Primordial black holes beyond spherical symmetry

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In collaboration with
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To appear

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What are they ?

- Primordial black holes (PBH) are black hole formed in the early universe
- Many different possible mechanisms for the formation $\rightarrow$ not standard stellar evolution

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- Because black evaporate through Hawking radiation, only a fraction might remains today
- Depending on their abundance, can affect the cosmological observables:

  → CMB characteristic, non-gaussianities for instance
- Source of gravitational waves: → affect the expected stochastic GW background
- Potential candidate for (part of) the dark matter
- Potential seed for the high redshift structure observed by JWST ($z \sim 14$)

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Key challenge to describe PBH

- Asymptotically FLRW + non-vacuum geometries
- No Killing horizon nor time-like Killing vector
- Understanding gravitational radiation and evaporation require new tools
- Testing these tools require exact solutions for PBH
- Key targets: formation criteria, environmental effects, lifetime

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Mass loss from Hawking radiation
Building exact solutions for PBH models

Spherically symmetric case: Temperature and mass loss

Kodama symmetry

- Consider a spherically symmetric spacetime

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a\mathrm{d}x^b + R^2\mathrm{d}\Omega^2 = -f(t,r)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{f(t,r)} + R^2(t,r)\mathrm{d}\Omega^2 \tag{1}$$

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- Any spherically symmetric spacetime exhibits a hidden Killing-Yano symmetry

$$\nabla_{(\mu}Y_{\nu)\alpha} = 0 \qquad Y_{\mu\nu}dx^\mu dx^\nu = R^2(t,r)\sin\theta d\theta \wedge d\varphi \tag{2}$$

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- One can introduce the so-called Kodama vector canonically defined as

$$k^\mu\partial_\mu = \varepsilon^{\mu\nu\alpha\beta}\nabla_\nu Y_{\alpha\beta}\,\partial_\mu = \left(R'\partial_t + \dot{R}\partial_r\right) \qquad \rightarrow \qquad \nabla_\mu k^\mu = 0 \tag{3}$$

[Kodama '80]

Three main properties

- Kodama norm directly related to expansions of light rays : allow to identify the horizon !

$$k_\alpha k^\alpha \propto \theta_\ell\theta_n \qquad \rightarrow \qquad \text{vanishes on the horizons} \tag{4}$$

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$$\mathcal{M} = \frac{R}{2}\left(1 - g^{\mu\nu}\nabla_\mu R\nabla_\nu R\right) = \frac{R}{2}\left(1 + |k^\mu k_\mu|\right) \qquad \mathcal{C} = \frac{\mathcal{M} - \mathcal{M}_{\text{FLRW}}}{R} \tag{5}$$

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- Hayward-Kodama surface gravity κ_{HK} and temperature for spherically symmetric and dynamical horizon

$$\frac{1}{2}k^\mu\nabla_{[\mu}k_{\alpha]} = \kappa_{\mathrm{HK}}K_\alpha \tag{6}$$

[Hayward '11]

Spherically symmetric case: Static versus dynamical horizons

Schwarzschild

- Consider the asymptotically flat and stationary Schwarzschild black hole

$$f(r) = 1 - \frac{2m}{r} \qquad R(t,r) = r \qquad \rightarrow \qquad T = \frac{\kappa}{2\pi} = \frac{1}{8M} \tag{7}$$

  with horizon at $r_h = 2M$.
- Mass loss through evaporation assuming Stefan law:

$$\frac{\mathrm{d}M}{\mathrm{d}t} = -4\pi\sigma r_h^2 T^4 \qquad \rightarrow \qquad \frac{\mathrm{d}M}{\mathrm{d}t} = -\frac{\hbar c^4}{15360 G^2}\frac{1}{M^2} \qquad \rightarrow \qquad M(t) \propto (\tau_0 - \tau)^{1/3} \tag{8}$$

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- Result for a vacuum asymptotically flat black hole ... preferred notion of asymptotic time ! what about dynamical geometry ?

FLRW cosmology

- Consider the case of a flat FLRW geometry

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j \tag{9}$$

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An example of a dynamical black hole embedded in cosmology

Vaidya-de Sitter black hole

- Consider the simplest model of dynamical black hole embedded in a de Sitter universe

$$\mathrm{d}s^2 = -\left[1 - \frac{2GM(u)}{c^2 r} - \frac{\Lambda r^2}{3}\right] c^2 \mathrm{d}u^2 - c\mathrm{d}u\mathrm{d}r + r^2 \mathrm{d}\Omega^2 \tag{11}$$

with energy momentum tensor describing null dust following a radial trajectory

$$T_{\mu\nu} = \pm \frac{c^2}{4\pi r^2} \frac{\mathrm{d}M(u)}{\mathrm{d}u} \partial_\mu u \partial_\nu u \tag{12}$$

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- Constraints of PBH abundance today should be understood with much more care !

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Next target

- What about non-spherically symmetric asymptotically FLRW black hole ?

Going beyond spherical symmetry: The problematic

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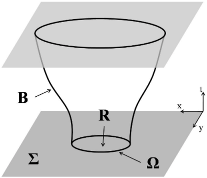
- Can we identify exact solutions of General Relativity describing axi-symmetric PBH models to test the above definitions ?

A generalized Kodama vector

- Consider a region of spacetime $\mathcal{V}$ with boundary $\partial\mathcal{V} = \Sigma_i \cup \mathcal{B} \cup \Sigma_f$ and 2-sphere $\Omega = \Sigma \cup \mathcal{B}$



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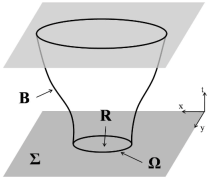

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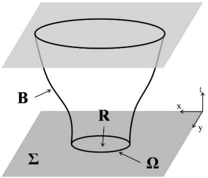

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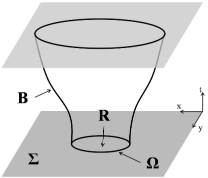

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- Metric on Ω

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu \tag{14}$$

such that $q_{\mu\nu} n^\mu = q_{\mu\nu} s^\mu = 0$

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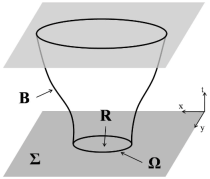
- Consider a region of spacetime $\mathcal{V}$ with boundary $\partial\mathcal{V} = \Sigma_i \cup \mathcal{B} \cup \Sigma_f$ and 2-sphere $\Omega = \Sigma \cup \mathcal{B}$



- Unit normal vector $n_\mu \mathrm{d}x^\mu$ to Σ, i.e. $n^\mu n_\mu = -1$, to Σ
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- Metric on Ω

$$q_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu - s_\mu s_\nu \tag{14}$$

such that $q_{\mu\nu} n^\mu = q_{\mu\nu} s^\mu = 0$

- Bending of Ω within the hypersurface Σ or within the hypersurface $\mathcal{B}$ which are respectively defined by

$$K_{\mu\nu}(n) = D_\mu n_\nu \qquad K_{\mu\nu}(s) = D_\mu s_\nu \tag{15}$$

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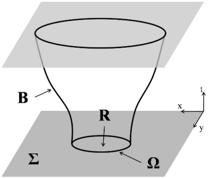

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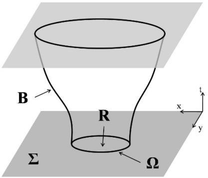

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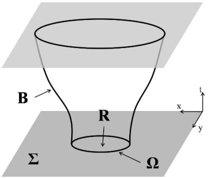

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- The vector $H_\perp^\mu \partial_\mu$ is the generalization of the Kodama vector beyond spherical symmetry

A generalized Kodama vector

Localize the horizon

- Norm of the GKV vanishes on the apparent horizons [Anco '05]

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Crucial to compute the energy of dynamical spacetime ! [Ashfar '17]

Generalized compaction function and temperature

Generalized notion of compaction function beyond spherical symmetry

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- Extend the proposal of Hayward to this generalized Kodama vector

$$\frac{1}{2}H^\mu_\perp \nabla_{[\mu} H^\perp_{\alpha]} = \kappa H^\perp_\alpha \tag{27}$$

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Concrete example beyond spherical symmetry?

Application to an axi-symmetric dynamical black hole

Kerr-Vaidya black hole

- Consider the Kerr-Vaidya model which provides the simplest rotating evaporating black hole model

$$\mathrm{d}s^2 = -\left(1 - \frac{2M(u)r}{\rho^2}\right)\mathrm{d}u^2 + 2\mathrm{d}u\mathrm{d}r + \rho^2\mathrm{d}\theta^2 - \frac{4aM(u)rsin^2\theta}{\rho^2}\mathrm{d}\phi\mathrm{d}u$$
$$- 2a\sin^2\theta\mathrm{d}\phi\mathrm{d}r + \frac{(r^2+a^2)^2 - a^2\Delta\sin^2\theta}{\rho^2}\sin^2\theta\mathrm{d}\phi^2 \quad (28)$$

  where $\rho^2 = r^2 + a^2\cos^2\theta$ and $\Delta(u,r) = r^2 - 2rM(u) + a^2$

- Generalized surface gravity gives:

$$\kappa = \frac{(r_+ - M)\left[4r_+^2\rho_+^2 + a^2(a^2+3r_+^2)\sin^2\theta\right] + a^4 r_+^2\sin^4\theta\ddot{M}_+}{4r_+(a^2+r_+^2)^{3/2}\rho_+^2} \quad (29)$$

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What about asymptotically flat axi-symmetric compact objects ?

Building asymptotically FLRW axisymmetric black hole in GR

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Focus on GR coupled to a scalar field or perfect fluid

- Known solution-generating method for spherically symmetric black hole:

  $\rightarrow$ Buchdal and Fonarev methods
- Can we extend these solution-generating method to axi-symmetry ?

Extended Fonarev method for axi-symmetric PBH

- Consider the static and axisymmetric vacuum solution $\bar{g}$ of the massless Einstein-Scalar system

$$d\bar{s}^2 = \bar{g}_{aa}(dx^a)^2 + \bar{h}_{ij}dx^i dx^j, \tag{30}$$

- Key assumption: no $a-$dependence.

$$\partial_a \bar{g}_{\mu\nu} = 0 = \bar{g}_{ia}, \tag{31}$$

- Then, one can construct an a-dependent extension $(\tilde{g}, \tilde{\phi})$ that solves the self-interacting Einstein-Scalar system with potential $V(\tilde{\phi}) = V_0 e^{\xi_3 \tilde{\phi}}$, and which takes the form

$$d\tilde{s}^2 = e^{2\mu(a)}[(\bar{g}_{aa})^{\beta}(dx_a)^2 + (\bar{g}_{aa})^{1-\beta}\bar{h}_{ij}dx^i dx^j], \tag{32}$$

$$\tilde{\phi} = \xi_0 \ln(\bar{g}_{aa}) + \frac{\xi_1}{\kappa}\mu(a), \tag{33}$$

with conformal factor

$$\mu(a) = \xi_2 \ln(Ca + B). \tag{34}$$

- The parameter space defined by V_0, ξ_1, ξ_2, and ξ_3 is constrained to follow

$$\xi_1 = -\xi_3 = \frac{\beta}{\xi_0} \tag{35}$$

$$(\beta^2 - 2\xi_0^2\kappa)\xi_2 = 2\xi_0^2\kappa \tag{36}$$

$$(2\xi_0^2\kappa - \beta^2)^2 V_0 = \mp\, 2\xi_0^2 C^2(\beta^2 - 6\xi_0^2\kappa), \tag{37}$$

- Systematic method to construct the first axi-symmetric PBH solution in GR [**BA**, Cisterna, Hassaine, Vennin '25]

A first exact solution of axi-symmetric PBH

Axi-symmetric seed solution: The Zippoy-Voorees naked singularity

- Exact vacuum axi-symmetric solution of GR : quadrupolar deformation of Schwarzschild

$$ds^2_{\rm ZV} = -f^{\delta} dt^2 + f^{-\delta} \left[\left(\frac{f}{g} \right)^{\delta^2} g \left(\frac{dr^2}{f} + r^2 d\theta^2 \right) + f r^2 \sin^2\theta d\varphi^2 \right] , \tag{38}$$

where the metric functions (f, g) are given by

$$f = \left(1 - \frac{2M}{r} \right) , \quad g = \left(1 - \frac{2M}{r} + \frac{M^2 \sin^2\theta}{r^2} \right) . \tag{39}$$

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First exact asymptotically FLRW and axi-symmetric black hole solution in GR

- Exact vacuum axi-symmetric solution of GR : quadrupolar deformation of Schwarzschild

$$d\tilde{s}^2 = a^2(t)\left\{-f^{\delta\beta}dt^2 + f^{-\delta\beta}\left[\left(\frac{f}{g}\right)^{\delta^2} g\left(\frac{dr^2}{f} + r^2 d\theta^2\right) + f r^2\sin^2\theta d\varphi^2\right]\right\}, \tag{40}$$

$$\tilde{\phi} = \delta\xi_0 \ln\left(1 - \frac{2M}{r}\right) + \frac{\xi_1\xi_2}{\kappa}\ln(Ct + D), \tag{41}$$

where the conformal factor reads

$$a(t) = (Ct + B)^{\xi_2}. \tag{42}$$

[**BA**, Cisterna, Hassaine, Vennin '25]

- Turning on the dynamics transforms a naked singularity into a dynamical black hole !
- Provide a whole new family of exact solutions to study axi-symmetric PBH

Conclusion

Take away messages

- PBH dynamical trapped regions in a non-vacuum and asymptotically FLRW
- Not harmless to use results from stationary and asymptotically flat black holes in GR
  $\rightarrow$ lifetime estimate and PBH abundance today
- Studying their phenomenology requires new geometrical tools

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- Provide a notion of quasi-local energy and compaction function beyond spherical symmetry relying on a preferred flow
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- Explore a whole new family of asymptotically FLRW and axi-symmetric compact objects in GR
- First exact asymptotically FLRW and axi-symmetric black hole solution in GR: time-dependent Zipoy-Vorhees BH
- Relaxing stationarity : naked singularity $\rightarrow$ covered by dynamical horizon

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Second result : New solution-generating technique

- Explore a whole new family of asymptotically FLRW and axi-symmetric compact objects in GR
- First exact asymptotically FLRW and axi-symmetric black hole solution in GR: time-dependent Zipoy-Vorhees BH
- Relaxing stationarity : naked singularity $\rightarrow$ covered by dynamical horizon

Work in progress

- Extending the method to rotating black holes
- New solution as testbed for perturbative approaches: gravitational radiation / Hawking radiation
- Confronting the new compacting function to numerical simulations of collapse

Conclusion

Thank you