Evolution and spectrum in non-normal dynamics: a (black hole) gravitational case

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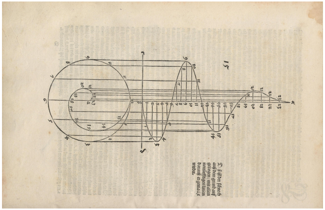

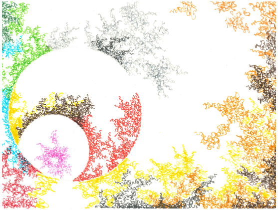

Théorie, Univers et Gravitation -TUG

Saclay, 14-16 October 2025

Scheme

1. The general problem: linear “non-normal” wave equation
2. Brief overview of non-normal operators and non-modal analysis
   - Spectral instability
   - Non-modal transient growths
   - Pseudo-resonances
   - Some elements of non-modal analysis
3. A gravitational case: hyperboloidal approach to scattering on black holes
   - BH QNM instability
   - “Free” evolutions on BHs and non-modal transient growths
   - “Driven” evolutions on BHs (and pseudo-resonances?)
4. Conclusions and Perspectives

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4. Conclusions and Perspectives

General problem: non-normal dynamics

Setting: dissipative linear wave equation

Linear wave equation with dissipation (in the bulk/through boundaries) and source:

$$\begin{cases} \Box_g \phi + k^a \nabla_a \phi + V\phi = S(x,t) \\ \text{Possibly "leaky" Boundary Conditions} \end{cases}$$

Wave dynamics with non-selfadjoint time generator

Cast in "Schrödinger form" (1st-order reduction in time), with $u = (\phi, \partial_t \phi)$:

$$\boxed{\partial_t u(t,x) = iLu(t,x) + S(t,x)}$$

with L **non-selfadjoint** operator acting on **appropriate Hilbert (Banach) space**.

Goal

Discussion of qualitative **non-selfadjoint dynamical and spectral phenomena**.

General problem: non-normal dynamics

Hyperboloidal **non-normal** (linear) evolution problem driven by an external source

$$\partial_t u(t,x) = iLu(\tau,x) + S(t,x) \quad , \quad [L, L^\dagger] \neq 0$$

Dynamics and spectral theory: characteristic "non-normal" phenomena

- Spectral problem of L: **Eigenvalue instabilities**

$$Lv_n(x) = \omega_n v_n(x) \quad , \quad L^\dagger w_n = \overline{\omega}_n w_n \quad , \quad (L^t \alpha_n(x) = \omega_n \alpha_n(x))$$

- Source-less dynamics: **Non-modal transient growths**

$$(\partial_t - iL)u(\tau,x) = 0$$

- Source-driven dynamics: **Pseudo-resonances**

$$(\partial_t - iL)u(\tau,x) = S(\tau,x)$$

A gravitational setting: GR perturbation theory

GR perturbation theory (sketch!): same background wave operator [cf. L. Sberna's talk]

Writing

$$g_{ab} = g_{ab}^{(0)} + \epsilon h_{ab}^{(1)} + \epsilon^2 h_{ab}^{(2)} + O(\epsilon^3)$$

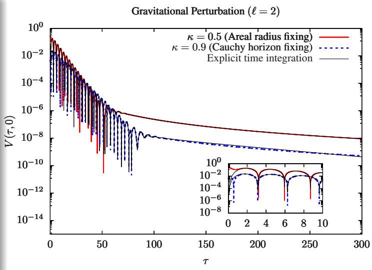
Hierarchical structure:

$$\begin{aligned} \delta G_{ab} \cdot h^{(1)} &= 0 \\ \delta G_{ab} \cdot h^{(2)} &= \delta^2 G_{ab}[h^{(1)}, h^{(1)}] \,, \end{aligned}$$

Hierarchy of evolution problems

$$\begin{aligned} (\partial_\tau - iL)\, u^{(1)} &= 0 \\ (\partial_\tau - iL)\, u^{(2)} &= S(\tau, x; u^{(1)}) \end{aligned}$$

- Ring-downs, QNMs (2nd-order QNMs...).
- Self-force calculations.
- ...
- "**Wave**-Mean Flow" (asymptotic) PDEs.



Scheme

Spectral Theorem. Normal and 'non-normal' operators

Normal operators: Spectral Theorem

- **Normality**: denoting the adjoint matrix by $L^\dagger$, then L is normal iff

$$[L, L^\dagger] = LL^\dagger - L^\dagger L = 0$$

  Matrix examples: symmetric, hermitian, orthogonal, unitary...

- **Spectral Theorem** ("moral statement"):

$$L \ \text{ is normal iff is unitarily diagonalisable}$$

Note: this depends on the adjoint $L^\dagger$, then on the Hilbert space (scalar product).

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Normal modes: key notion for "modal (harmonic) analysis"

The eigenvectors $\hat{v}_n$ of L, i.e. $L\hat{v}_n = \omega_n \hat{v}_n$:

- **Orthonormal set**: $\langle \hat{v}_i, \hat{v}_i \rangle_G = \delta_{ij}$, **Complete set**: $\mathrm{Id} = \sum_n^\infty |\hat{v}_n\rangle\langle\hat{v}_n|$

- Spectral resolution of (homogenerous) evolution problem, $u(t=0) = u_o(x)$

$$u(t,x) = \sum_{n=0}^{\infty} e^{i\omega_n t} a_n v_n(x) \quad , \quad a_n = \langle \hat{v}_n, u_0 \rangle_G$$

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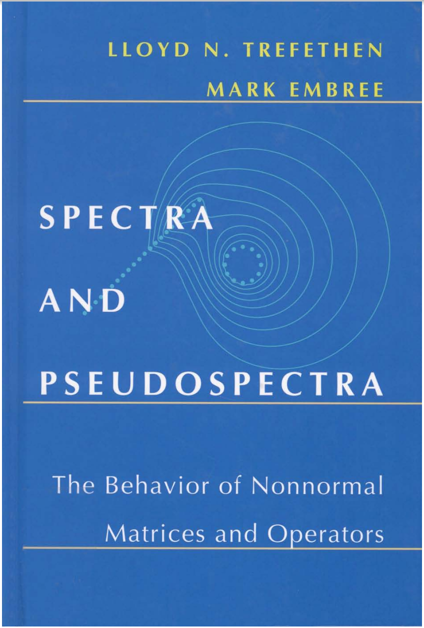
'Non-normal' operators, $[L, L^\dagger] \neq 0$: no Spectral Theorem

- **No spectral theorem**: no "normal modes" (no Hilbert basis).
- Eigenfunctions of L non-orthornormal and not complete.
- "Non-modal" effects associated with non-normal operators:
  - **Eigenvalue instabilities**.
  - Non-modal (linear) **transient growths**.
  - **Pseudo-resonances**.
  - ...

Spectral Theorem. Normal and 'non-normal' operators



Plan

Spectral (in)stability

Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$$

acting on functions in $L^2([0,1])$, with homogeneous Dirichlet conditions (Chebyshev finite-dimensional matrix approximates).

Spectral (in)stability

Example of spectral instability

$$L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c + \epsilon E_{\text{Random}} \quad , \quad a, b, c \in \mathbb{R}, \; ||E_{\text{Random}}|| = 1$$

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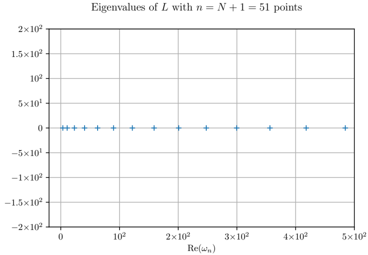

$$a = -1, \ b = 0, \ c = 1, \ \epsilon = 0$$

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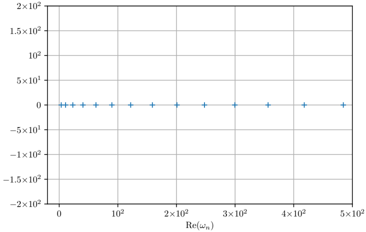

$$a = -1,\ b = 0,\ c = 1,\ \epsilon = 10^{-5}$$

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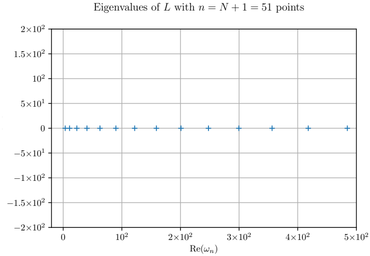

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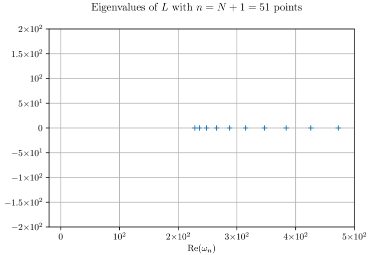

$$a = -1, \ b = 30, \ c = 1, \ \epsilon = 0$$

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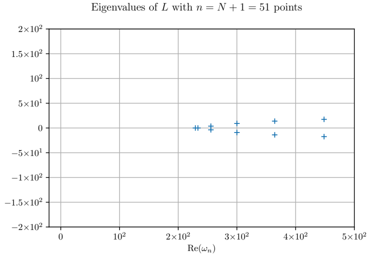

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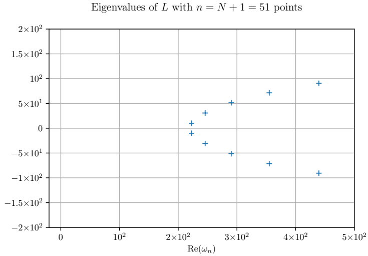

$$a = -1, \; b = 30, \; c = 1, \; \epsilon = 10^{-8}$$

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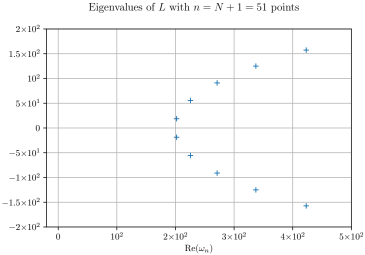

$$a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-6}$$

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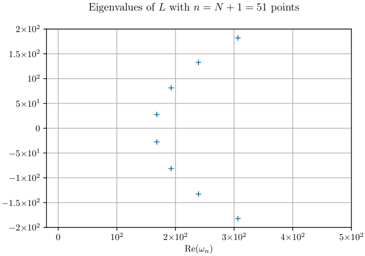

$$a = -1, \ b = 30, \ c = 1, \ \epsilon = 10^{-4}$$

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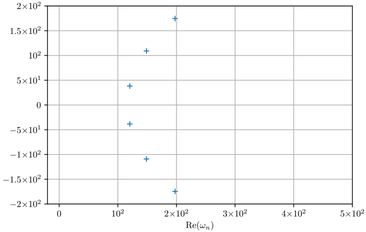
Eigenvalues of L with $n = N + 1 = 51$ points



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Spectral (in)stability

Right- and left-eigenvectors, respectively v_i and w_i, of L

$$L v_i = \omega_i v_i \quad , \quad L^\dagger w_i = \bar{\omega}_i w_i \quad (\Leftrightarrow w_i^\dagger L = \omega_i w_i^\dagger)$$

Perturbation theory of eigenvalues [cf. Kato 80, ...; e.g. Trefethen, Embree 05]:

$$L(\epsilon) \quad = \quad L + \epsilon \delta L \quad , \qquad ||\delta L|| = 1 \ .$$

$$|\omega_i(\epsilon) - \omega_i| = \epsilon \frac{|\langle w_i, \delta L \ v_i(\epsilon)\rangle|}{|\langle w_i, v_i\rangle|} \leq \epsilon \frac{||w_i|| \ ||\delta L \ v_i||}{|\langle w_i, v_i\rangle|} + O(\epsilon^2) \leq \epsilon \frac{||w_i|| \ ||v_i||}{|\langle w_i, v_i\rangle|} + O(\epsilon^2).$$

Eigenvalue condition number: $\kappa(\omega_i)$

$$\boxed{\kappa(\omega_i) = \frac{||w_i|| ||v_i||}{|\langle w_i, v_i\rangle|}}$$

Pseudospectrum

Pseudospectrum

Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [e.g Trefethen & Embree 05]:

$$\begin{aligned}
\sigma_\epsilon(L) &= \boxed{\{\omega \in \mathbb{C},\ \text{such that } \omega \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon\}} \\
&= \{\omega \in \mathbb{C},\ \text{such that } ||Lv - \omega v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1\} \\
&= \{\omega \in \mathbb{C}, \text{such that } ||(\omega I - L)^{-1}|| > \epsilon^{-1}\}
\end{aligned}$$

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\end{aligned}$$

Normal case: bounds on the norm of the resolvent $R_L(\omega) = (\omega I - L)^{-1}$

Given $\omega \in \mathbb{C}$ and $\sigma(L)$ the spectrum of L, it holds

$$||(\omega I - L)^{-1}||_2 = \frac{1}{\text{dist}(\omega, \sigma(L))}$$

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\end{aligned}$$

Non-normal case: bad control on the resolvent $R_L(\omega)$. **Pseudospectrum**

The norm of the resolvent can become very large far from the spectrum:

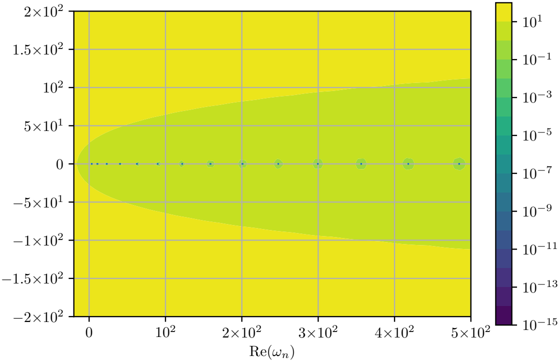
$$||(\omega I - L)^{-1}||_2 \leq \frac{\kappa}{\mathrm{dist}(\omega, \sigma(L))}$$

where κ is a "condition number" assessing the lack of proportionality of 'left' and 'right' eigenvectors of L, and can become very large in the non-normal case.

Pseudospectrum

Pseudospectrum of: $L = a\dfrac{d^2}{dx^2} + b\dfrac{d}{dx} + c + \epsilon E_{\mathrm{Random}}$

Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -50$



$a = -1, \; b = 0, \; c = 1, \; \epsilon = 0$

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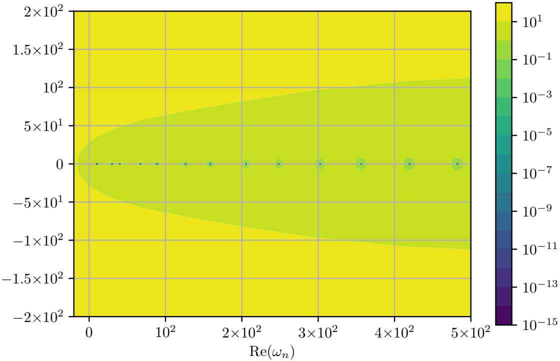
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = 1$



$a = -1,\ b = 0,\ c = 1,\ \epsilon = 10^{1}$

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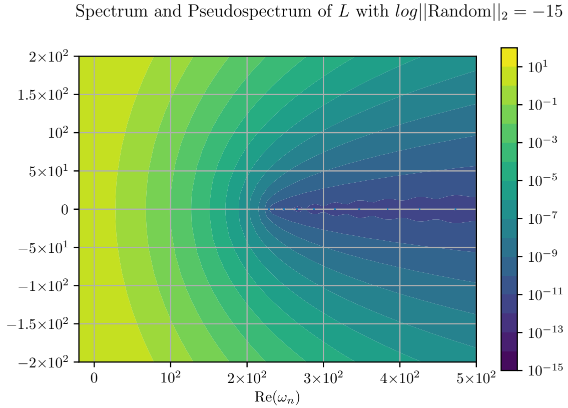

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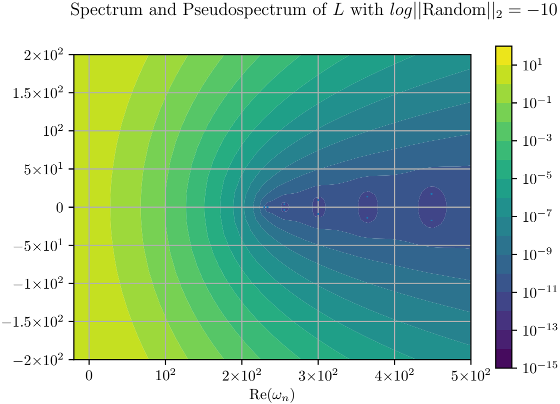

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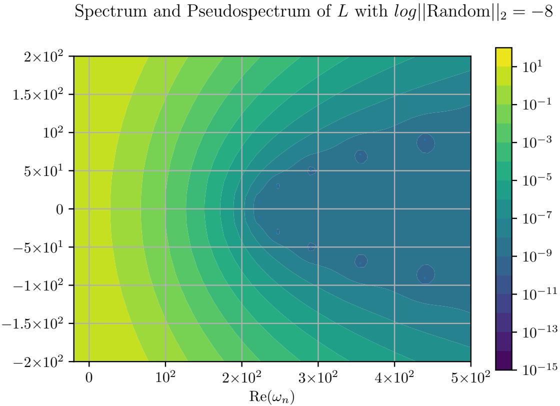

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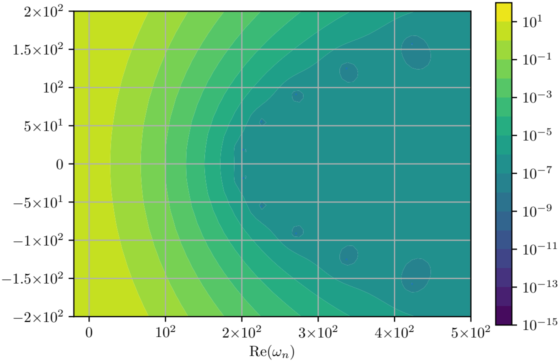
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -6$



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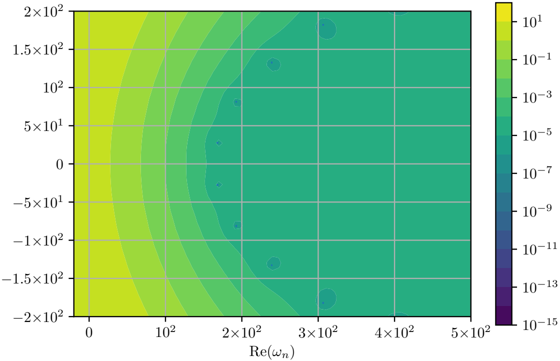
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -4$



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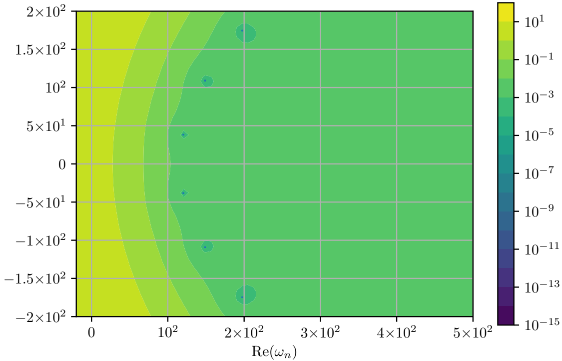
Spectrum and Pseudospectrum of L with $log||\mathrm{Random}||_2 = -2$



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Pseudospectrum

The 'role' of random perturbations [Sjöstrand 19; Hager 05, Montrieux, Nonnenmacher, Vogel,...]

Random perturbations improve the analytical behaviour of $R_L(\omega)$!!!

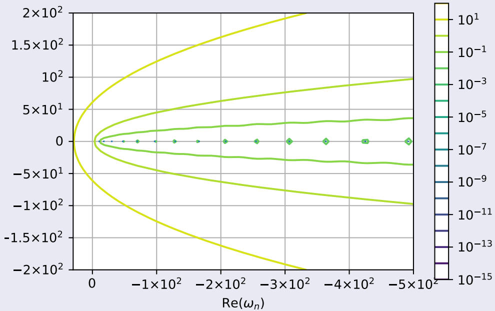
The relevance of the scalar product: assessing large/small

The illustrative operator: $L = a\frac{d^2}{dx^2} + b\frac{d}{dx} + c \quad , \quad a, b, c \in \mathbb{R}$ [Gasperin & JLJ 22]

- Non-selfadjoint in standard $L^2([0,1])$ for $b \neq 0$.
- Formally normal!
- Non-normal: domain of $L^\dagger L$ and $LL^\dagger$ different.
- But actually self-adjoint...

Cast in Sturm-Liouville form: selfadjoint for appropriate scalar product $\langle \cdot, \cdot \rangle_w$!!!

Pseudospectrum using the L^2-inner-product



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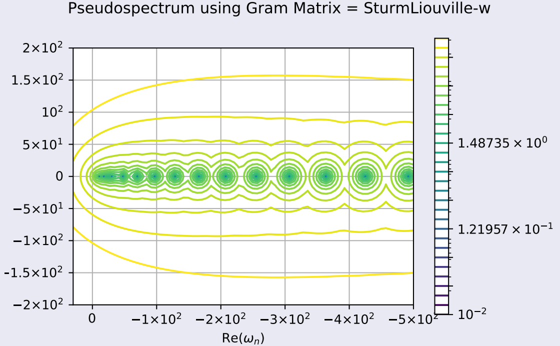
Cast in Sturm-Liouville form: selfadjoint in appropriate scalar product $\langle \cdot, \cdot \rangle_w$



Plan

Non-modal transient growths

Superposition of two non-orthogonal (eigen-)vectors: growth dynamics

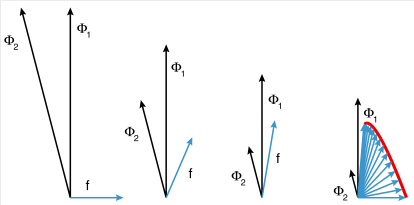
Given two QNMs frequencies $\omega_1 = \omega_1^R + i\omega_1^I$ and $\omega_2 = \omega_2^R + i\omega_2^I$

$$u_1(t,x) = e^{i\omega_1 t} v_1(x) \qquad , \qquad u_2(t,x) = e^{i\omega_2 t} v_2(x)$$

with $\omega_1^I > 0$ and $\omega_2^I > 0$, consider the superposition

$$\boxed{u(t,x) = a_1 u_1(t,x) + a_2 u_2(t,x)} \; .$$

Example [Schmid 07]: $f = \Phi_1 - \Phi_2$



Non-modal transient growths

Notation

Define:

$$\begin{aligned} a_1 &= e^{i\varphi_1}|a_1| \\ a_2 &= e^{i\varphi_2}|a_2| \\ \langle u_1, u_2\rangle &= e^{i\delta_{12}}|\langle u_1, u_2\rangle| \\ \cos\hat{\theta}_{12} &= \frac{|\langle u_1, u_2\rangle|}{||u_1||\ ||u_2||} \end{aligned}$$

and compute the norm of the $u(t, x)$.

Non-modal transient growths

Transient growth of non-orthogonal vector superpositiom

The norm $||u||(t)$ evolves according to:

$$\begin{aligned} ||u||^2(t) &= |a_1|^2 e^{-2\omega_1^I t} + |a_2|^2 e^{-2\omega_2^I t} \\ &+ \boxed{2|a_1||a_2|\cos\hat{\theta}_{12} e^{-(\omega_1^I+\omega_2^I)t} \cos\left((\omega_2^R + \omega_1^R)t + \Phi_{12}\right)} \end{aligned}$$

where we have imposed $||u_1|| = ||u_2|| = 1$ and with

$$\Phi_{12} = (\varphi_2 - \varphi_1) + \delta_{12}$$

Non-modal transient growth: a genuinely non-normal effect

For "stable" QNM frequencies ($\omega_1^I > 0, \omega_2^I > 0$):

- If u_1 and u_2 are orthogonal then, $||u||(t)$ is decreasing.
- If u_1 and u_2 are not orthogonal, an initial transient growth can happen due to the third term.
- This phenomenon depends on the choice of scalar product (norm).

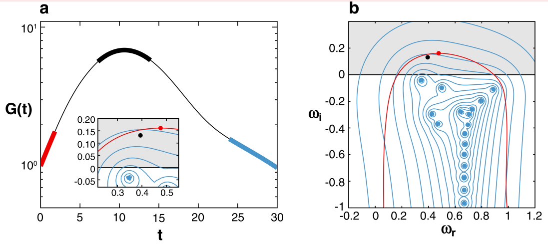
Non-modal transient growths

Growth function $G(t)$: maximum possible amplification

$$G(t) = \sup_{u_0 \neq 0} \frac{||u(t)||}{||u_0||} = \sup_{u_0 \neq 0} \frac{||e^{itL} u_0||}{||u_0||} = ||e^{itL}||$$

Optimal excitation u_0: eigenfunction of the maximum (generalised) eigenvalue in the Singular Value Decomposition (eigenfunction of the $\max \sigma[(e^{itL})^\dagger e^{itL})]$.

Growth factor $G(t) = ||e^{itL}||$ and pseudospectrum (eg. Poiseuille flow [Schmid 07])



Non-modal transient growths

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Optimal excitation u_0: eigenfunction of the maximum (generalised) eigenvalue in the Singular Value Decomposition (eigenfunction of the $\max \sigma[(e^{itL})^\dagger e^{itL})]$.

Pseudospectrum $\sigma_\epsilon(L)$: (kind of) Fourier-transform of $G(t)$

Given $\epsilon > 0$, the ϵ-pseudospectrum $\sigma_\epsilon(L)$ of L is defined as [e.g Trefethen & Embree 05]:

$$\begin{aligned} \sigma_\epsilon(L) &= \boxed{\{\omega \in \mathbb{C}, \text{ such that } \omega \in \sigma(L + \delta L) \text{ for some } \delta L \text{ with } ||\delta L|| < \epsilon\}} \\ &= \{\omega \in \mathbb{C}, \text{ such that } ||Lv - \omega v|| < \epsilon \text{ for some } v \text{ with } ||v|| = 1\} \\ &= \boxed{\{\omega \in \mathbb{C}, \text{such that } ||(\omega I - L)^{-1}|| = ||R_L(\omega)|| > \epsilon^{-1}\} =} \end{aligned}$$

Plan

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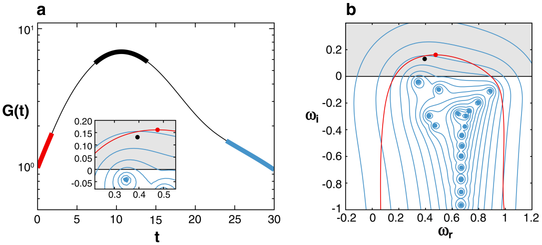
4. Conclusions and Perspectives

Pseudo-resonances



Resonances and Pseudo-resonances

If we force the system with an external source $S(\omega)$ at a (real) frequency ω, then:

- For **normal** L:
  If ω is close to a (complex) resonant frequency ω_n, where the resolvent ("Green function") "diverges", we have a strong response: **resonance**.
- For **non-normal** L:
  If there is no resonant frequency ω_n, but the norm of the resolvent (pseudospectrum) is large at that ω, we have still a strong response: **pseudoresonance**.

Pseudo-resonances

Driving the (non-normal) linear dynamics with an external force $S(t,x)$ [JLJ 22]

Consider the linear equation driven by a harmonic source:

$$(\partial_t - iL)u(t,x) = S(t,x) \quad , \quad S(t,x) = e^{i\omega t}s(x)$$

Then, the solution can be written in terms of the resolvent $R_L(\omega)$

$$u(t,x) = \frac{1}{i}e^{i\omega t}\big((\omega - L)^{-1}s\big)(x) = \frac{1}{i}e^{i\omega t}\big(R_L(\omega)s\big)(x)$$

Maximising over all initial data we get

$$\begin{aligned} R_{\max}(\omega) &= \sup_{s\neq 0}\frac{||u||}{||s||} = e^{-\mathrm{Im}(\omega)t}\sup_{s\neq 0}\frac{||(\omega - L)^{-1}s||}{||s||} = e^{-\mathrm{Im}(\omega)t}||(\omega - L)^{-1}|| \\ &= e^{-\mathrm{Im}(\omega)t}||R_L(\omega)|| \end{aligned}$$

And, finally, maximising over **real frequencies** ω, we obtain:

$$R_{\max} = \sup_{\omega\in\mathbb{R}} R_{\max}(\omega) = \sup_{\omega\in\mathbb{R}}||(\omega - L)^{-1}|| = \sup_{\omega\in\mathbb{R}}||R_L(\omega)||$$

Conclusion:

If ϵ-pseudospectra lines with small ϵ (i.e. large values of $||R_L(\omega)||$) approach the real line, then **pseudo-resonant phenomena** can be expected.

Plan

Keldysh asymptotic QNM expansions [Besson & JLJ 25]

Homogeneous evolution problem with "non-normal" time generator L

$$\left\{ \begin{array}{l} \partial_t u(\tau, x) = iLu(t,x) + S(t,x) \ , \\ u(t=0,x) = u_0(x) \quad , \quad ||u_0|| < \infty \ , \end{array} \right.$$

Keldysh asymptotic QNM expansions [Besson & JLJ 25]

Dual spectral problems: vectors, covectors (bi-orthonormal bases) $\langle \alpha_i, v_j \rangle = \delta_{ij}$

$$Lv_n = \omega_n v_n \quad , \quad L^t \alpha_n = \omega_n \alpha_n \quad , \quad v_n \in \mathcal{H}, \alpha_n \in \mathcal{H}^*$$

If a scalar product available: spectral and adjoint spectral problem

$$L\hat{v}_n = \omega_n \hat{v}_n \quad , \quad L^\dagger \hat{w}_n = \overline{\omega}_n \hat{w}_n \quad , \quad \hat{v}_n, \hat{w}_n \in \mathcal{H}$$

Keldysh asymptotic QNM expansions [Besson & JLJ 25]

Dual spectral problems: vectors, covectors (bi-orthonormal bases) $\langle \alpha_i, v_j \rangle = \delta_{ij}$

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Keldysh expansion $\langle \alpha_i, v_j \rangle = \langle \hat{w}_i, \hat{v}_j \rangle_G = \delta_{ij}, ||\hat{v}_i||_G = ||\hat{w}_i||_G = 1$ [Besson & JLJ 25]

$$\begin{aligned} u(t,x) &= \sum_{n=0}^{N_{\mathrm{QNM}}} e^{i\omega_n t} \langle \alpha_n, u_0 \rangle v_n(x) + E_{N_{\mathrm{QNM}}}(t; u_0) \\ &= \sum_{n=0}^{N_{\mathrm{QNM}}} e^{i\omega_n t} \kappa_n \langle \hat{w}_n, u_0 \rangle_G \hat{v}_n(x) + E_{N_{\mathrm{QNM}}}(t; u_0) \end{aligned}$$

with $\quad ||E_{N_{\mathrm{QNM}}}(t;u_0)|| \leq C(N_{\mathrm{QNM}}, L) e^{-a_{N_{\mathrm{QNM}}} t} ||u_0|| \ ,$

Non-modal analysis

Beyond spectral analysis: some elements

- **QNM spectrum** $\sigma(L) = \lim_{\epsilon\to 0} \sigma_\epsilon(L)$: possibility of spectral instabilities.
- **Numerical range** $W(L)$ **and numerical abscisa** $\omega(L)$ (in $\lim_{\epsilon\to\infty} \sigma_\epsilon(L)$ limit):

$$
\begin{aligned}
W(L) &= \{\langle u, Lu\rangle, \text{with } ||u|| = 1, u \in H\} \\
\omega(L) &= \sup \mathrm{Im}\big(W(L)\big) \\
\omega(L) &= \frac{d}{dt}\,||e^{tL}||\Big|_{t=0}
\end{aligned}
$$

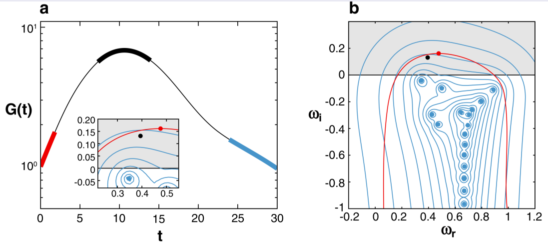
- **Intermediate/maximum transient, Kreiss constant** $\mathcal{K}(L)$:

$$
\begin{aligned}
\sup_{t\geq 0} G(t) = \sup_{t\geq 0} ||e^{tL}|| &\geq \mathcal{K}(L) \\
\mathcal{K}(L) &= \sup_{\mathrm{Im}(z)>0} \{|\mathrm{Im}(z)|\cdot||(L - zI)^{-1}||\}
\end{aligned}
$$

- **Pseudo-resonances**: $$R_{\max}(\omega) = \sup_{s\neq 0} \frac{||u^{(2)}||}{||s||} = e^{-\mathrm{Im}(\omega)t}||R_L(\omega)||$$

Non-modal analysis

Growth factor $G(t) = ||e^{itL}||$ and pseudospectrum (eg. Poiseuille flow [Schmid 07])



Spectral instability, Transients and Pseudoresonances [Trefethen et al. 93, Tref. & Embrée 05, ...]

- **Late times**: QNM spectrum $\sigma(L)$.
- **Transients** (no source, $u^{(1)}$): consequence of non-orthogonality of QNMs.
  - **Initial times**: numerical range $W(L)$ and spectral abscisa $\omega(L)$.
  - **Intermediate/maximum**: pseudospectrum $\sigma_\epsilon(L)$ and Kreiss constant $\mathcal{K}(L)$.
- **Pseudo-resonances** (source present, $u^{(2)}$): $R_{\max}(\omega)$ (with $\omega \in \mathrm{Re}(\omega)$).

Non-modal analysis

Dynamics and spectral theory: characteristic "non-normal" phenomena

- Spectral problem of L: **Eigenvalue (QNM) instabilities**

  [JIJ, Macedo, Al Sheikh 21; ...;]

$$Lv_n(x) = \omega_n v_n(x) \quad , \quad L^\dagger w_n = \overline{\omega}_n w_n \quad , \quad (L^t \alpha_n(x) = \omega_n \alpha_n(x))$$

- Source-less dynamics: **Non-modal transient growths**

  [JLJ 22, Boyanov, Destounis et al. 23, Carballo & Withers 24, Chen, Wu & Guo 24, Carballo, Pantelidou & Withers 25, Besson, Carballo, Pantelidou & Withers 25]

$$(\partial_\tau - iL)u(\tau, x) = 0$$

- Source-driven dynamics: **Pseudo-resonances**

  [JLJ 22, Boyanov, Destounis et al. 23]

$$(\partial_\tau - iL)u(\tau, x) = S(\tau, x)$$

Scheme

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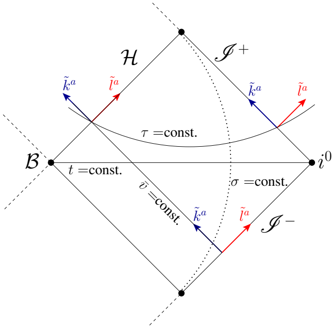
Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to scattering on BHs

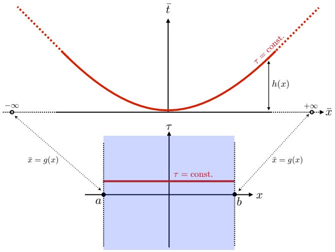
- Wave equation with purely outgoing boundary conditions.
- Outgoing BCs naturally imposed at $\mathscr{I}^+$.
- Outgoing BCs actually "incorporated" at $\mathscr{I}^+$:
  - Geometrically: null cones outgoing.
  - Analytically: BCs encoded into a singular operator, **"BCs as regularity conditions"**.
- **QNM eigenfunctions** not diverging at $x \to \infty$: actually **integrable**. Key to Hilbert space.



Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

Hyperboloidal approach to scattering on BHs: QNMs

- B. Schmidt [Schmidt 93; cf. also Friedman & Schutz 75]
- Analysis in the conformally compactified picture [Friedrich; Frauendiener,...]. Micro. Analysis [Vasy 13]
- Framework for BH perturbations [Zenginoglu 11].
- QNMs of asymp. AdS spacetimes [Warnick 15].
- QNM definition as operator eigenvalues [Bizoń...; Bizoń, Chmaj & Mach 20].
- Schwarzschild QNMs [Ansorg & Macedo 16]. (cf. also Reissner-Nordström [Macedo, JLJ, Ansorg 18]).
- "Gevrey" [Gajic & Warnick 20; Galkowski & Zworski 21].



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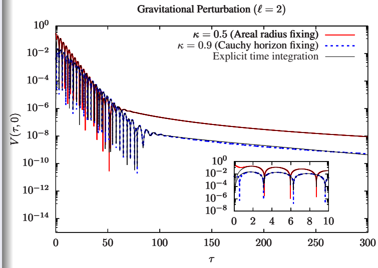
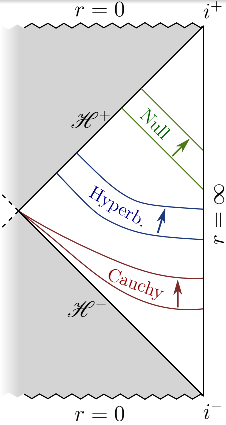


Hyperboloidal slices: geometric outgoing BCs at $\mathscr{I}^+$

BH QNM instability: Spacetime asymptotics

- Asymptotically flat [L. Al Sheikh Ph.D thesis 22] [JLJ, R. P. Macedo, L. Al Sheikh 21; E. Gasperin, JLJ 22; ...; K. Destounis et al 21; V. Boyanov et al 22; JLJ 22].
- Asymptotically de Sitter [S. Sarkar, M. Rahman, S. Chakraborty 23; JLJ, R. P. Macedo, L. Al Sheikh 21]
- Asymptotically Anti-de Sitter
  - Hyperboloidal slicing [D. Areán, D. García-Fariña, K. Landsteiner 23]
  - Null slicing [B. Cownden, C. Pantelidou, M. Zilhão 23].
  - Structural assessments [V. Boyanov et al 23].



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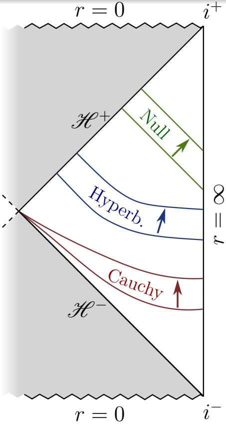

Wave problem in spherically symmetric asymptotically flat case

As starting point, consider the problem for a $\phi_{\ell m}$ mode in tortoise coordinates:

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \quad , \quad t \in]-\infty, \infty[\ , \ r^* \in]-\infty, \infty[$$

Hyperboloidal approach to BH QNM [Warnick 15, Ansorg & Macedo 16, ...]

Starting point: (scalar) wave equation in "tortoise" coordinates

On a stationary spacetime (with timelike Killing ∂_t):

$$\left(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial r_*^2} + V_\ell\right)\phi_{\ell m} = 0 \ ,$$

Dimensionless coordinates: $\bar{t} = t/\lambda$ and $\bar{x} = r_*/\lambda$ (and $\bar{V}_\ell = \lambda^2 V_\ell$),

Hyperboloidal approach [..., Zenginoğlu 08, 11,..., Macedo 24]

$$\left\{ \begin{array}{lcl} \bar{t} & = & \tau - h(x) \\ \bar{x} & = & f(x) \end{array} \right. \ .$$

- $h(x)$: implements the hyperboloidal slicing, i.e. $\tau = \mathrm{const.}$ is a horizon-penetrating hyperboloidal slice Σ_τ intersecting future $\mathscr{I}^+$.
- $f(x)$: spatial compactification between $\bar{x} \in [-\infty, \infty]$ to $[a, b]$.
- Timelike Killing: $\lambda\partial_t = \partial_{\bar{t}} = \partial_\tau$.

Hyperboloidal approach to BH QNM [Warnick 15, Ansorg & Macedo 16, ...]

First-order reduction: $\psi_{\ell m} = \partial_\tau \phi_{\ell m}$

$$\boxed{\partial_\tau u_{\ell m} = iL u_{\ell m}} \quad , \quad \text{with} \quad u_{\ell m} = \begin{pmatrix} \phi_{\ell m} \\ \psi_{\ell m} \end{pmatrix}$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

$$L_1 = \frac{1}{w(x)}\left(\partial_x\left(p(x)\partial_x\right) - q(x)\right) \qquad \text{(Sturm-Liouville operator)}$$

$$L_2 = \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)$$

with $w(x) = \dfrac{f'^2 - h'^2}{|f'|} > 0$, $p(x) = \dfrac{1}{|f'|}$, $q(x) = |f'|\, V_\ell$, $\gamma(x) = \dfrac{h'}{|f'|}$.

Hyperboloidal approach to BH QNM [Warnick 15, Ansorg & Macedo 16, ...]

Spectral problem

Taking Fourier transform, dropping (ℓ, m) (convention $u(\tau, x) \sim u(x)e^{i\omega\tau}$):

$$\boxed{L\, u_n = \omega_n\, u_n} \ .$$

where

$$\boxed{L = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 \end{array}\right)}$$

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L_2 &= \frac{1}{w(x)}\left(2\gamma(x)\partial_x + \partial_x\gamma(x)\right)
\end{aligned}
$$

Hyperboloidal approach: **No boundary conditions**

It holds $p(a) = p(b) = 0$, L_1 is “singular”: **BCs “in-built” in** L.

Hyperboloidal approach to BH QNM [Warnick 15, Ansorg & Macedo 16, ...]

A physically motivated scalar product: "energy (H^1) scalar product)

Natural scalar product (where $\tilde{V}_\ell := q(x) > 0$):

$$\langle u_1,u_2 \rangle_{_E} = \frac{1}{2}\int_a^b \Big(w(x)\bar{\psi}_1\psi_2 + p(x)\partial_x\bar{\phi}_1\partial_x\phi_2 + \tilde{V}_\ell\bar{\phi}_1\phi_2 \Big) dx \ ,$$

associted with the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||^2_{_E} = \langle u, u \rangle_{_E} = \int_{\Sigma_\tau} T_{ab}(\phi, \partial_\tau\phi) t^a n^b d\Sigma_\tau \ ,$$

Spectral problem of a non-selfadjoint operator

- Full operator L: not selfadjoint.
- L_2: dissipative term encoding the energy leaking at $\mathscr{I}^+$.
- L selfadjoint in the non-dissipative $L_2 = 0$ case.

Non-normal dynamics: $\boxed{\partial_\tau u_{\ell m} = i L u_{\ell m}}$

Hyperboloidal approach to BH QNM [Warnick 15, Ansorg & Macedo 16, ...]

Adjoint operator for the "energy scalar problem"

$$L^\dagger = \frac{1}{i}\left(\begin{array}{c|c} 0 & 1 \\ \hline L_1 & L_2 + L_2^\partial \end{array}\right)$$

where

$$L_2^\partial = 2\frac{\gamma}{w}\Big(\delta(x-a) - \delta(x-b)\Big)$$

Loss of "self-adjointness" happens at the boundaries (as expected)

Plan

Spectral instability in BH QNMs

BH QNMs as an proper eigenvalue problem [Warnick 15, Ansorg & Macedo 16, ...]

$$\boxed{L\, u_n = \omega_n\, u_n} \ .$$

The 'definition' versus the 'instability' problem

Different norms for two different questions: the key role of the norm $||\cdot||$

"Definition" versus "Instability" problem

- **Instability problem**: given a norm, assess spectral instability.
  For instance, "energy scalar product":

$$\langle u_1,u_2\rangle_{_E}=\frac{1}{2}\int_a^b\Big(w(x)\bar{\psi}_1\psi_2+p(x)\partial_x\bar{\phi}_1\partial_x\phi_2+\tilde{V}_\ell\bar{\phi}_1\phi_2\Big)dx \ ,$$

  associated with the "total energy" of ϕ on Σ_t, defining the **"energy norm"**

$$||u||^2_{_E}=\langle u,u\rangle_{_E}=\int_{\Sigma_\tau}T_{ab}(\phi,\partial_\tau\phi)t^a n^b d\Sigma_\tau \ ,$$

Spectral instability in BH QNMs

BH QNMs as an proper eigenvalue problem [Warnick 15, Ansorg & Macedo 16, ...]

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The 'definition' versus the 'instability' problem

Different norms for two different questions: the key role of the norm $||\cdot||$

"Definition" versus "Instability" problem

- **Definition problem**: given an operator, search norm to control eigenvalue instability. In AdS, Sobolev H^p-norms [Warnick 15] from:

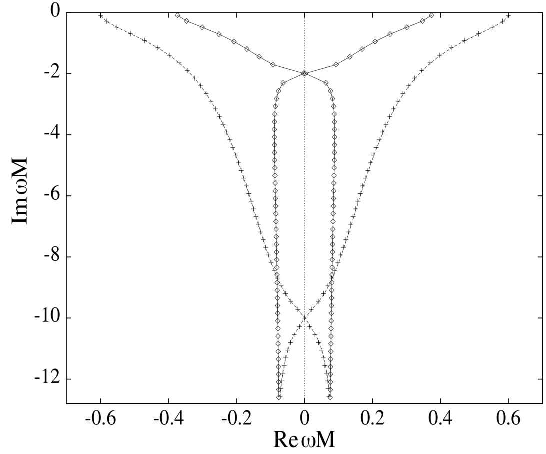
$$\langle u_1, u_2\rangle_{_{H^p}} = \Big\langle \begin{pmatrix}\phi_1\\ \psi_1\end{pmatrix}, \begin{pmatrix}\phi_2\\ \psi_2\end{pmatrix} \Big\rangle_{_{H^p}} = \sum_{j=0}^{p} \Big\langle \begin{pmatrix}\partial_x^j\phi_1\\ \partial_x^j\psi_1\end{pmatrix}, \begin{pmatrix}\partial_x^j\phi_2\\ \partial_x^j\psi_2\end{pmatrix} \Big\rangle_{_E} ,$$

  leading to

$$\Big|\Big|\begin{pmatrix}\phi\\ \psi\end{pmatrix}\Big|\Big|^2_{_{H^p}} := \sum_{j=0}^{p} \Big|\Big|\begin{pmatrix}\partial_x^j\phi\\ \partial_x^j\psi\end{pmatrix}\Big|\Big|^2_{_E}$$

Spectral instability in BH QNMs

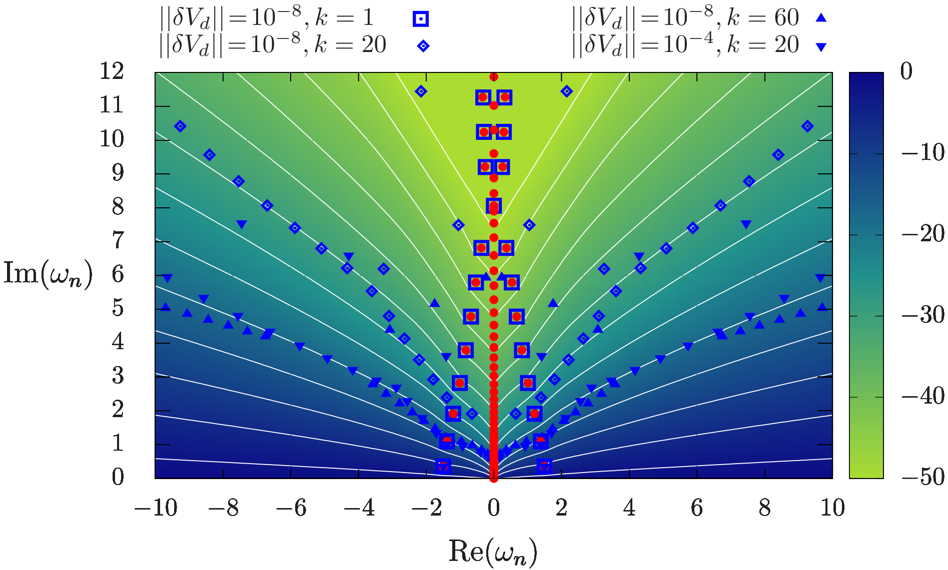
Schwarzschild gravitational QNMs



Schwarzschild QNMs ($\ell = 2$ diamonds, $\ell = 3$ crosses) [e.g. Kokkotas & Schmidt 99; ...]

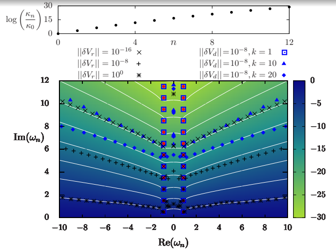
QNM frequencies ω_n and asymptotics in the complex plane

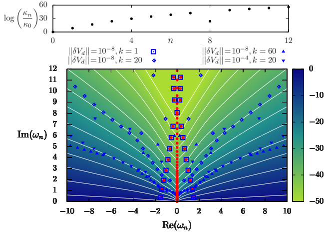
Spectral instability in BH QNMs

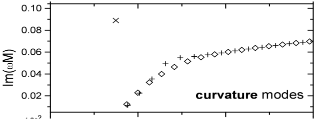


QNMs in Schwarzschild and in perturbed Schwarzschild [JLJ, Macedo, Al Sheikh 21]

Spectral instability in BH QNMs



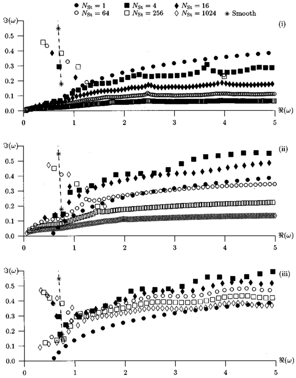




Black Hole and Neutron Star QNMs

Comparison with:

- Nollert's high-frequency Schwarzschild perturbations.
- Nollert's remark on Neutron Stars (w-modes) curvature QNMs.



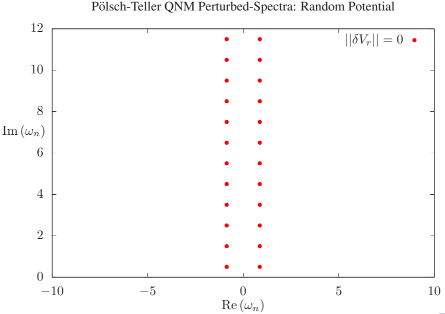
Spectral instability in BH QNMs

Pöschl-Teller potential [JLJ, Macedo & Al Sheikh 21] (toy-model in [Bizoń, Chmaj & Mach 20])

$$V(x) = V_o \operatorname{sech}^2(x)$$

Particularly simple form (scalar field in de Sitter, $m^2 = V_o$ [Bizoń, Chmaj & Mach 20])

- Integrable potential (QNM completeness [Beyer 99] with $m^2 = V_o$!).
- QNM frequencies: $\omega_n^\pm = \pm\frac{\sqrt{3}}{2} + i\left(n + \frac{1}{2}\right)$
- Here, eigenfunctions are Jacobi polynomials: $\phi_n(\bar{x}) = P_n^{(s_n^\pm, s_n^\pm)}(\bar{x})$.



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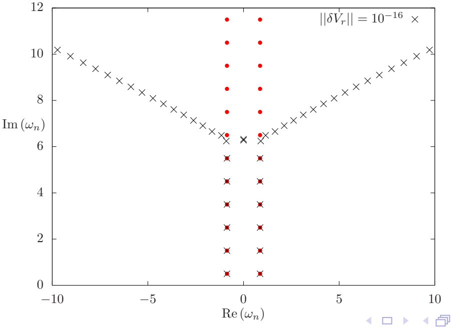

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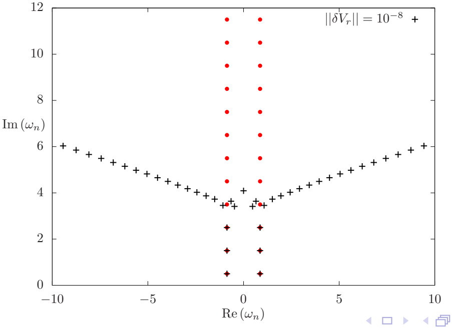

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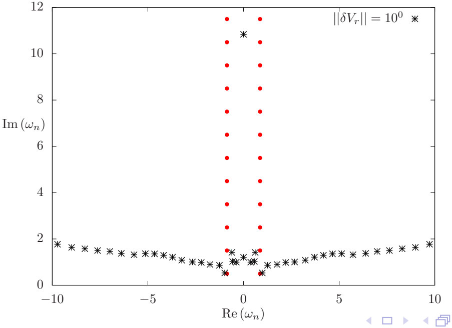

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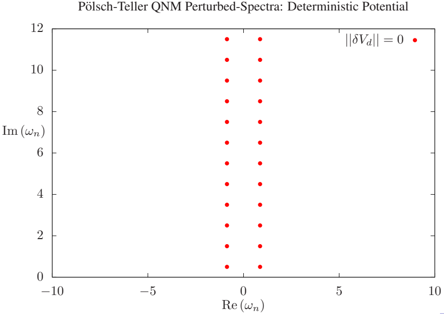

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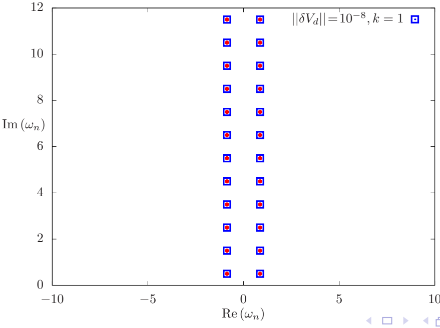

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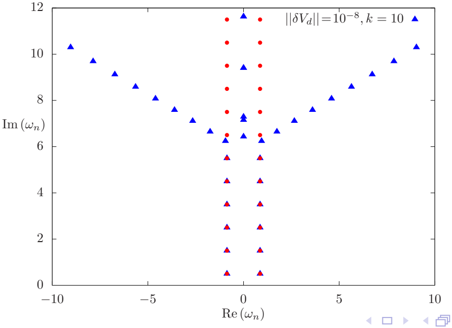

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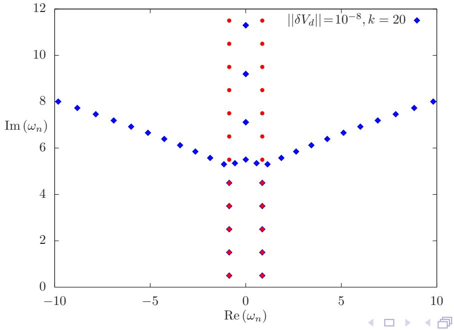
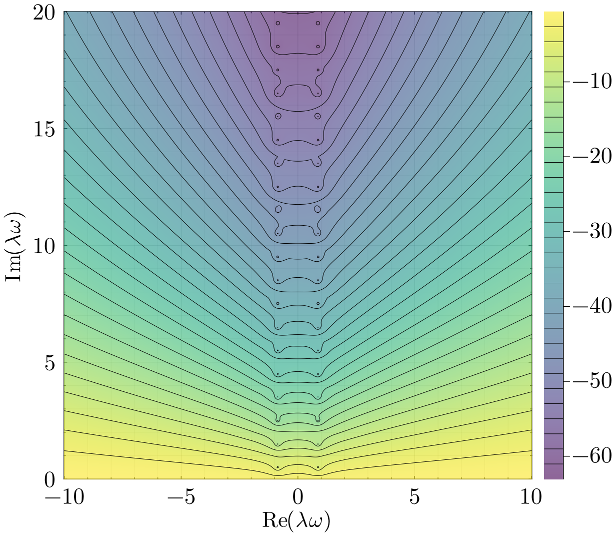
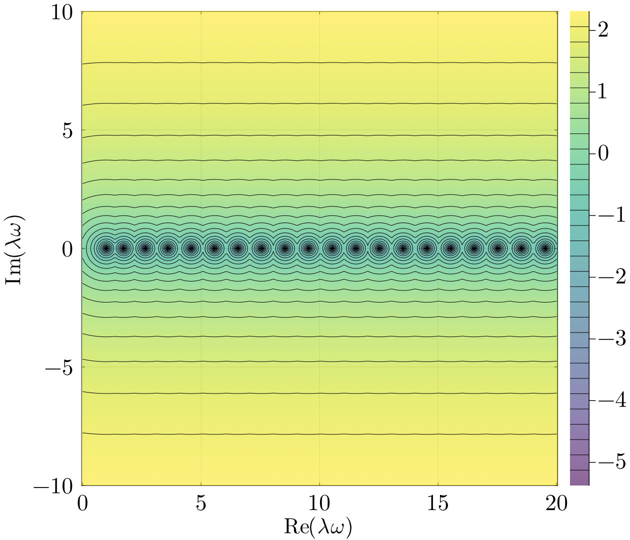


Spectral instability in BH QNMs



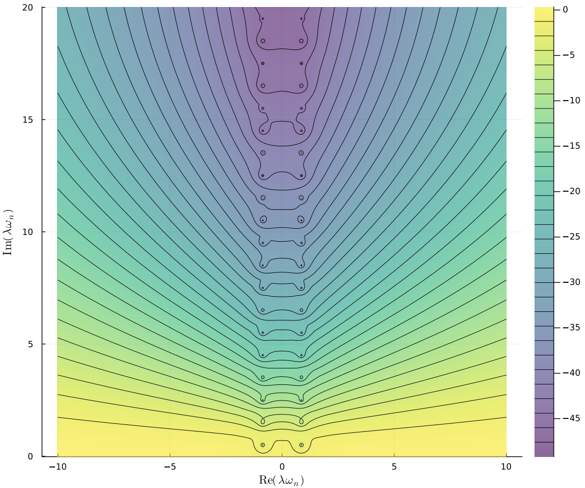
Pöschl-Teller, energy norm

Spectral instability in BH QNMs



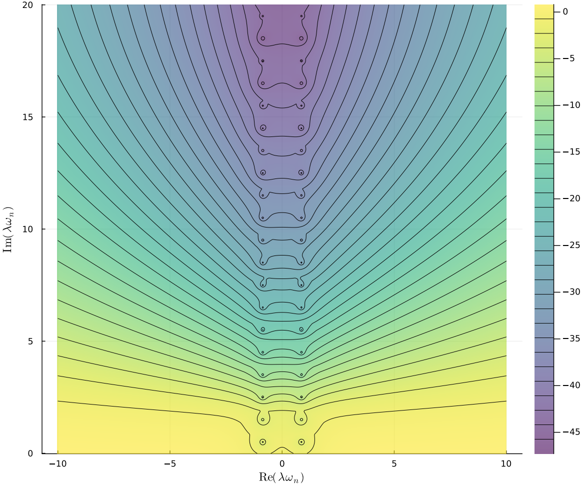
Pöschl-Teller, energy norm: $L_2 = 0$

Spectral instability in BH QNMs



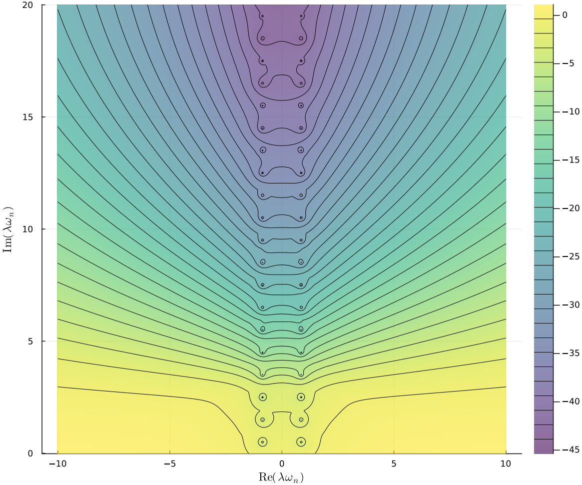
Pöschl-Teller, H^1 case

Spectral instability in BH QNMs



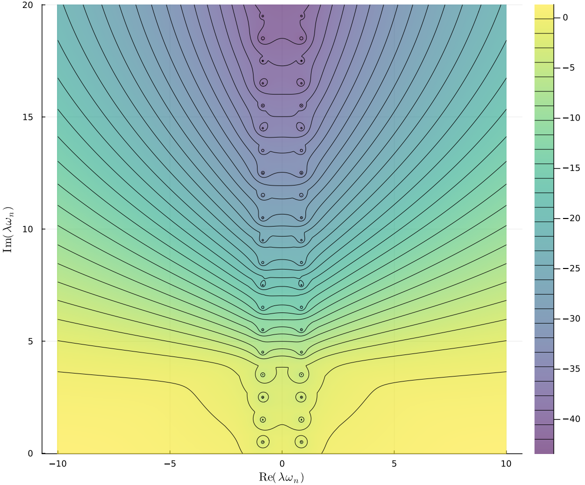
Pöschl-Teller, H^2 case

Spectral instability in BH QNMs



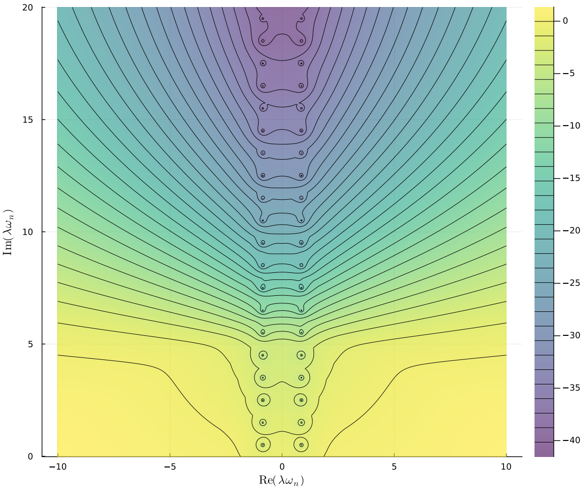
Pöschl-Teller, H^3 case

Spectral instability in BH QNMs



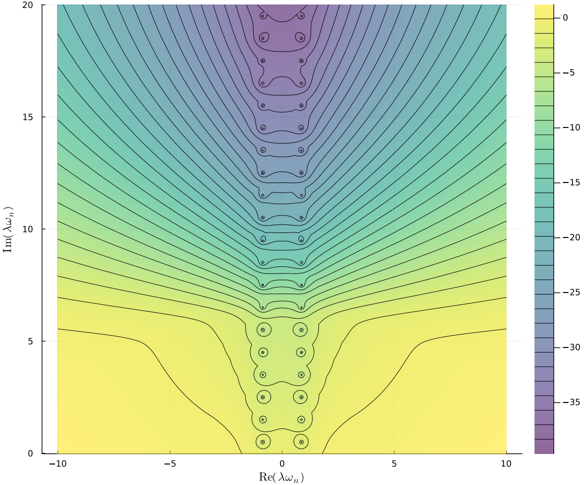
Pöschl-Teller, H^4 case

Spectral instability in BH QNMs



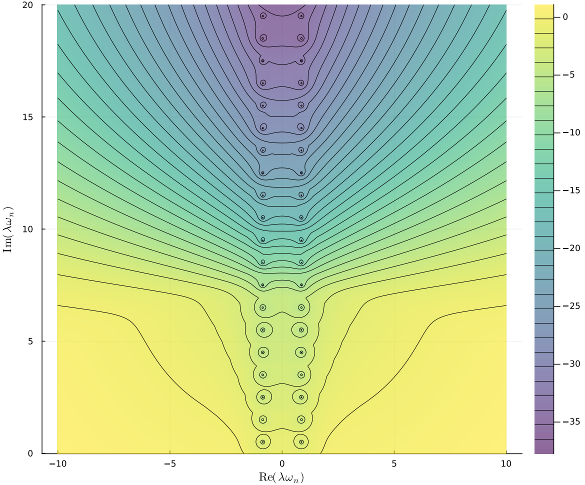
Pöschl-Teller, H^5 case

Spectral instability in BH QNMs



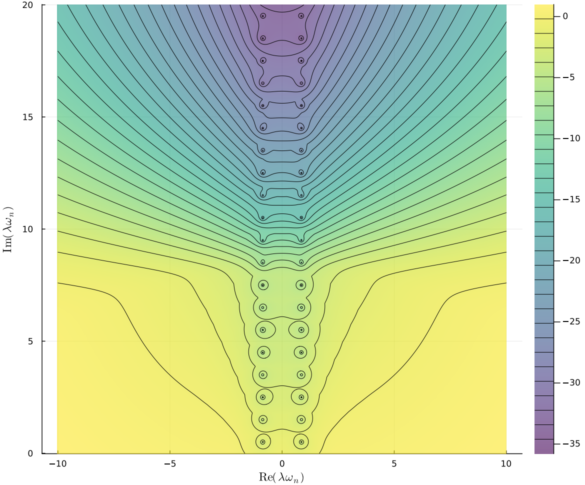
Pöschl-Teller, H^6 case

Spectral instability in BH QNMs



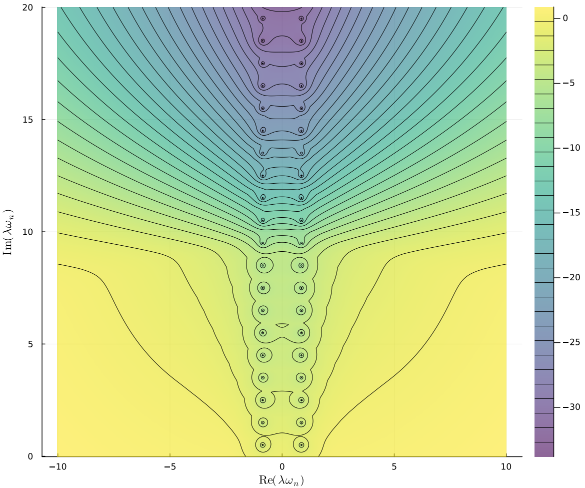
Pöschl-Teller, H^7 case

Spectral instability in BH QNMs



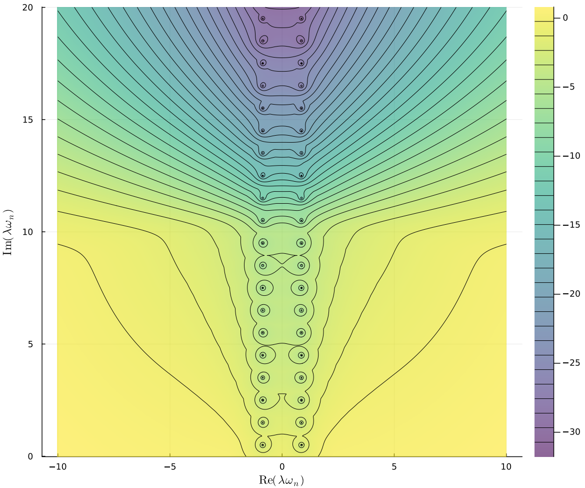
Pöschl-Teller, H^8 case

Spectral instability in BH QNMs



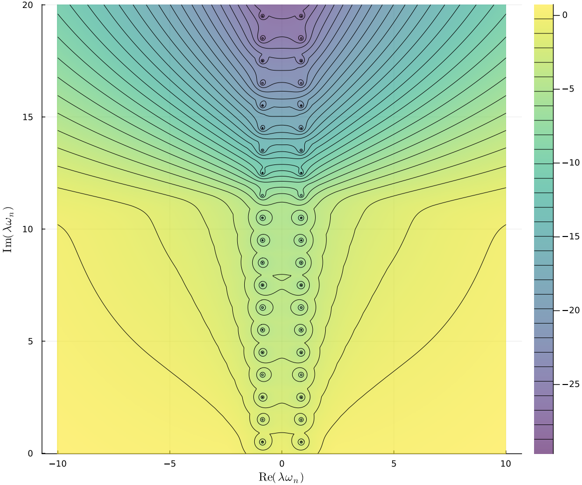
Pöschl-Teller, H^9 case

Spectral instability in BH QNMs



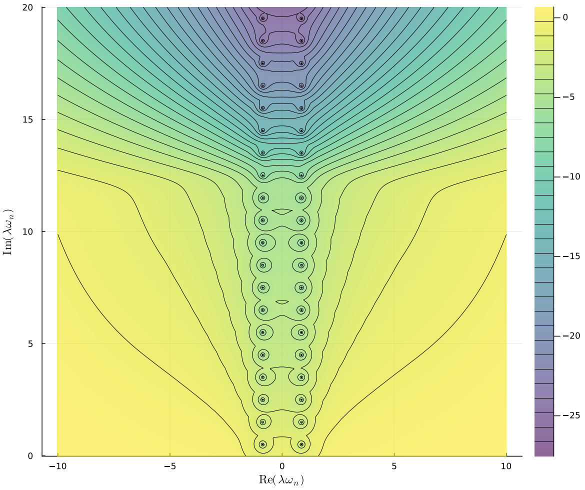
Pöschl-Teller, H^{10} case

Spectral instability in BH QNMs



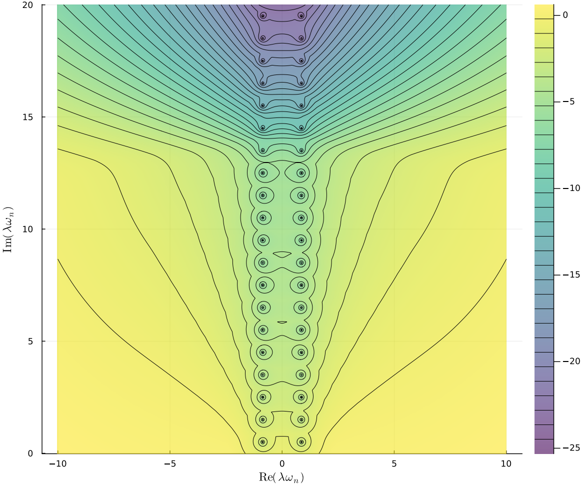
Pöschl-Teller, H^{11} case

Spectral instability in BH QNMs



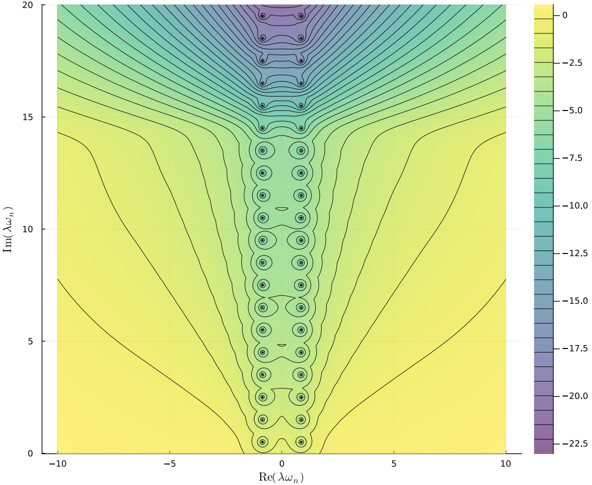
Pöschl-Teller, H^{12} case

Spectral instability in BH QNMs



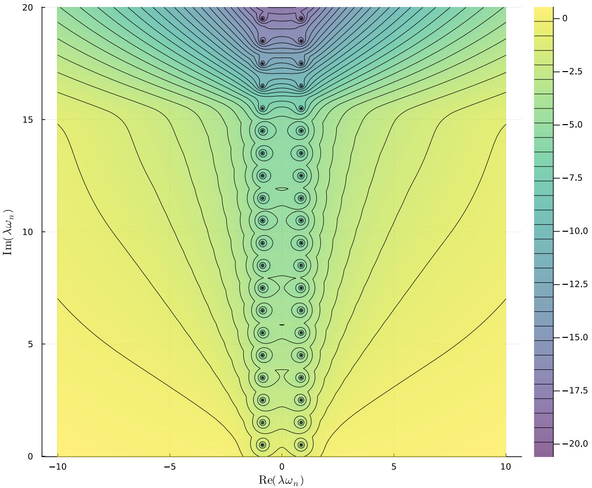
Pöschl-Teller, H^{13} case

Spectral instability in BH QNMs



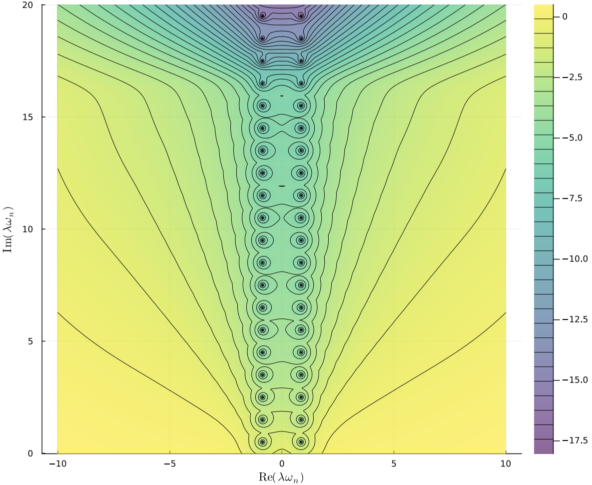
Pöschl-Teller, H^{14} case

Spectral instability in BH QNMs



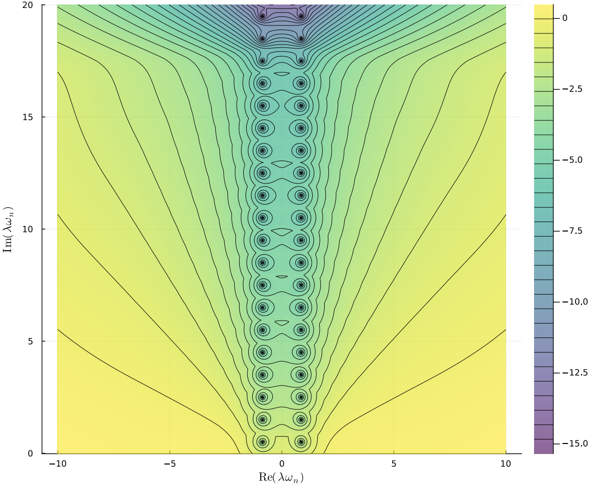
Pöschl-Teller, H^{15} case

Spectral instability in BH QNMs



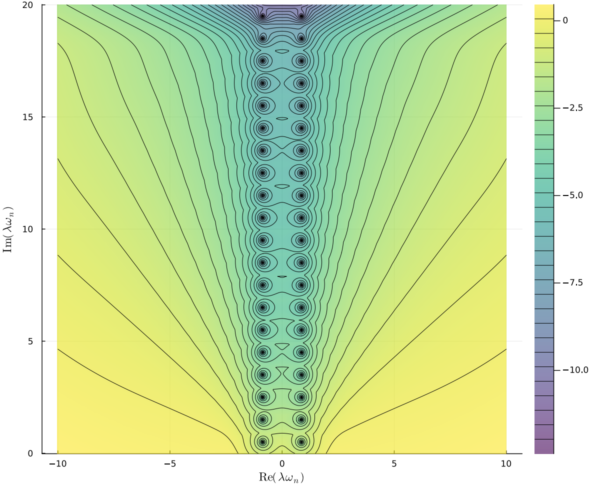
Pöschl-Teller, H^{16} case

Spectral instability in BH QNMs



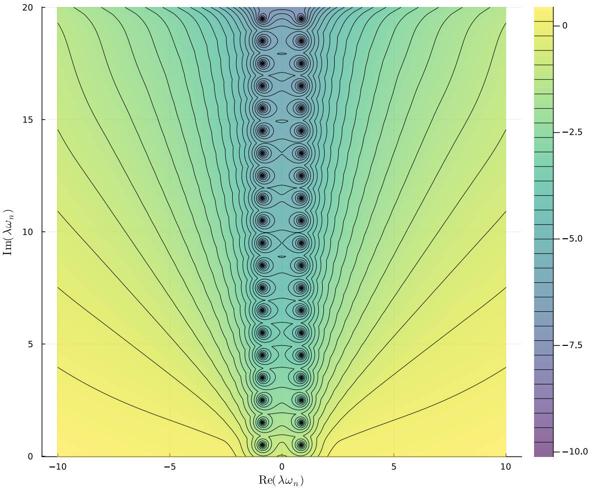
Pöschl-Teller, H^{17} case

Spectral instability in BH QNMs



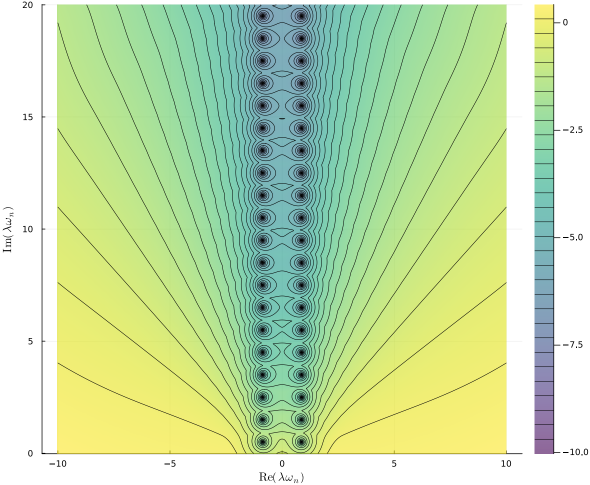
Pöschl-Teller, H^{18} case

Spectral instability in BH QNMs



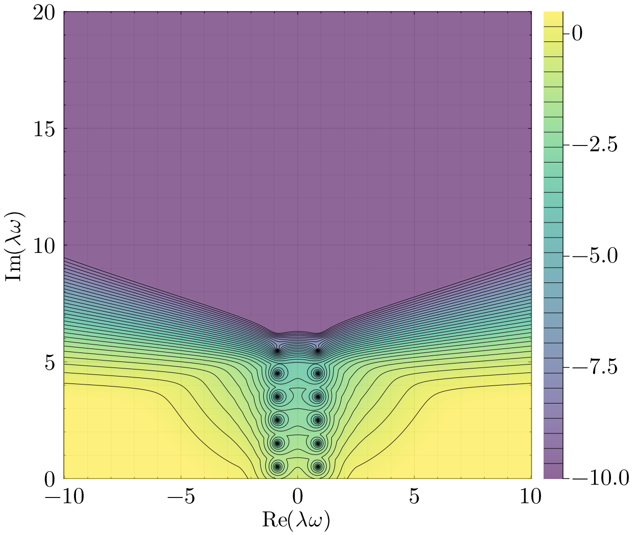
Pöschl-Teller, H^{19} case

Spectral instability in BH QNMs



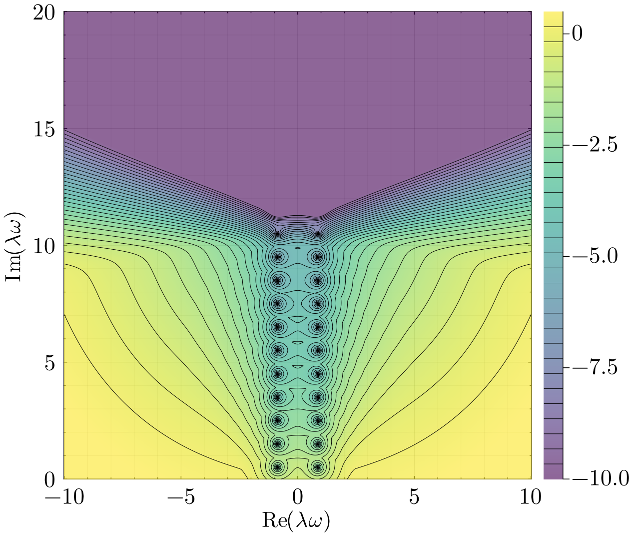
Pöschl-Teller, H^{20} case

Spectral instability in BH QNMs



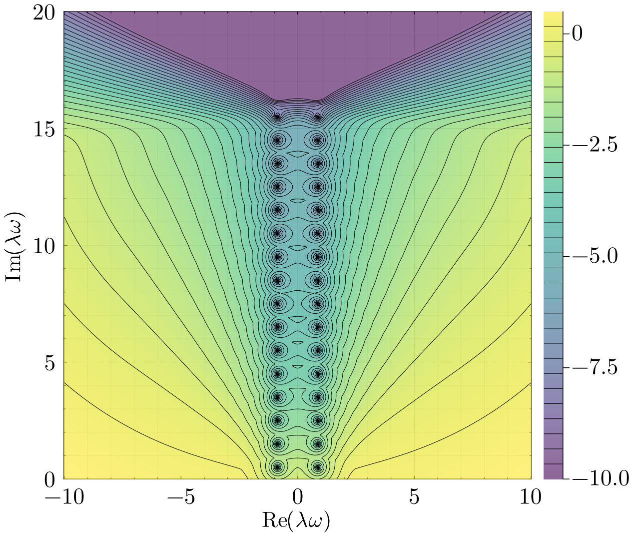
Pöschl-Teller, H^5 case

Spectral instability in BH QNMs



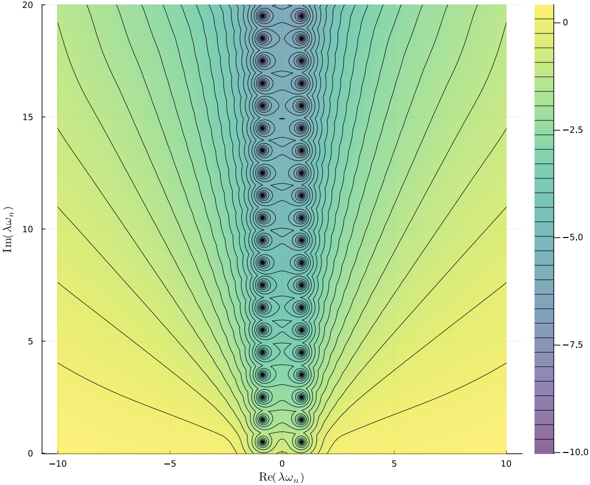
Pöschl-Teller, H^{10} case

Spectral instability in BH QNMs



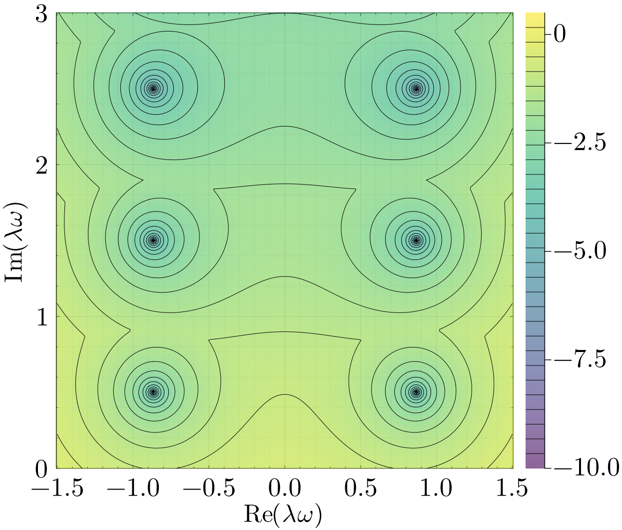
Pöschl-Teller, H^{15} case

Spectral instability in BH QNMs



Pöschl-Teller, H^{20} case

Spectral instability in BH QNMs



Pöschl-Teller, H^{20} case

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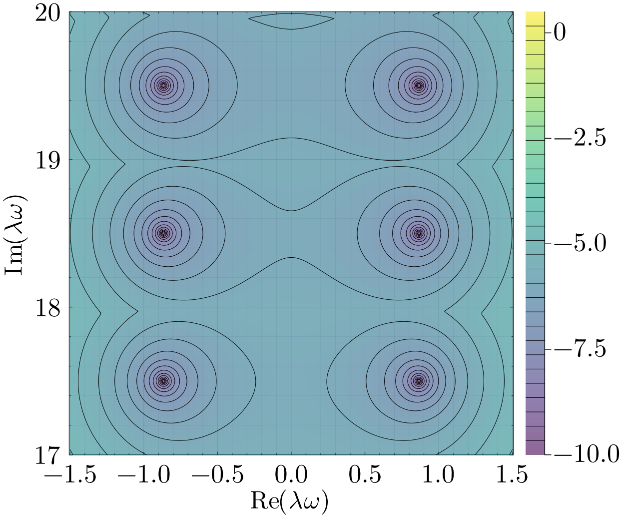


Pöschl-Teller, H^{20} case

Plan

1. The general problem: linear "non-normal" wave equation
2. Brief overview of non-normal operators and non-modal analysis
   - Spectral instability
   - Non-modal transient growths
   - Pseudo-resonances
   - Some elements of non-modal analysis
3. A gravitational case: hyperboloidal approach to scattering on black holes
   - BH QNM instability
   - "Free" evolutions on BHs and non-modal transient growths
   - "Driven" evolutions on BHs (and pseudo-resonances?)
4. Conclusions and Perspectives

Keldysh QNM decomposition [Besson & JLJ 25]

Dual spectral problems: vectors and covectors (reminder)

$$L v_n = \omega_n v_n \quad , \quad L^t \alpha_n = \omega_n \alpha_n \quad , \quad v_n \in \mathcal{H}, \alpha_n \in \mathcal{H}^*$$

If a scalar product available: spectral and adjoint spectral problem

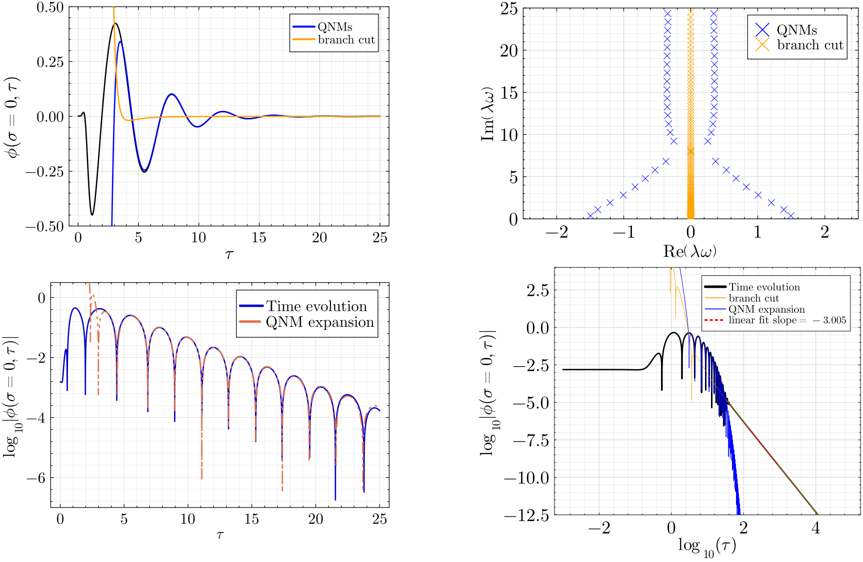
$$L \hat{v}_n = \omega_n \hat{v}_n \quad , \quad L^\dagger \hat{w}_n = \overline{\omega}_n \hat{w}_n \quad , \quad \hat{v}_n, \hat{w}_n \in \mathcal{H}$$

Keldysh expansion (reminder) [Besson & JLJ 25]

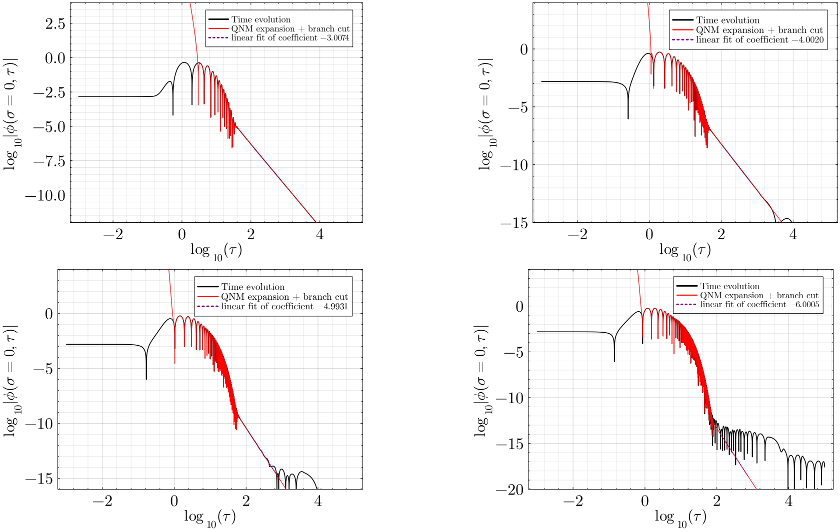
$$\begin{aligned} u(\tau, x) &= \sum_{n=0}^{N_{\mathrm{QNM}}} e^{i\omega_n \tau} \langle \alpha_n, u_0 \rangle v_n(x) + E_{N_{\mathrm{QNM}}}(\tau; u_0) \\ &= \sum_{n=0}^{N_{\mathrm{QNM}}} e^{i\omega_n \tau} \kappa_n \langle \hat{w}_n, u_0 \rangle_G \hat{v}_n(x) + E_{N_{\mathrm{QNM}}}(\tau; u_0) \end{aligned}$$

with $\quad ||E_{N_{\mathrm{QNM}}}(\tau; u_0)|| \leq C(N_{\mathrm{QNM}}, L) e^{-a_{N_{\mathrm{QNM}}} \tau} ||u_0|| \ ,$

Keldysh QNM decomposition [Besson & JLJ 25]

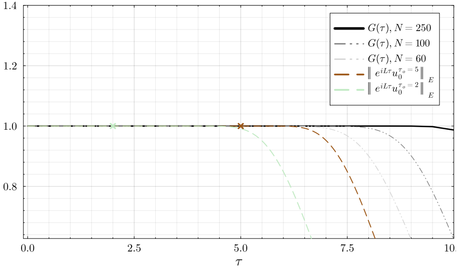


Keldysh QNM decomposition [Besson & JLJ 25]

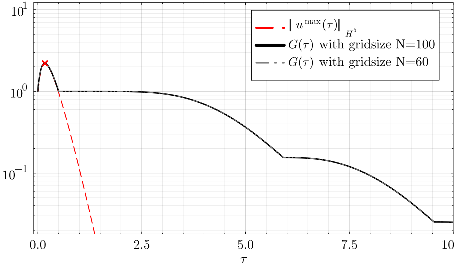


Non-normal transient growths and distributions [Besson & JLJ 25]

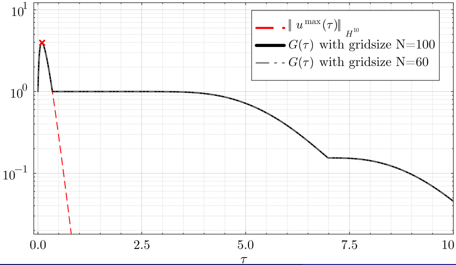
Energy norm



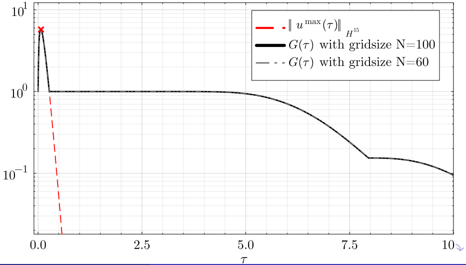
Sobolev H^5-norm



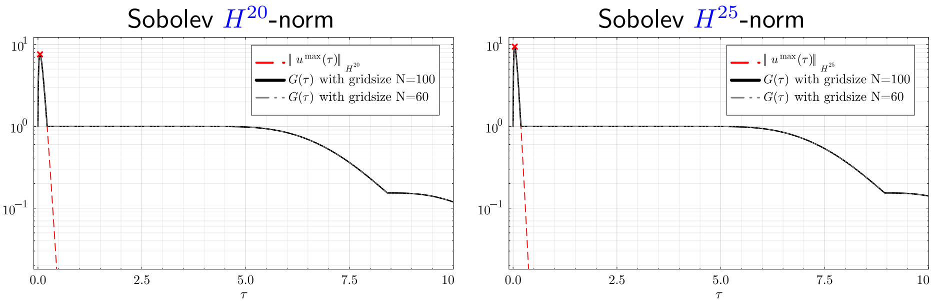
Sobolev H^{10}-norm



Sobolev H^{15}-norm



Non-normal transient growths and distributions [Besson & JLJ 25]



H^p growth transients: distributions at large p

In the limit $p \to \infty$:

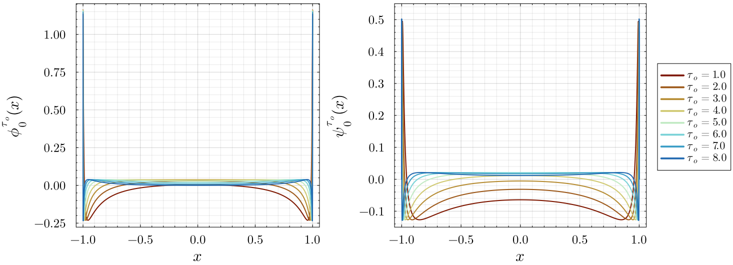
$$\tau_{\max} \sim \frac{1}{p} \quad , \quad G_{\max} \sim p$$

$$\boxed{\lim_{p\to\infty} G(\tau) \sim \delta(\tau)}$$

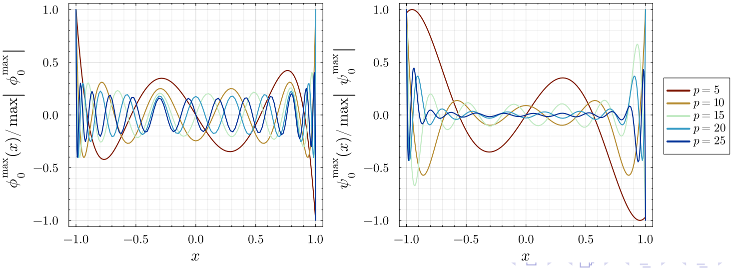
Distributional (in time) 'impulsive disturbance': key in "response function" in linear response theory.

Optimal initial data

$u_0^{\tau_o}(x)$ as a function of x (energy norm)

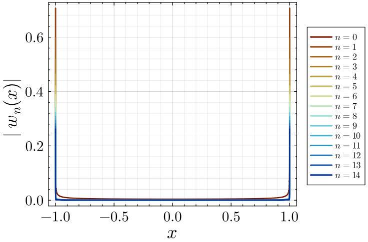


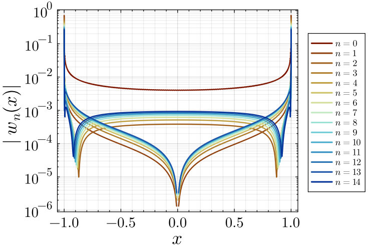
$u_0^{\max}(x)$ as a function of x for different H^p norms



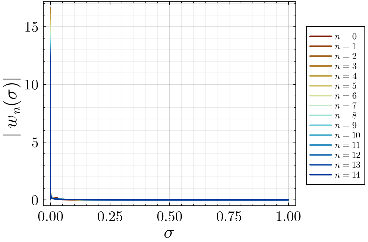
Co-modes $\hat{w}_n$ of $L^\dagger$: distributions peaked at the boundary

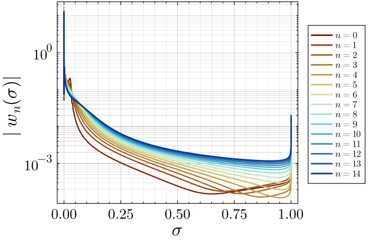
eigenfunctions $\hat{w}_n$ of $L^\dagger$: Pöschl-Teller case





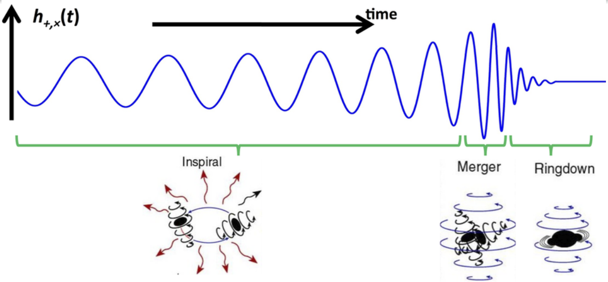
eigenfunctions $\hat{w}_n$ of $L^\dagger$: Schwarzschild case



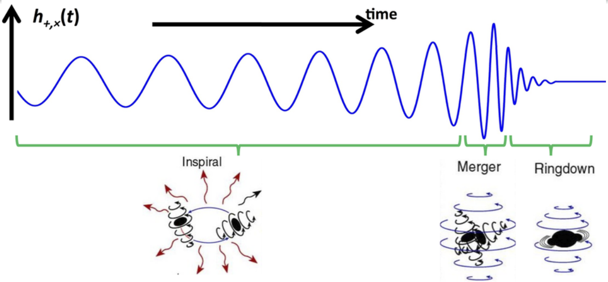


Plan

BBH dynamics: A "Wave-Mean Flow" approach



BBH dynamics: A "Wave-Mean Flow" approach



Non-linear dispersive hydrodynamics effective picture: scattering on solitons

Effective separation of slow degrees of freedom $u(t, x)$. In a "sketchy" manner:

$$\begin{cases} \left(-\Box + V_{\text{even,odd}}(t, x; u)\right) \Psi_{\text{even,odd}} = S_{\text{even,odd}}(t, x; u) \\ \partial_t u = F(t, x; u, u_x, u_{xx}, \ldots) \end{cases}$$

[note the affinity with De Amicis, Cannizzaro, Carullo & Sberna 25, cf. L. Sberna]

- Wave (Ψ): **non-normal linear wave dynamics (fast DoFs)**.
- Mean flow (u): "integrable" background dynamics (slow DoFs).

BBH dynamics: A "Wave-Mean Flow" approach

Bottom-up asymptotic hierarchy to BBH merger dynamics: a "f-Airy tale"

Asymptotic BBH Model	Mathematical/Physical Framework	Key Structures/Mechanisms
Fold-caustic model	Geometric Optics Catastrophe (singularity) Theory	Arnol'd-Thom's Theorem Classification of Stable Caustics
Airy function model	Fresnel's Diffraction Semiclassical Theory asymptotic ODE theory	**Universal Diffraction Patterns in Caustics** linear ODE turning points
Painlevé-II model	Painlevé Transcendents and Integrability Self-force calculations and EMRBs	Painlevé property Non-linear Turning Points
KdV-like model (**Wave-Mean Flow**)	Inverse Scattering Transform and Integrability Dispersive Non-linear PDEs Critical Phenomena in Dispersive PDEs	Painlevé test, **Lax pairs** Darboux transformations **Scatt. on Solitons**, Soliton Resolution **Universal Wave Patterns** Dubrovin's Conjecture
Propagation models on (anti)-Self-Dual backgrounds	Ward's Conjecture and Integrability Twistorial techniques	(anti-)Self-Dual DoF Scattering on Instantons, Tunneling Penrose Transform, 'Twistor' BBH data

BBH dynamics: A "Wave-Mean Flow" approach

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The hierarchical BBH program: "Wittgenstein's ladder" [JLJ, Krishnan & Sopuerta 23]

Resulting proposal:
"Wave-Mean Flow" approach with "fast" degrees of freedom **"linearly"** propagating/interacting on a "slow" degrees of freedom **integrable** background.

BBH dynamics: A "Wave-Mean Flow" approach

Bottom-up asymptotic hierarchy to BBH merger dynamics: a "f-Airy tale"

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Ablowitz and Segur (1981): on Integrability and Linearity

"Certain nonlinear problems have a surprisingly simple underlying structure, and can be solved by essentially linear methods".

Application to Simplicity and Universality in BBH dynamics?

BBH dynamics: A "Wave-Mean Flow" approach

"Top-down" separation of (slow) background and (fast) dynamics?

Full (Conformal) Einstein equations

[Friedrich...; Frauendiener...; Valiente-Kroon; here: Frauendiener, Stevens & Thwala 25]

Semi-linear system, with "Wave-Mean Flow" structure:
"Subsystem 1" + "Subsystem 2".

Subsystem 1, "Slow" degrees of freedom: transport equations

$$e_a(c_b^\mu) - e_b(c_a^\mu) = \hat{\Gamma}_{ab}{}^c c_c^\mu - \hat{\Gamma}_{ba}{}^c c_c^\mu,$$

$$e_a(\hat{\Gamma}_{bc}{}^d) - e_b(\hat{\Gamma}_{ac}{}^d) = (\hat{\Gamma}_{ab}{}^e - \hat{\Gamma}_{ba}{}^e)\hat{\Gamma}_{ec}{}^d - \hat{\Gamma}_{bc}{}^e\hat{\Gamma}_{ae}{}^d + \hat{\Gamma}_{ac}{}^e\hat{\Gamma}_{be}{}^d$$

$$+\Theta K_{abc}{}^d - 2\eta_{c[a}\hat{P}_{b]}{}^d + 2\delta_{[a}{}^d\hat{P}_{b]c} - 2\hat{P}_{[ab]}\delta_c{}^d,$$

$$\hat{\nabla}_a \hat{P}_{bc} - \hat{\nabla}_b \hat{P}_{ac} = b_e K_{abc}{}^e,$$

Subsystem 2, "Fast" degrees of freedom: symmetric hyperbolic system

$$\hat{\nabla}_e K_{abc}{}^e = b_e K_{abc}{}^e,$$

Scheme

1. The general problem: linear "non-normal" wave equation

2. Brief overview of non-normal operators and non-modal analysis
   - Spectral instability
   - Non-modal transient growths
   - Pseudo-resonances
   - Some elements of non-modal analysis

3. A gravitational case: hyperboloidal approach to scattering on black holes
   - BH QNM instability
   - "Free" evolutions on BHs and non-modal transient growths
   - "Driven" evolutions on BHs (and pseudo-resonances?)

4. Conclusions and Perspectives

Perspectives

Conclusions

- **Nonselfajoint (non-normal)** early/intermediate **dynamics** are **not captured by spectrum**: rather **"non-modal analysis"**.
  It requires to cast the problem in a proper **Hilbert (Banach) space.**
- **Characteristic "non-normal effects"**: eigenvalue (QNM) instability, growth transients, pseudo-resonances.
- **Application in GR**: BH QNM instability, (RN superradiance [Carballo et al. 25], low-regularity) transients, pseudo-resonances (ECO bootstrap instability?)

Perspectives

- **Non-normal dynamics/non-modal approach to BBH merger-ringdown**: Transients? Pseudo-resonances? BH QNM instability?
- **Non-normal dynamics** tools in the (hyperboloidal) **"wave" dynamics** in the **"wave-mean flow"** approach to strong gravity **(BBH) dynamics**.
- **Application to other gravity ("dissipative") scenarios**: cosmological settings, transition to **turbulence in gravity** [Lehner], near-horizon geometries, fundamental dissipation [Pérez, Sudarsky], ...