Quantum gravity and asymptotic symmetries: charge ambiguities and cocycles

Simone Speziale

TUG '25 - Paris, 15-10-25

based on *General covariance and boundary symmetry algebras* with Antoine Rignon-Bret

Centre de Physique Théorique

Symmetries in General Relativity

Symmetries are one of the most useful concepts in the whole of physics
they map solutions to new solutions, relate solutions and observers,
can be associated to conservation laws and integrability…

In general relativity, the situation is subtle: the only symmetry of the theory is *diffeomorphism invariance* (namely invariance under general coordinate transformations),
and this is a **gauge symmetry**

- maps solutions to physically indistinguishable solutions,
- is a degenerate direction of the symplectic 2-form
- has vanishing Noether charges

In special circumstances, diffeomorphisms can be turned into **physical symmetries:**

- if there are *isometries* (only for special solutions admitting Killing vectors)
- if there are *boundaries* (and specific boundary conditions)

Isometries as special diffeomorphisms

To highlight how isometries emerge as a special case of diffeomorphism invariance, let us consider a matter Lagrangian $\mathcal{L}_{\rm M}(\psi, g)$

If the metric is also a dynamical field, like in a general covariant theory,

$$\delta\mathcal{L}_{\rm M} = \frac{\delta\mathcal{L}_{\rm M}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}_{\rm M}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \partial_\mu\tilde{\theta}^\mu(\delta)$$

Isometries as special diffeomorphisms

To highlight how isometries emerge as a special case of diffeomorphism invariance, let us consider a matter Lagrangian $\mathcal{L}_{\rm M}(\psi, g)$

If the metric is also a dynamical field, like in a general covariant theory,

$$\delta\mathcal{L}_{\rm M} = \frac{\delta\mathcal{L}_{\rm M}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}_{\rm M}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \partial_\mu\tilde{\theta}^\mu(\delta)$$

Then specializing the variation to be a diffeomorphism:

$$\begin{aligned}\delta_\xi\mathcal{L}_{\rm M} &= \frac{\delta\mathcal{L}_{\rm M}}{\delta\psi}\delta_\xi\psi + \frac{\delta\mathcal{L}_{\rm M}}{\delta g_{\mu\nu}}\delta_\xi g_{\mu\nu} + \partial_\mu\tilde{\theta}^\mu(\delta_\xi) \\ &= \frac{\delta\mathcal{L}_{\rm M}}{\delta\psi}\pounds_\xi\psi + \frac{\delta\mathcal{L}_{\rm M}}{\delta g_{\mu\nu}}\pounds_\xi g_{\mu\nu} + \partial_\mu\tilde{\theta}^\mu(\delta_\xi) \\ &= \pounds_\xi\mathcal{L}_{\rm M} + \partial_\mu\tilde{\theta}^\mu = \partial_\mu(\xi^\mu\mathcal{L}_{\rm M} + \tilde{\theta}^\mu)\end{aligned}$$

⇨ **every diffeomorphism is a symmetry**

(And further analysis shows that it is a gauge symmetry)

Isometries as special diffeomorphisms

To highlight how isometries emerge as a special case of diffeomorphism invariance, let us consider a matter Lagrangian $\mathcal{L}_{\mathrm{M}}(\psi, g)$

If the metric is not a dynamical field, like in special relativity,

$$\delta \mathcal{L}_{\mathrm{M}} = \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \psi}\delta\psi + \cancel{\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g_{\mu\nu}}}\delta g_{\mu\nu} + \partial_\mu \tilde{\theta}^\mu(\delta)$$

Isometries as special diffeomorphisms

To highlight how isometries emerge as a special case of diffeomorphism invariance, let us consider a matter Lagrangian $\mathcal{L}_{\mathrm{M}}(\psi, g)$

If the metric is not a dynamical field, like in special relativity,

$$\delta \mathcal{L}_{\mathrm{M}} = \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \psi} \delta \psi + \cancel{\frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g_{\mu\nu}}} \delta g_{\mu\nu} + \partial_\mu \tilde{\theta}^\mu(\delta)$$

Then specializing the variation to be a diffeomorphism:

$$\begin{aligned} \delta_\xi \mathcal{L}_{\mathrm{M}} &= \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \psi} \delta_\xi \psi + \partial_\mu \tilde{\theta}^\mu(\delta_\xi) \\ &= \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \psi} \pounds_\xi \psi + \partial_\mu \tilde{\theta}^\mu(\delta_\xi) \\ &= \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta \psi} \pounds_\xi \psi + \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g_{\mu\nu}} \pounds_\xi g_{\mu\nu} - \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g_{\mu\nu}} \pounds_\xi g_{\mu\nu} + \partial_\mu \tilde{\theta}^\mu \\ &= \partial_\mu(\xi^\mu \mathcal{L}_{\mathrm{M}} + \tilde{\theta}^\mu) - \frac{\delta \mathcal{L}_{\mathrm{M}}}{\delta g_{\mu\nu}} \pounds_\xi g_{\mu\nu}. \end{aligned}$$

⇨ **only isometries are symmetries**

(And further analysis shows that they are a proper, physical symmetry)

Isometries as special diffeomorphisms

To highlight how isometries emerge as a special case of diffeomorphism invariance, let us consider a matter Lagrangian $\mathcal{L}_{\rm M}(\psi, g)$

If the metric is not a dynamical field, like in special relativity,

$$\delta\mathcal{L}_{\rm M} = \frac{\delta\mathcal{L}_{\rm M}}{\delta\psi}\delta\psi + \frac{\delta\mathcal{L}_{\rm M}}{\delta g_{\mu\nu}}\delta g_{\mu\nu} + \partial_\mu\tilde\theta^\mu(\delta)$$

Then speci

to be a diff

arXiv > gr-qc > arXiv:2508.21817

General Relativity and Quantum Cosmology

[Submitted on 29 Aug 2025 (v1), last revised 9 Oct 2025 (this version, v2)]

An Introduction to Gravitational Wave Theory

Simone Speziale, Danièle A. Steer

Introduction to the theoretical foundations of gravitational waves: from general relativity notes prepared for the MaNiTou summer school on gravitational waves. Draft chapter for published by ISTE.

$$\pounds_\xi g_{\mu\nu} + \partial_\mu\tilde\theta^\mu$$

⇨ **only iso**

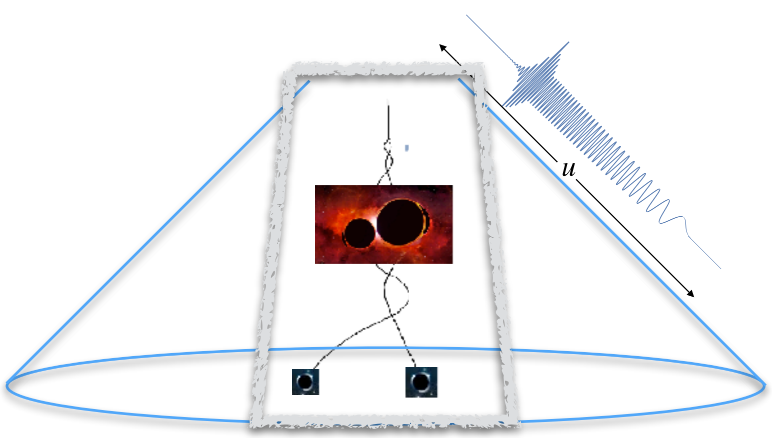
(And further analysis shows that they are a proper, physical symmetry)

Boundaries in General Relativity

According to general relativity, the gravitational force cannot be screened, nor confined

Therefore there are no boundaries in the same sense as in electromagnetism, e.g. the plates of a capacitor

There is however a meaningful notion of boundaries as hypersurfaces where specific boundary conditions are imposed, be it for mathematical or physical reasons



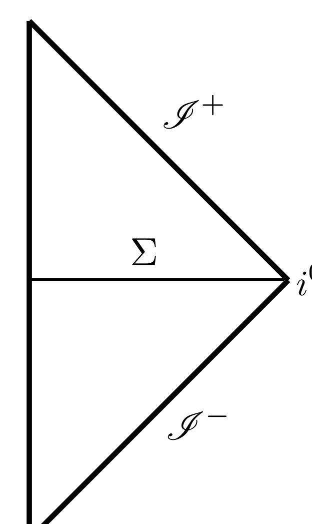
For example when *modelling isolated systems*, we assume a flat background metric asymptotically far away from the sources

⇨ ***asymptotic boundary conditions***

These asymptotic boundary conditions are preserved by a set of residual diffeomorphisms at the boundary: **asymptotic symmetries** (E.g. Poincare at spatial infinity, BMS at null infinity)

These residual diffeos are proper symmetries in every sense:

- maps solutions to physically distinguishable solutions,
- are non-degenerate directions of the symplectic 2-form
- have non-vanishing Noether charges

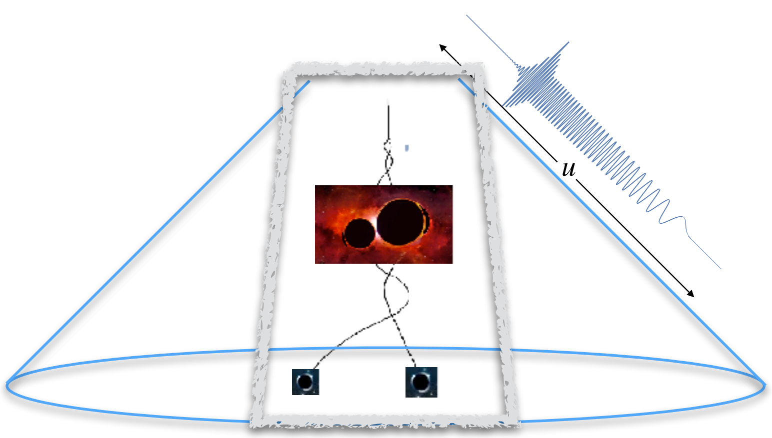


Boundaries in General Relativity

According to general relativity, the gravitational force cannot be screened, nor confined

Therefore there are no boundaries in the same sense as in electromagnetism,
e.g. the plates of a capacitor

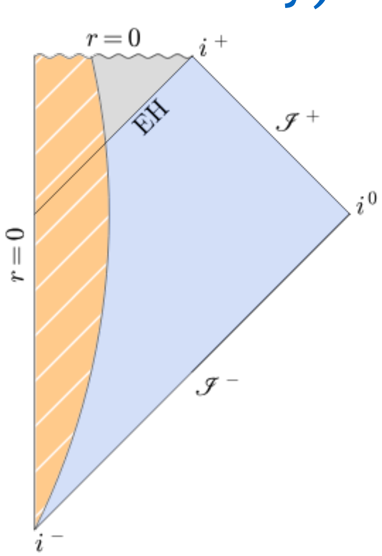
There is however a meaningful notion of boundaries as hypersurfaces where specific boundary conditions are imposed, be it for mathematical or physical reasons



For example when *modelling isolated systems*,
we assume a flat background metric
asymptotically far away from the sources
⇨ ***asymptotic boundary conditions***

These asymptotic boundary conditions are preserved by a set of residual diffeomorphisms at the boundary: **asymptotic symmetries** (E.g. Poincare at spatial infinity, BMS at null infinity)

Another example of physically motivated spacetime boundaries are **horizons**,
and there as well we have a notion of boundary symmetries



In general, we talk about **boundary symmetries**,
as residual diffeomorphisms preserving the boundary conditions

Boundary symmetries

In many cases, the boundary conditions lead to the identification of a ***universal structure*** shared by all solutions at the boundary

Accordingly, one can find in the literature two alternative definitions of boundary/asymptotic symmetries:

- As residual diffeomorphisms compatible with the boundary conditions, modulo bulk diffeos
- As isometries of the universal structure

The procedure and the technical steps in each case can look quite different, but **the result is typically the same**

In the rest of the talk I will focus on the case of BMS symmetries

BMS symmetry in a nut-shell

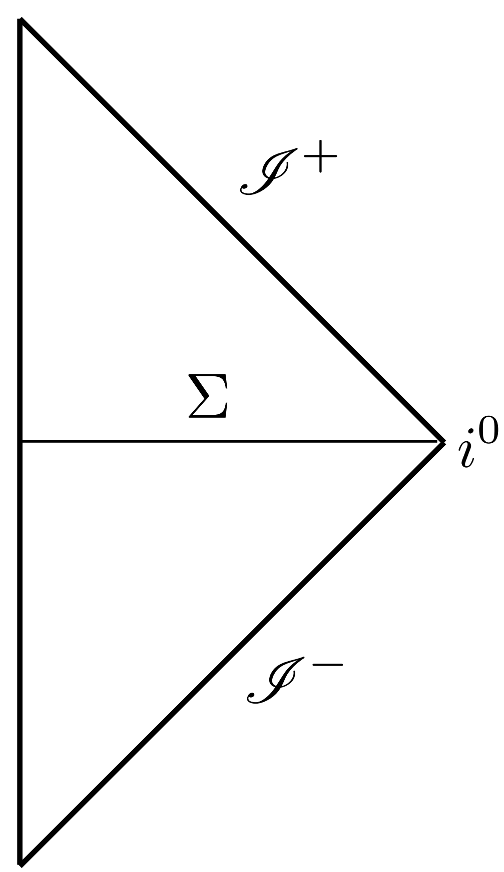
In Cartesian coordinates, the ten Killing vectors of Minkowski take the form $\xi = (a^\mu + b^\mu{}_\nu x^\nu)\partial_\mu$

Consider now Minkowski in retarded null coordinates $x^\mu = (u, r, x^A), \qquad u := t - r$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & -1 & 0 \\ & 0 & 0 \\ & & r^2 q_{AB} \end{pmatrix}$$

As $r \to \infty$ at constant u, we reach future null infinity $\mathcal{I}^+$, `Scri'-plus

Can be visualized on a Penrose diagram:



BMS symmetry in a nut-shell

In Cartesian coordinates, the ten Killing vectors of Minkowski take the form $\xi = (a^\mu + b^\mu{}_\nu x^\nu)\partial_\mu$

Consider now Minkowski in retarded null coordinates $x^\mu = (u, r, x^A), \qquad u := t - r$

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & -1 & 0 \\ & 0 & 0 \\ & & r^2 q_{AB} \end{pmatrix}$$

In these coordinates, the Killing vectors $\pounds_\xi \eta_{\mu\nu} = 0$ read

$$\xi = f\partial_u + Y^A\partial_A - \frac{r}{2}\mathscr{D}Y\partial_r - \frac{1}{r}\partial^A f\partial_A + \frac{1}{2}\mathscr{D}^2 f\partial_r, \qquad f = T + \frac{u}{2}\mathscr{D}Y.$$

where: $T(x^A)$ harmonic function on the sphere with $l = 0, 1$
corresponding to translations
$Y^A(x^B)$ conformal Killing vector of the sphere
corresponding to Lorentz transformations

BMS symmetry in a nut-shell

For an asymptotically flat metric at null infinity, we require

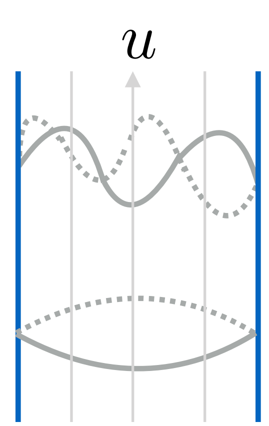
$$g_{\mu\nu} = \begin{pmatrix} -1 & -1 & 0 \\ & 0 & 0 \\ & & r^2 q_{AB} \end{pmatrix} + O(r^{-1})$$

An asymptotic symmetry preserves the leading, flat metric: $\mathcal{L}_\xi g_{\mu\nu} = O(r^{-1})$

The result is similar to the previous one, with **two** key differences:
the bulk extension is now free

$$\xi = f\partial_u + Y^A\partial_A + \Omega\left(\frac{1}{2}\mathscr{D}Y\partial_\Omega - \partial^A f\partial_A\right) + O(\Omega^2) \qquad f = T + \frac{u}{2}\mathscr{D}Y$$

where: $T(x^A)$ harmonic function on the sphere **of arbitrary spin: Super-translations**
corresponding to translations
$Y^A(x^B)$ conformal Killing vector of the sphere
corresponding to Lorentz transformations



From harmonic coordinates to null infinity

Blanchet-Compere-Faye-Oliveri-Seraj '20

While the origin of super-translations is clear from a geometric perspective,
it is also instructive to understand how it arise from the point of view of the PM expansion:

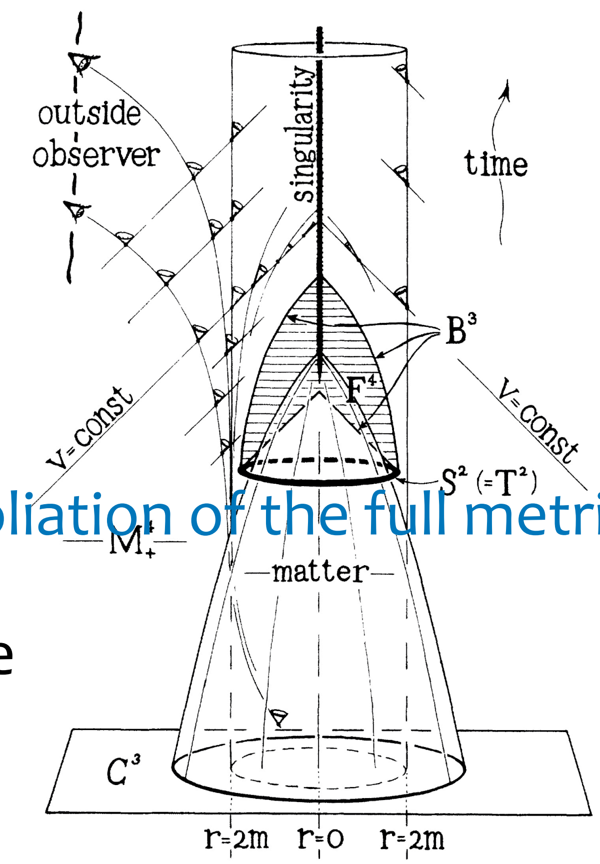
$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} + \dots$$

background
outgoing null radiation:

first correction,
e.g. Schwarzschild:

as we move away from the flat background,
the null cones bend according to the curvature,
and so do the outgoing null directions

$$u = t - r \; - 4GM \ln\left(\frac{r}{2M} - 1\right) + G^2 \dots$$



The Bondi gauge is defined so that one coordinates describes a null foliation of the full metric

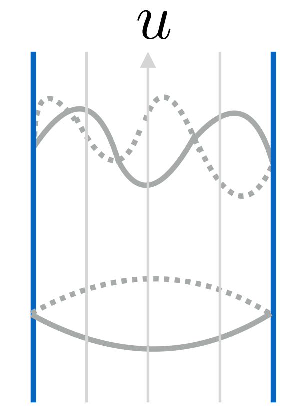
When changing coordinates from harmonic gauge to the Bondi gauge
one gets an ambiguity: super-translations

- In non-radiative spacetimes the ambiguity can be removed
- In radiative spacetimes, the ambiguity cannot be removed without removing the radiation!

The BMS algebra

The BMS algebra is an infinite-dimensional extension of the Poincare algebra:

$$\xi = \left(T + \frac{u}{2}\mathscr{D}Y\right)\partial_u + Y^A\partial_A \quad \begin{cases} T = \sum_{l=0,1} T_{lm}Y_{lm} & P^4 = \mathrm{SO}(3,1) \ltimes T^4 \\ T = \sum_{l=0}^{\infty} T_{lm}Y_{lm} & \mathrm{BMS} = \mathrm{SO}(3,1) \ltimes \mathbb{R}^S \end{cases}$$



$$[\xi_1, \xi_2] = f_{12}\partial_u + Y^A_{12}\partial_A$$

$$x^a = (u, x^A), \qquad x^A = (\theta, \phi)$$

$$T_{12} = \frac{1}{2}T_1\mathscr{D}Y_2 + Y_1[T_2] - 1 \leftrightarrow 2, \qquad Y^A_{12} = [Y_1, Y_2]^A$$

contains the usual translation dependence
of angular momentum of special relativity:

$$[P_a, R_b] = -\epsilon_{ab}{}^c P_c$$

But now enhanced to a **super-translation dependence:**
not just as many Lorentz subgroups as choices of origin,
but as many as choices of `cuts' of Scri

The phase space of asymptotically flat solutions of general relativity should contain
canonical generators associated with this symmetry,
and by Noether's theorem, flux-balance laws associated with them

Canonical generators for the BMS algebra

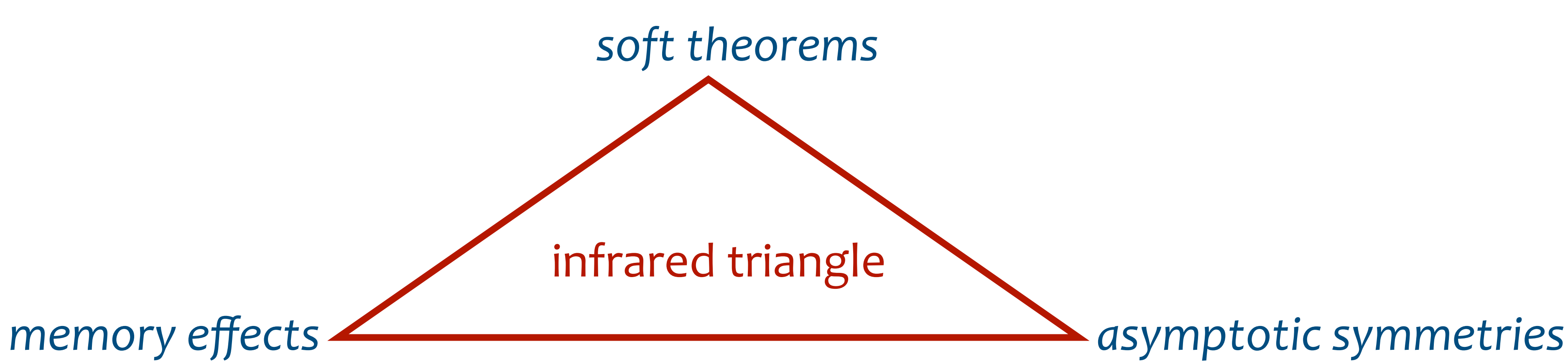
In special relativity, the canonical generators of the Poincare algebra are of great utility: energy, momentum, and relativistic angular momentum (boosts/CoM and angular momentum)

Similarly, we expect the canonical generators of the BMS algebra to be of great utility in general relativity : energy, momentum, **super-momentum,** and relativistic angular momentum

Would such canonical generators provide a realization of the algebra?

$$\{Q_{\xi_1}, Q_{\xi_2}\} \stackrel{?}{=} Q_{[\xi_1,\xi_2]}$$

This question has a direct relevance to **quantum gravity**, because as showed in Strominger '14, super-translations show up as Ward identities for the soft graviton theorem



Charge algebra and cocycles

If the symmetries transformations correspond to **Hamiltonian vector fields,** then it is guaranteed that there exist canonical generators associated to them, and that they realize the algebra (up to at most a central extension)

$$\pounds_\xi \omega = d i_\xi \omega = 0 \quad \Rightarrow \quad i_\xi \omega = d h_\xi$$

$$\begin{aligned} \pounds_{\xi_1} i_{\xi_2} \omega &= d i_{\xi_1} i_{\xi_2} \omega = d\{h_{\xi_1}, h_{\xi_2}\} \\ &= [\pounds_{\xi_1}, i_{\xi_2}]\omega = i_{[\xi_1,\xi_2]} = d h_{[\xi_1,\xi_2]} \end{aligned}$$

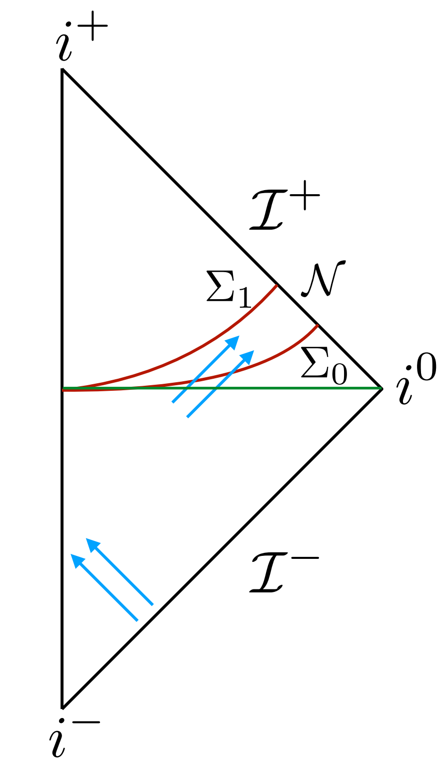
$$\Rightarrow \qquad \{h_{\xi_1}, h_{\xi_2}\} = h_{[\xi_1,\xi_2]} + X_{(\xi_1,\xi_2)}, \qquad dX_{(\xi_1,\xi_2)} = 0$$

Charge algebra and cocycles

If the symmetries transformations correspond to **Hamiltonian vector fields,** then it is guaranteed that there exist canonical generators associated to them, and that they realize the algebra (up to at most a central extension)

The problem is that in the presence of radiation the system is dissipative, and some of the symmetry transformations are **not Hamiltonian vector fields**

Different phase spaces and the problem of `integrability'



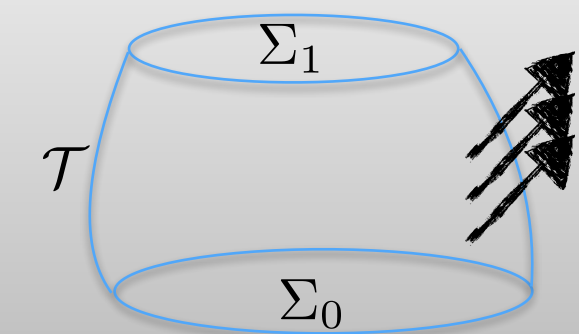
- Different phase spaces (*Cauchy*, *partial Cauchy*, *radiative*)

Symplectic 2-form: $\Omega_\Sigma = \int_\Sigma \omega \qquad \Omega_{\mathcal{N}} = \int_{\mathcal{N}} \omega$

The symplectic 2-form is closed on-shell: $d\omega \approx 0$ Stokes's theorem: $\int_{\Sigma_0} \omega = \int_{\Sigma_1} \omega + \int_{\mathcal{T}} \omega$

In the presence of outgoing or incoming flux,
the symplectic form is *not conserved in time*

$\int_{\mathcal{T}} \omega$ ~~$= 0$~~



This situation makes also the notion of HVF tricky:
there are non-HVF which carry useful physical information!
e.g. symmetries producing normal deformations of the corner S

⇨ $-I_\xi \Omega \neq \delta H_\xi$

Shown explicitly in Iyer-Wald '94

Charge algebra and cocycles

If the symmetries transformations correspond to **Hamiltonian vector fields,** then it is guaranteed that there exist canonical generators associated to them, and that they realize the algebra (up to at most a central extension)

The problem is that in the presence of radiation the system is dissipative, and some of the symmetry transformations are **not Hamiltonian vector fields**

In this case one cannot obtain canonical generators in the usual sense, and an additional prescription is needed

For the BMS case, such a prescription was identified in **two complementary approaches, with the same end result** :

1. Ashtekar-Streubel '81 (generalized in Ashtekar-SSp '24)
2. Wald-Zoupas '99 (with key aspects clarified in Grant-Prabhu-Shezhad '21, Odak-Rignon-Bret-SSp '22 and Ashtekar-SSp '24)

Charge algebra and cocycles

If the symmetries transformations correspond to **Hamiltonian vector fields,**
then it is guaranteed that there exist canonical generators associated to them,
and that they realize the algebra (up to at most a central extension)

The problem is that in the presence of radiation the system is dissipative,
and some of the symmetry transformations are **not Hamiltonian vector fields**

In this case one cannot obtain canonical generators in the usual sense,
and an additional prescription is needed

For the BMS case, such a prescription was identified in **two complementary approaches,**
with the same end result :

1. Ashtekar-Streubel '81 (generalized in Ashtekar-SSp '24)
2. Wald-Zoupas '99 (with key aspects clarified in Grant-Prabhu-Shezhad '21, Odak-Rignon-Bret-SSp '22 and Ashtekar-SSp '24)

Without using this prescription, Barnich-Troessaert '11 computed the charge algebra,
and found a *field-dependent 2-cocycle*

$$\{Q^{\rm BT}_\xi, Q^{\rm BT}_\chi\}_* \hat{=} Q^{\rm BT}_{[\![\xi,\chi]\!]} + K^{\rm BT}_{(\xi,\chi)}$$

$$K^{\rm BT}_{(\xi,\chi)} = \oint_S k^{\rm BT}_{(\xi,\chi)}, \qquad k^{\rm BT}_{(\xi,\chi)} = \frac{1}{32\pi}\left[f_\xi\left(C^{AB}\mathscr{D}_A\mathscr{D}_B\mathscr{D}_C Y^C_\chi + \partial^A f_\chi \partial_A \mathcal{R}\right) - (\xi \leftrightarrow \chi)\right]\epsilon_S$$

Charge algebra and cocycles

If the symmetries transformations correspond to **Hamiltonian vector fields,** then it is guaranteed that there exist canonical generators associated to them, and that they realize the algebra (up to at most a central extension)

The problem is that in the presence of radiation the system is dissipative, and some of the symmetry transformations are **not Hamiltonian vector fields**

In this case one cannot obtain canonical generators in the usual sense, and an additional prescription is needed

For the BMS case, such a prescription was identified in **two complementary approaches, with the same end result** :

1. Ashtekar-Streubel '81 (generalized in Ashtekar-SSp '24)
2. Wald-Zoupas '99 (with key aspects clarified in Grant-Prabhu-Shezhad '21, Odak-Rignon-Bret-SSp '22 and Ashtekar-SSp '24)

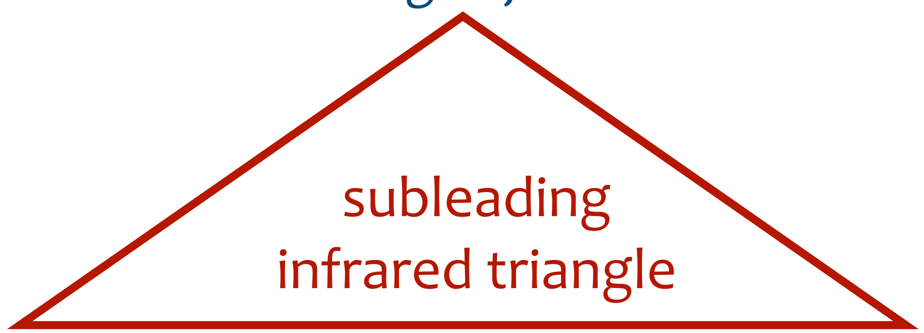
If the prescription is correctly implemented, the field-dependent cocycle is removed and covariance restored.
Furthermore, the calculations show that the cocycle was due precisely to the non-covariance of the BT11 prescription, in the sense of background-dependence

Remark: $\{Q_\xi, Q_\chi\}_* := \delta_\chi Q_\xi - I_\xi \mathcal{F}_\chi = I_\xi I_\chi \Omega_\Sigma + I_\chi \mathcal{F}_\xi - I_\xi \mathcal{F}_\chi$

Canonical generators for the extended BMS algebra

This is not the end of the story, because there exist other infrared triangles !

sub-leading soft theorems Cachazo-Strominger '14



new memory effects
Pasterski-Strominger-Zhiboedov '16

new asymptotic symmetries
Barnich-Troessart '10

It is also possible to find a covariant prescription for these charges,
and this requires tapping into a subtle feature of the covariant phase space:
the freedom to add corner terms to the symplectic 2-form
(already present in the ADM formulation)

The new prescription for the extended BMS symmetry proves and extends
the validity of an ansatz made by Campiglia-Peraza '20 and used by Donnay-Nguyen-Ruzziconi '22

An independent justification for the extra corner terms is still missing

Cocycles also related to quantum anomalies ➩ Baulieu-Wetzstein '25

Conclusions

Asymptotic and boundary symmetries are a very active line of research in general relativity
Adami, Ashtekar, Campiglia, Chandrasekaran, Compere, Donnay, Fiorucci, Flanagan, Freidel, Geiller, Giribert, Godazgar, Grumiller, Herfray, Khera, Laddha, Lewandowski, Mason, Nguyen, Pasterski, Peraza, Perry, Pino, Pope, Prabhu, Pranzetti, Raclariu, Rignon-Bret, Ruzziconi, Seraj, Strominger, Wieland, Zwikel…
(and more generally in gauge theories as well as modified theories of gravity)
with implications for

- classical GR (new boundary conditions, new solution spaces…)
- observables (new memory effects…)
- quantum gravity (relation to soft theorems, integrability, twistor theory…)

Finding consistent phase space realizations for the symmetries is a **delicate task**, typically plagued by issues of coordinate dependence/lack of covariance, and these issues show up directly in field-dependent 2-cocycles in the charge algebra

A realization of the charges that respects the covariance criteria is guaranteed to be free of field-dependent 2-cocycles