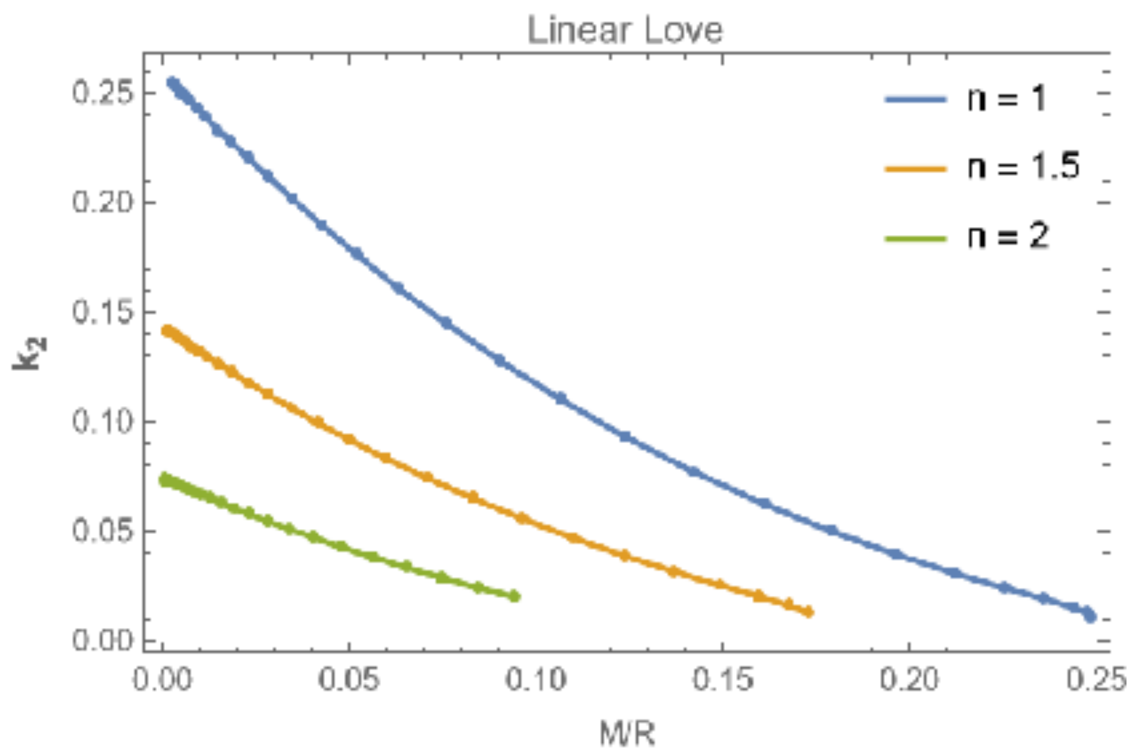
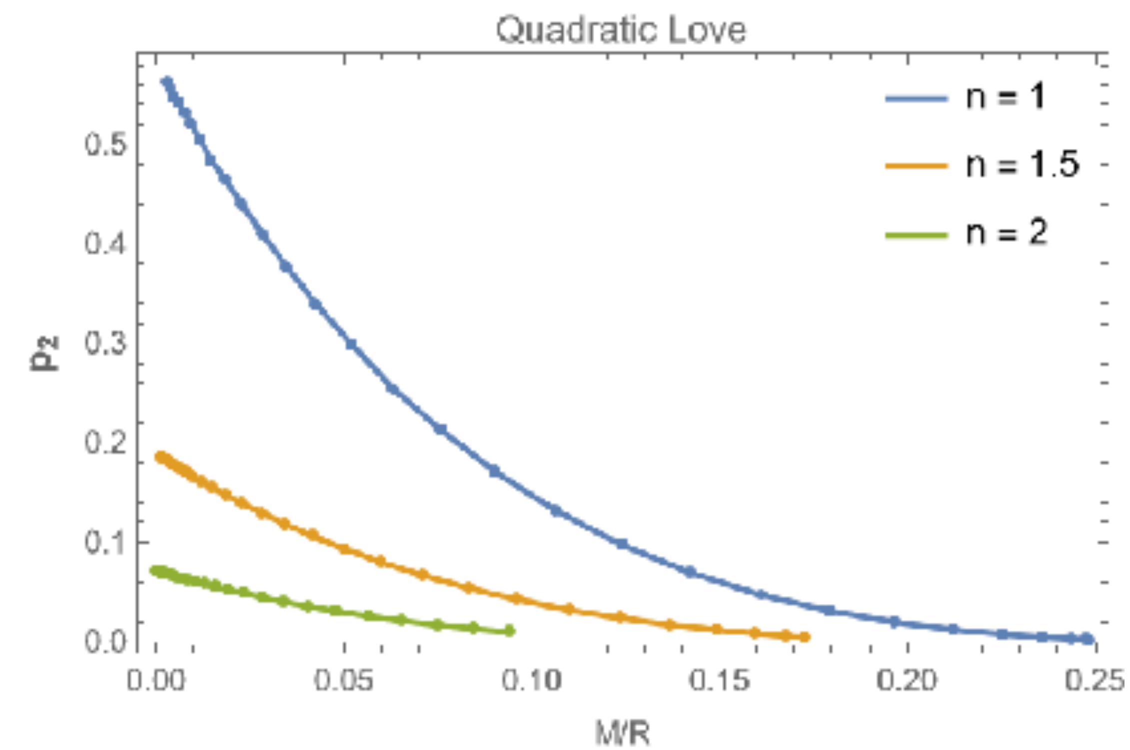
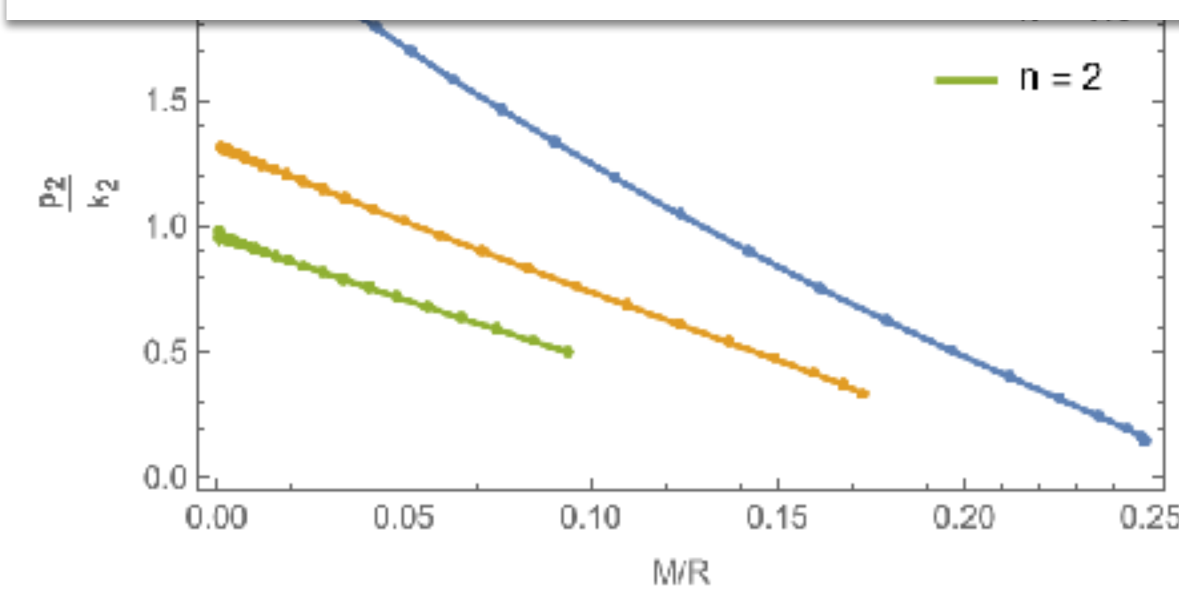
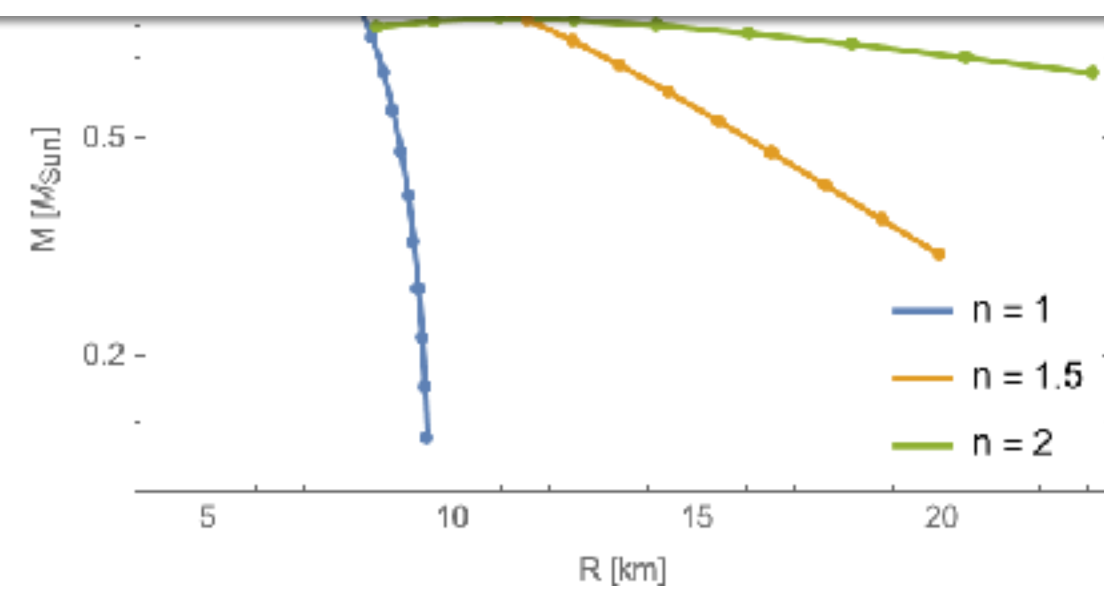




For Sch. BH ($C \to 1/2$) we have finite polynomials $\Rightarrow$ $p_{\ell\ell_1\ell_2} = 0$

[Riva, Santoni, Savic, FV '23;
Iteanu, Riva, Santoni, Savic, FV '24;]





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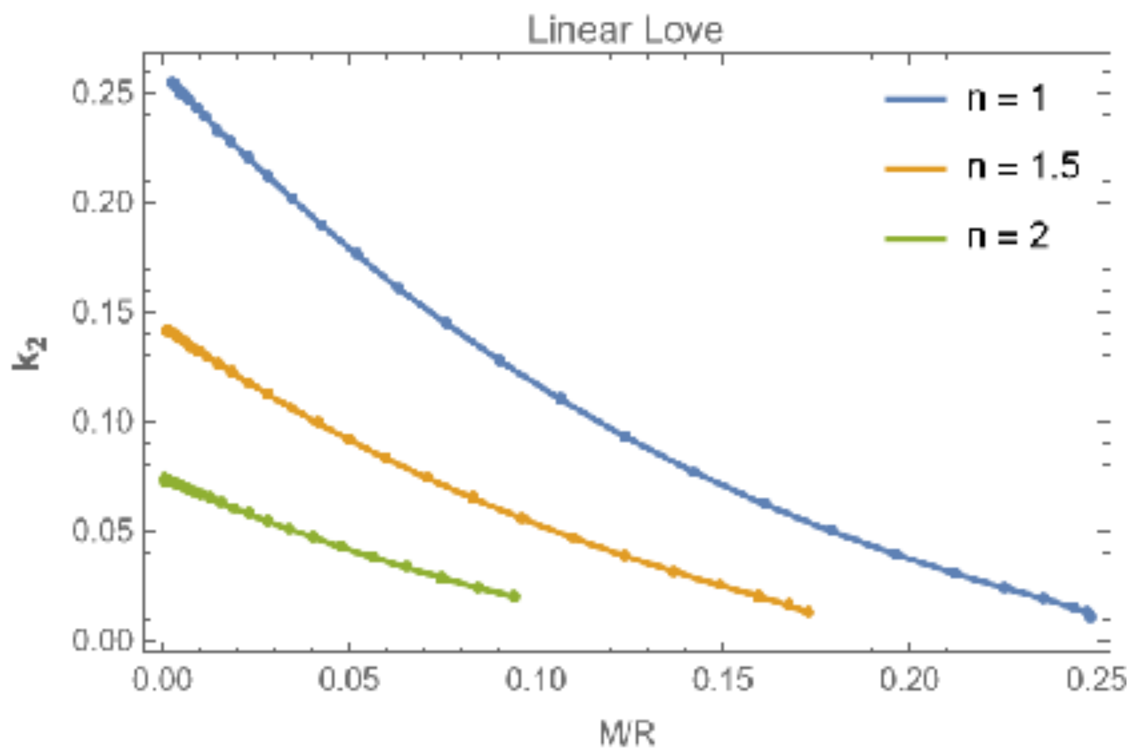

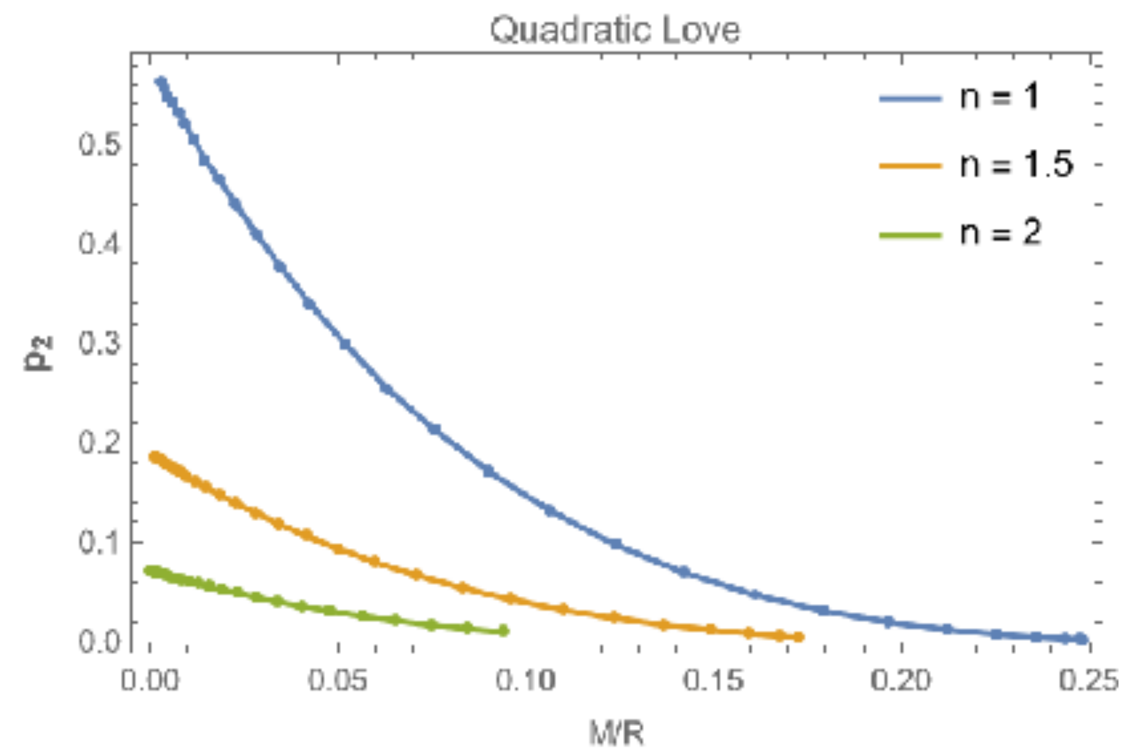

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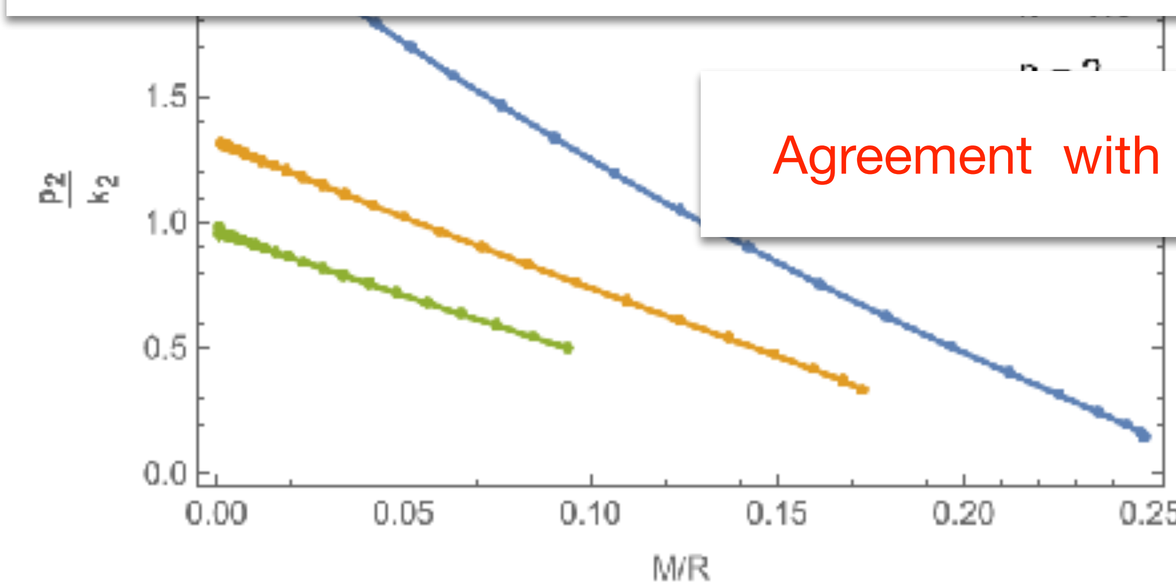
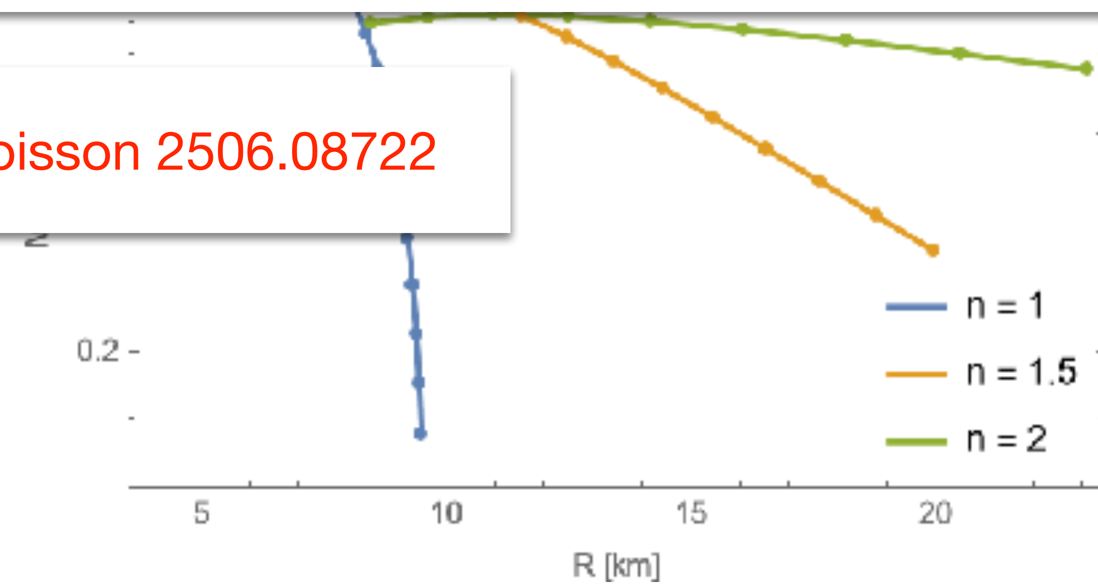


Agreement with Pitre & Poisson 2506.08722



Conclusion

- Obtained even static <u>quadratic tidal Love numbers</u> of neutron stars by matching the worldline effective field theory with perturbation theory at second-order
- WEFT unambiguously defines TNL and reconstructs perturbatively NS perturbation theory
- Linear and quadratic Love numbers don't run: $\lambda_\ell, \lambda_{\ell\ell_1\ell_2} \not\supset \log(r), \ldots$
- Future: Extend to odd modes, possibly all multipoles, study observability