Examining the Anomalous Nature of Chiral Effects in Thermodynamics

Diego Saviot

Laboratoire d'Annecy de Physique Théorique

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Relativistic hydrodynamics

Description with local fluid velocity u, temperature T, chemical potential μ.

Derivative expansion of a current J associated to a charge:

$$J^\mu = n\, u^\mu + \sigma_E E^\mu + \sigma_B B^\mu + \sigma_\Omega \Omega^\mu + \ldots$$

$$\Omega^\mu = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} u_\mu \partial_\rho u_\sigma \quad \text{vorticity}$$

[Son et al. '09] $\partial_\mu J^\mu = \mathcal{A} \sim E \cdot B \quad \rightarrow \quad \sigma_B,\, \sigma_\Omega \neq 0.$

Chiral Effects

Massless Dirac fermion, axial current $\mathrm{j}_5^\mu = \bar{\psi}\gamma_5\gamma^\mu\psi$

$$\langle \mathrm{j}_5^\mu \rangle \supset \frac{\mu}{2\pi^2}\mathrm{B}^\mu + \left(\frac{\mu^2}{2\pi^2} + \frac{\mathrm{T}^2}{6}\right)\Omega^\mu$$

Chiral Separation Effect

Chiral Vortical Effect

Reminder on the Chiral anomaly

$$\langle \nabla_\mu j_5^\mu \rangle = \frac{1}{16\pi^2} F_{\mu\nu} \tilde{F}^{\mu\nu} + \frac{1}{384\pi^2} R_{\mu\nu\rho\sigma} \tilde{R}^{\mu\nu\rho\sigma}$$

Literature: unchanged by temperature and chemical potential

Our goal

Compute both the anomaly and the chiral effects from the same path-integral to understand their relation

Setting up the formalism

Partition function

- Temperature

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int_{\text{AP b.c.}} \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int_0^\beta dt \int d^3x\, \bar{\psi} i \not{D} \psi}$$

(Fourier: $q_t = (2n+1)\pi T$)

Partition function

- Temperature

$$Z = \mathrm{Tr}\, e^{-\beta H} = \int_{\text{AP b.c.}} \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int_0^\beta dt \int d^3x\, \bar{\psi} i \not{D} \psi}$$

(Fourier: $q_t = (2n+1)\pi T$)

- (Electro-) Chemical potential

$$H - \mu_0 Q \quad \rightarrow \quad \mathcal{V}^\mu = V^\mu + \mu_0 \delta^t_\mu$$

Partition function

- Fluid velocity and curved spacetime

$$Z = \mathrm{Tr}\, e^{-\int d\Sigma n_\mu T^{\mu\nu} \beta_\nu}, \qquad \beta^\mu = \frac{1}{T} u^\mu$$

Specific choice of coordinates to get a path-integral representation (in curved space).

Partition function

- Fluid velocity and curved spacetime

$$Z = \mathrm{Tr}\, e^{-\int d\Sigma n_\mu T^{\mu\nu} \beta_\nu}, \qquad \beta^\mu = \tfrac{1}{T} u^\mu$$

Specific choice of coordinates to get a path-integral representation (in curved space).

- e.g in flat space, with velocity:

$$g_{\mu\nu} = \begin{pmatrix} 1 - \vec{v}^2 & \vec{v} \\ \vec{v} & -\mathbb{1}_3 \end{pmatrix}, \quad u^\mu = \frac{1}{\sqrt{1-\vec{v}^2}} \begin{pmatrix} 1 \\ \vec{0} \end{pmatrix}$$

Global equilibrium

- Temperature: β_μ is Killing

$$\nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = \frac{1}{T_0} \partial_t g_{\mu\nu} = 0$$

- Electro-chemical potential:

$$T \partial_\mu \frac{\mu_{ec}}{T} = 0$$

Computing the chiral anomaly

Computing the chiral anomaly without temperature

Infinitesimal chiral transformation $\delta\psi(x) = i\alpha(x)\gamma_5\psi$

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i\int d^4x\, \bar{\psi} i\not{D}\psi} = \int J[\alpha]\mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i\int d^4x\, \bar{\psi}(i\not{D} - (\not{\partial}\alpha)\gamma_5)\psi}$$

$$\downarrow \frac{\delta}{\delta\alpha(x)}$$

$$0 = \mathcal{A}(x) - \langle \partial_\mu j_5^\mu \rangle$$

Computing the chiral anomaly with temperature

Infinitesimal chiral transformation $\delta\psi(x) = i\alpha(x)\gamma_5\psi$, $\alpha(t = i\beta_0) = \alpha(t = 0)$

$$Z = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int_0^{\beta_0} dt \int d^3x\, \bar{\psi} i \not{D} \psi} = \int J[\alpha] \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{-\int_0^{\beta_0} dt \int d^3x\, \bar{\psi}(i\not{D} - (\not{\partial}\alpha)\gamma_5)\psi}$$

$$J[\alpha] = \frac{\det_{\beta_0}\left(i\not{D}\right)}{\det_{\beta_0}\left(i\not{D} - (\not{\partial}\alpha)\gamma_5\right)}$$

Computation

$$\log J[\alpha] = \mathrm{Tr}\left(-(\partial\!\!\!/\alpha)\gamma_5(iD\!\!\!\!/)^{-1}\right)$$

$$\langle j_5^\mu \rangle = \frac{1}{\beta_0}\sum_{n\in\mathbb{Z}}\int\frac{d^{3\text{-}\epsilon}q}{(2\pi)^{3\text{-}\epsilon}}\mathrm{tr}\,\gamma^\mu\gamma_5\sum_{k\in\mathbb{N}}\left[\Delta_{n,q}iD\!\!\!\!/\right]^k\Delta_{n,q}$$

$$D_\mu \equiv \nabla_\mu + \omega_\mu + i\delta^i_\mu V_i$$

Truncate in same way as previously in $j_5^\mu = n\,u^\mu + \sigma_E E^\mu + \sigma_B B^\mu + \sigma_\Omega \Omega^\mu + \ldots$

Possible from the presence of mass scales T, μ_{ec}

Computation

- Contribution from the spin-connection ω

$$\langle j_5^\mu \rangle \supset T \sum_n \int d^3q \,\mathrm{tr}\, \gamma^\mu \gamma_5 \Delta_{n,q} i\not{\omega} \Delta_{n,q}$$
$$= \left(\frac{\mu_{ec}^2}{2\pi^2} + \frac{T^2}{6} \right) \Theta^\mu \qquad \Theta^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu u^\lambda \omega_{\lambda,\rho\sigma}$$

Computation

- Contribution from the spin-connection ω

$$\langle j_5^\mu \rangle \supset T \sum_n \int d^3q \, \mathrm{tr}\, \gamma^\mu \gamma_5 \Delta_{n,q} i\not{\omega} \Delta_{n,q}$$

$$= \left(\frac{\mu_{ec}^2}{2\pi^2} + \frac{T^2}{6} \right) \Theta^\mu \qquad \Theta^\mu = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} u_\nu u^\lambda \omega_{\lambda,\rho\sigma}$$

- Contribution from the electromagnetic field

$$\langle j_5^\mu \rangle \supset \frac{1}{2\pi^2} \mu_{ec} B^\mu$$

Anomaly

$$\mathcal{A} = \nabla_\mu \langle j_5^\mu \rangle = \left(\frac{\mu_{ec} \partial_\mu \mu_{ec}}{\pi^2} + \frac{T \partial_\mu T}{3} \right) \Theta^\mu + \left(\frac{\mu_{ec}^2}{2\pi^2} + \frac{T^2}{6} \right) \nabla_\mu \Theta^\mu$$
$$+ \frac{1}{2\pi^2} \left(\partial_\mu \mu_{ec} - \mu_{ec} a_\mu \right) B^\mu$$

The anomaly at finite temperature and density is modified.

Global equilibrium: $\mathcal{A} = 0$

Conclusion

- $\langle \nabla_\mu j_5^\mu \rangle_{\text{quantum + statistical}} \neq \langle \nabla_\mu j_5^\mu \rangle_{\text{quantum}}$
- $T^2 \Omega^\mu$ not from usual $R\tilde{R}$
- more general operator than vorticity
- interesting for condensed matter
  (better understanding of what qualifies as analog gravity)
- direction to explore: impact on baryogenesis?

Thank you for your attention!