Wheeler-DeWitt equation for flat minisuperspace models and emergence of time

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- H.P., N. Toumbas, B. de Vaulchier, Nucl. Phys. B 973 (2021), 115600
- A. Kehagias, H.P., N. Toumbas, JHEP 12 (2021) 165
- E. Kaimakkamis, H.P., K. Sil, N. Toumbas, Class.Quant.Grav. 42 (2025) 11, 115003
- E. Kaimakkamis, H.P., N. Toumbas, V. Franken to appear

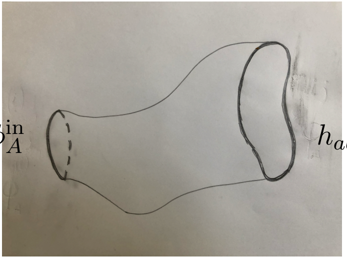
Wavefunction of the Universe

■ The probability amplitude to observe the **3D metric h_{ab} and fields ϕ_A (fonctions of x^1, x^2, x^3)** is [Hartle, Hawking, '83]

$$\psi[h_{ab}, \phi_A] = \int \mathcal{D}g_{\mu\nu}\, \mathcal{D}\Phi_A\, e^{\frac{i}{\hbar}\int d^4x\, L}$$

where **$g_{\mu\nu}$ and Φ_A** vary from initial values that define the state, **up to final values h_{ij}, ϕ_A**

h_{ab}^{in}, ϕ_A^{in}



h_{ab}, ϕ_A

■ No time in $\psi[h_{ab}, \phi_A]$

$\Longrightarrow$ **How does time emerges from the quantum theory?**

$$\psi[h_{ij}, \phi_A] = \int \mathcal{D}g_{\mu\nu}\, \mathcal{D}\Phi_A \; e^{\frac{i}{\hbar}\int d^4x\, L}$$

■ **Field redefinitions to $\tilde{g}_{\mu\nu}$, $\tilde{\Phi}_A$**

$\mathcal{D}g_{\mu\nu}\, \mathcal{D}\Phi_A \neq \mathcal{D}\tilde{g}_{\mu\nu}\, \mathcal{D}\tilde{\Phi}_A$ **because there is a Jacobian**

$\Longrightarrow$ **Different wavefunctions**

$\Longrightarrow$ **Different quantum theories?**

■ Minisuperspace models where the Universe is homogeneous and close

$$ds^2 = -N(x^0)^2\, \mathrm{d}x^{0\,2} + g_{ab}(x^0)\, \mathrm{d}x^a \mathrm{d}x^b\,, \qquad \Phi_A(x^0)$$

Denote collectively q^i, $i = 0, \dots, n-1$,

$$L(N, \vec{q}, \dot{\vec{q}}) = N\left(\frac{1}{2N^2}\,\gamma_{ij}(\vec{q})\,\dot{q}^i\dot{q}^j - v(\vec{q})\right)$$

- Classically $$\frac{H}{N} = -\frac{\partial L}{\partial N} = 0 \quad \text{on-shell}$$ $$= \frac{1}{2}\,\gamma^{ij}(\vec{q})\,\pi_i\pi_j + v(\vec{q})$$

NB: $\pi_i = M_{ik}(\vec{q})\,\pi_l\, M^{-1,kl}(\vec{q})$

- Canonical quantization: $$\pi_i \longrightarrow -i\hbar\,\frac{\mathrm{d}}{\mathrm{d}q^i}$$

$$-i\hbar\,\frac{\mathrm{d}\psi}{\mathrm{d}q^i} \quad \neq \quad -i\hbar\, M_{ik}(\vec{q})\,\frac{\mathrm{d}}{\mathrm{d}q^i}\left(M^{-1,kl}(\vec{q})\,\psi\right)$$

Different **Wheeler-DeWitt equations *i.e.* wavefunctions** [DeWitt, '67]

■ **Fix gauge of time** [H.P., Toumbas, de Vaulchier, '21]

$$\psi(\vec{q}_{\rm f}) = \int \frac{\mathcal{D}N \, \prod_{k=0}^{n-1} \mathcal{D}q^k}{\text{Vol(Diff)}} \; e^{\frac{i}{\hbar} S[N,\vec{q}]} = \int_0^{+\infty} \mathrm{d}\ell \, U(\vec{q}_{\rm i}, \vec{q}_{\rm f}, \ell)$$

where ℓ is cosmological time and $\quad U(\vec{q}_{\rm i}, \vec{q}_{\rm f}, \ell) = \int_{\vec{q}_{\rm i}}^{\vec{q}_{\rm f}} \prod_{k=0}^{n-1} \mathcal{D}q^k \; e^{\frac{i}{\hbar} S}$

■ **Restriction:** The target space *i.e.* the space parametrized by $\vec{q}$ is flat

E.g. R^α inflationary models such as R^2-Starobinsky model, JT-gravity.

$\Longrightarrow$ **Field redefinition $q^i = q^i(\vec{Q})$, such that the new field $\vec{Q}(t)$ have a quadratic kinetic term** [E. Kaimakkamis, H.P., K. Sil, N. Toumbas, '24] [E. Kaimakkamis, H.P., N. Toumbas, V. Franken, '25]

$$\prod_{k=0}^{n-1} \mathcal{D}q^k = \prod_{k=0}^{n-1} \mathcal{D}Q^k \prod_{t\in[0,\ell]} J\big(\vec{Q}(t)\big) \,, \quad \text{where} \quad J(\vec{Q}) = \left| \det \frac{\partial q^i}{\partial Q^j} \right|$$

- Discretize time, like in the derivation of the Schrödinger equation

$$\Longrightarrow \qquad i\hbar \frac{\partial}{\partial \ell} U(q_{\rm i}, q_{\rm f}, \ell) = \frac{1}{J} \left[-\frac{\hbar^2}{2} \nabla^2 (JU) + vJU \right]$$

- Integrate over ℓ $\quad \Longrightarrow \quad$ **Wheeler-DeWitt equation**

$$H\psi \equiv \left\{ -\frac{\hbar^2}{2} \left[\nabla^2 + \frac{2}{J} \nabla^i J \nabla_i + \frac{1}{J} \nabla^2 J \right] + v \right\} \psi = 0$$

This corresponds to a particular ordering of the operators.

Equivalence of all quantum theories

■ **Do different choices of J *i.e.* different path-integral measures $\prod_{k=0}^{n-1}\mathcal{D}q^k$ lead to different predictions?**

• Define the **transition amplitudes** in terms of an **inner product of the Hilbert space. Ansatz with an extra function μ**

$$\langle\psi_1,\psi_2\rangle = \int_{\mathcal{T}} \prod_{k=0}^{n-1} \mathrm{d}q^k \, \sqrt{-\gamma}\, \mu(\vec{q})\, \psi_1(\vec{q})^* \psi_2(\vec{q})$$

• $\langle\psi_1|H\psi_2\rangle = \langle H^\dagger\psi_1|\psi_2\rangle$ defines the adjoint $H^\dagger$ by integration by parts

• **Imposing $H = H^\dagger$ determines**

$$\mu = J^2$$

- Rewrite

$$\langle \psi_1, \psi_2 \rangle = \int_{\mathcal{T}} \prod_{k=0}^{n-1} \mathrm{d}q^k \, \sqrt{-\gamma} \, \Big(J(\vec{q}) \, \psi_1(\vec{q}) \Big)^* \Big(J(\vec{q}) \, \psi_2(\vec{q}) \Big)$$

$$\equiv \int_{\mathcal{T}} \prod_{k=0}^{n-1} \mathrm{d}q^k \, \sqrt{-\gamma} \, \Psi_1(\vec{q})^* \Psi_2(\vec{q})$$

where

$$\Psi = J \, \psi$$

- **The Wheeler-DeWitt equation takes a universal form!**

$$-\frac{\hbar^2}{4} \nabla^2 \Psi + v \Psi = 0$$

In fact: μ and ψ depend on J (the path-integral measure), but not Ψ and the inner product.

$\Longrightarrow$ **The quantum theory is unique!**

Emergence of time

■ The wavefunction of the Universe does not depend on time.
How time emerges from the quantum theory?

[Kehagias, H.P., Toumbas, '21]

- Homogeneous, isotropic, closed Universe : **scale factor $a(t)$**
- filled with a **perfect fluid** of state equation

$$p = w\rho \qquad \text{where} \qquad -1 \leq w \leq 1$$

■ **The quantum probability to observe the scale factor in the range $[a, a + \mathrm{d}a]$ is**

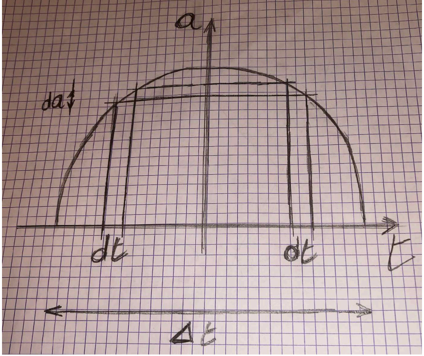
$$P(a)\,\mathrm{d}a = \mathrm{d}a\sqrt{-\gamma}\,|\Psi|^2$$

It contains the information to reconstruct the time evolution of the Universe. Let us recover the classical evolution.

■ Classical evolution

$$3\left(\frac{\dot{a}}{a}\right)^2 = \rho - \frac{3}{a^2}$$

- For $\boldsymbol{w > -\frac{1}{3}}$:



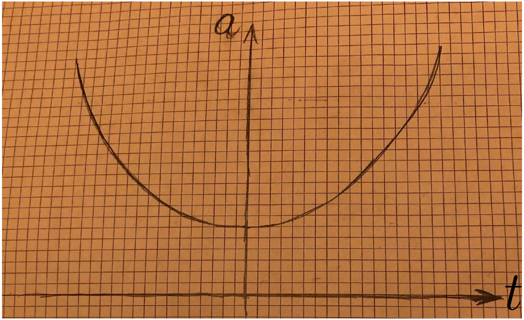
The probability to observe the scale factor in the range $[a, a + \mathrm{d}a]$ is

$$\left|\frac{2\mathrm{d}t}{\Delta t}\right| = P_{\mathrm{cl}}(a)\mathrm{d}a$$

It is the quantum probability $P(a)\mathrm{d}a$ in the $\hbar \to 0$ limit!

Time is the way to realize in a concrete way the probabilities derived from the quantum theory.

- For $w \leq -\frac{1}{3}$:



- The total duration Δt of the evolution is infinite

- The quantum probability *i.e.* wavefunction is not normalizable

- but $P(a) \to P_{\rm cl}(a)$ when $\hbar \to 0$.

Conclusion

■ The notion of wavefunction defined by Hartle-Hawking and DeWitt is highly ambiguous:

- **Infinite number of choices of path-integral measures.**
- **Infinite number of operator orderings in the Wheeler-DeWitt equation (the Hamiltonian)**

■ For each choice of path-integral measure, we have determined

- **the correct Wheeler-DeWitt equation,**
- **the correct inner product of the Hilbert space.**

■ **All quantum predictions are independent of the prescription.**

■ **The quantum probability to observe a 3D space is the probability to observe this 3D space during the cosmological evolution.**