A new integrable parametrization for deformations to the Kerr photon ring

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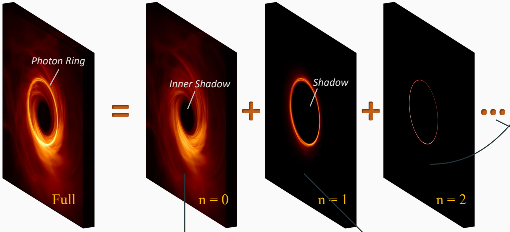
Introduction

- Observations of the vicinity of a BH constitute a new opportunity to **test General Relativity** in its strong field regime
- Modifications of gravity leave an imprint in the **black hole metric**
- This work: **new parametrization** of beyond-GR effects on the photon ring mixing concreteness of predictions and theoretical motivations
- Exciting opportunities to apply this framework to **future observations** in the context of Black Hole EXplorer (BHEX)

Black hole image: counting photon orbits

[Gralla, Lupsasca, Marrone '20]



Much harder to detect but less dependant on astrophysics [Lupsasca et al. '24]

critical curve:

- close to $n = 2$ subring (exponential behavior [Gralla, Lupsasca '20])
- analytical form for Kerr [Bardeen '73]
- **does not depend** on astrophysics and accretion disk physics

Complex GRMHD processes inside accretion disk

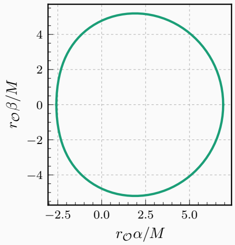
Imprint of BH metric with additional effects of astrophysical origin [Urso et al. '25]

Main goal of this work

Main goal: computation of the critical curve for beyond-GR black holes and check detectability of modifications of gravity with BHEX

Requirements

- parametrization beyond Kerr relevant for BHEX observations and other theoretical purposes
- computable critical curve
- observable quantifying deviations in the critical curve shape and size



First step: understand why Kerr is so special

Kerr black hole

Kerr metric: stationary axisymmetric black hole

$$\mathrm{d}s^2 = -\left(1 - \frac{2Mr}{\Sigma}\right)\mathrm{d}t^2 + \frac{\Sigma}{\Delta}\,\mathrm{d}r^2 + \Sigma\,\mathrm{d}\theta^2 + \left(r^2 + a^2 + \frac{2Mra^2}{\Sigma}\sin^2\theta\right)\sin^2\theta\,\mathrm{d}\phi^2 - \frac{4Mra\sin^2\theta}{\Sigma}\,\mathrm{d}t\,\mathrm{d}\phi$$

$$\Sigma = r^2 + a^2\cos^2\theta\,, \qquad \Delta = r^2 - 2Mr + a^2$$

Regions

- Outer horizon $r_+ = M + \sqrt{M^2 - a^2}$
- Inner horizon $r_- = M - \sqrt{M^2 - a^2}$
- Ergoregion $r_e^\pm = M \pm \sqrt{M^2 - a^2\cos^2\theta}$

Tetrad

$$l^\mu\partial_\mu = \frac{r^2 + a^2}{\Delta}\partial_t + \partial_r + \frac{a}{\Delta}\partial_\phi$$

$$n^\mu\partial_\mu = \frac{r^2 + a^2}{2\Sigma}\partial_t - \frac{\Delta}{2\Sigma}\partial_r + \frac{a}{2\Sigma}\partial_\phi$$

Integrability of the system

4D space: a geodesic $\mathcal{L}$ has **8 degrees of freedom** x^μ, p^μ

Symmetries of spacetime

- Stationarity: $\xi^\mu \partial_\mu = \partial_t$
- Axisymmetry: $\chi^\mu \partial_\mu = \partial_\phi$
- Rank-2 Killing tensor

$$K_{\mu\nu} = r^2 g_{\mu\nu} + 2\Sigma l_{(\mu} n_{\nu)}$$
$$\nabla_{(\mu} K_{\nu\rho)} = 0$$

Constants of motion on $\mathcal{L}$

- Energy $E = -\xi^\mu p_\mu$
- Angular momentum $L = \chi^\mu p_\mu$
- Hamiltonian $H = p_\mu p^\mu / 2$
- Carter constant [Carter '68]

$$K = K_{\mu\nu} p^\mu p^\nu$$

The system is integrable thanks to this additional symmetry $\longrightarrow$ **analytical computation of the critical curve**

Killing tower for a Kerr black hole

- Conformal closed Killing-Yano 2-form (principal tensor):

$$p_{\mu\nu}\, \mathrm{d}x^\mu \wedge \mathrm{d}x^\nu = a\cos\theta\sin\theta\,\mathrm{d}\theta \wedge \left[(r^2+a^2)\,\mathrm{d}\phi - a\,\mathrm{d}t\right] + r\,\mathrm{d}r \wedge \left(a\sin^2\theta\,\mathrm{d}\phi - \mathrm{d}t\right)$$

- Killing Yano tensor $f_{\mu\nu} = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}p^{\rho\sigma}$
- Killing tensor $K_{\mu\nu} = f_{\mu\alpha}f^{\alpha}{}_{\nu}$ and conformal Killing tensor $Q_{\mu\nu} = p_{\mu\alpha}p^{\alpha}{}_{\nu}$
- Recover Killing vectors: $\xi^\mu\partial_\mu = \frac{1}{3}\nabla_\nu p^{\nu\mu}\partial_\mu = \partial_t\,, \chi^\mu\partial_\mu = -K^\mu{}_\nu\xi^\nu\partial_\mu = \partial_\phi$
- Link to the Petrov type D of spacetime [Kubiznak, Frolov '07]

This tower is the reason for

- complete integrability of the geodesic motion
- integrability of the motion of spinning particles [Ramond '22]
- separability of the perturbation equations for spin s [Teukolsky '72; Kalnins et al. '89]

Roadmap for generalizing Kerr

No-hair theorem

Kerr is the unique asymptotically flat stationary vacuum solution of GR

Modified gravity

- Extensions of General Relativity [Deffayet et al. '11; Langlois, Noui '16]
- Disformal transformations [Anson et al. '21; Ben Achour et al. '20]
- *ad hoc* parametrizations of the metric functions [Johannsen, Psaltis '10]
- **This work**: focus on parametrizing without requiring a specific action

Preserving symmetries

- Plebanski-Demianski family that maintains symmetries (Petrov type D) [Plebanski, Demianski '76]
- Preserving separability of Hamilton-Jacobi [Carter '68]
- **This work**: unique spacetime preserving the Killing tower [Krtous et al. '07]

Existing parametrizations

$$\boxed{\mathrm{d}s^2 = g_{tt}\,\mathrm{d}t^2 + 2g_{t\phi}\,\mathrm{d}t\,\mathrm{d}\phi + g_{rr}\,\mathrm{d}r^2 + g_{\theta\theta}\,\mathrm{d}\theta^2 + g_{\phi\phi}\,\mathrm{d}\phi^2}$$ [Yagi et al. '24]

$$g_{tt} = -\frac{\tilde{\Sigma}(A_5 - a^2 A_2^2 \sin^2\theta)}{\rho^4}\,, \quad g_{t\phi} = \frac{aA_5(A_5 - A_0)\tilde{\Sigma}\sin^2\theta}{\rho^4}\,, \quad g_{\phi\phi} = \frac{\tilde{\Sigma}\sin^2\theta A_5(A_1^2 - a^2 A_5 \sin^2\theta)}{\rho^4}$$

$$\tilde{\Sigma} = r^2 + a^2\cos^2\theta + f(r) + g(\theta)\,, \quad g_{\theta\theta} = \tilde{\Sigma}\,, \quad g_{rr} = \frac{\tilde{\Sigma}}{A_5}$$

$$\rho^4 = a^4 A_2^2 A_5 \sin^4\theta + a^2\sin^2\theta(A_0^2 - 2A_0A_5 - A_1^2A_2^2) + A_1^2 A_5\,.$$

- Provides both radial and polar deformations
- General case: no integrability

- No parametrization in the literature has radial deformations, polar deformations and integrability

Kerr-off-shell family

Kerr off-shell: most general spacetime beyond Kerr
with **Killing tower preserved** [Krtous et al. '07]

$$\mathrm{d}s^2 = -\frac{\Delta_r(r)}{\Sigma}\left(\mathrm{d}\tau + y^2\mathrm{d}\varphi\right)^2 + \frac{\Delta_y(y)}{\Sigma}\left(\mathrm{d}\tau - r^2\mathrm{d}\varphi\right)^2 + \frac{\Sigma}{\Delta_r(r)}\mathrm{d}r^2 + \frac{\Sigma}{\Delta_y(y)}\mathrm{d}y^2\,, \quad \Sigma = r^2 + y^2$$

Observations

- Does not rely on a specific action
- Provides both radial and polar deviations
- Integrability makes the critical curve computation **analytical**

Theory

- Preserves the Killing tower of Kerr: expect the same properties (Teukolsky equation, etc.)
- Preserves the Petrov type D of Kerr
- Recover Kerr in Boyer-Lindquist with

$$\Delta_r = r^2 - 2Mr + a^2\,, \quad \Delta_y = a^2 - y^2\,,$$
$$y = a\cos\theta\,, \quad \varphi = \phi/a\,, \quad \tau = t - a\phi\,.$$

Concrete examples

Radial deformations

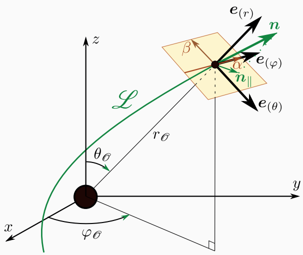
- $\Delta_r(r) = r^2 - 2Mr + a^2 + \alpha M^2$: Kerr-MOG [Moffat '15]
- $\Delta_r(r) = r^2 + a^2 - 2rMe^{-\ell M/r}$: Simpson-Visser regular model [Simpson, Visser '22]
- $\Delta_r(r) = r^2 - 2Mr + a^2 + qM^2 \log\left(\frac{r}{M}\right)$: logarithmic corrections

Polar deformations

- $\Delta_y(y) = a^2 - y^2 + py^4$: corrections that maintain the y-parity

Observation of the curve

- Conserved quantities: E and L from Killing vectors, K (generalized Carter constant) from the Killing tensor
- Geodesics labelled by $(\ell, k) = (L/E, K/E^2)$
- zero-angular-momentum observer at $(r_\mathcal{O}, y_\mathcal{O})$ with $r_\mathcal{O} \gg r_+$
- 2D coordinates (α, β) on the screen [Bardeen '73]



[Gourgoulhon '25]

- **Spherical photon orbits** form a family parametrized by the radius r_0:

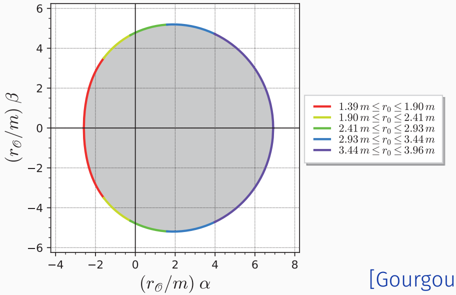
$$\ell(r_0) = r_0 \left(r_0 - \frac{4\Delta_r(r_0)}{\Delta_r'(r_0)} \right), \qquad k(r_0) = \frac{16 r_0^2 \Delta_r(r_0)}{\Delta_r'(r_0)}$$

Projection on the screen

Critical geodesics (infinitesimally close to spherical) impact the screen at large $r_{\mathcal{O}}$:

$$\alpha(r_0) = -\frac{\sqrt{\Delta_y(y_{\mathcal{O}})}}{r_{\mathcal{O}}}\left(1 + \frac{y_{\mathcal{O}}^2 + \ell(r_0)}{\Delta_y}\right), \quad \beta(r_0) = \pm\frac{1}{r_{\mathcal{O}}}\left(k(r_0) - \frac{(y_{\mathcal{O}}^2 + \ell)^2}{\Delta_y}\right)^{1/2}.$$

$\longrightarrow$ parametric analytical expression describing the critical curve

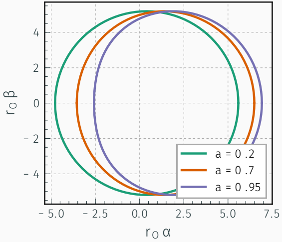


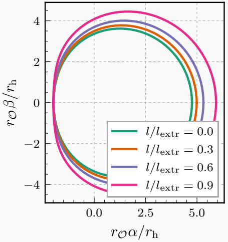
[Gourgoulhon '25]

Predicted photon rings

Kerr

- Recover the circle shape of Schwarzschild when $a \to 0$
- Asymmetric deformation when the spin increases





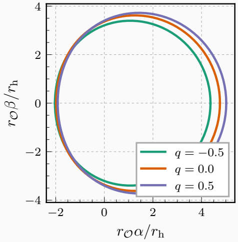
Simpson-Visser

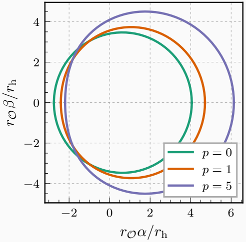
- New parameter ℓ affects the global scale of the curve
- No deformation of the Kerr shape

More examples

Log deviations

- Global modification of both the scale and the shape
- Effect barely visible at low values of q





Polar deformation

- Deformation similar to the one induces by the log term
- High values of p required to see differences

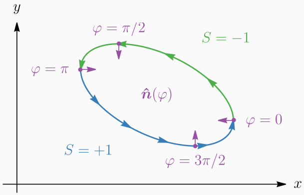
Building an observable

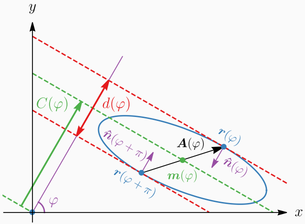
Quantitative description?

- Position $(\alpha = 0, \beta = 0)$ not available on a screen
- Interferometric observation does not yield directly $(\alpha(r_0), \beta(r_0))$
- Parametrization of closed convex curves [Gralla, Lupsasca '20]

$\longrightarrow$ describe the curve by

$$f(\varphi) = \frac{d(\varphi)}{2} + C(\varphi)$$

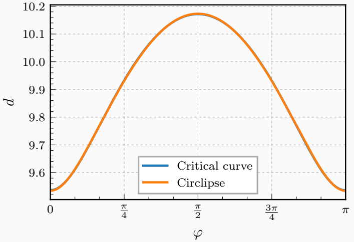




The circlipse parametrization

Parametrize the convex hull d by a circlipse shape [Gralla, Lupsasca '20]

$$f(\varphi) = \underbrace{R_0 + \sqrt{R_1^2 \sin^2\varphi + R_2^2 \cos^2\varphi}}_{d(\varphi)/2} + \underbrace{(X - \chi)\cos\varphi + \arcsin(\chi \cos\varphi)}_{C(\varphi)}$$



a	y_O	R_0	R_1	R_2	Residuals
0.95	0.6	9.23	0.94	0.30	10^{-4}
0.5	0	9.51	0.82	0.72	10^{-4}

$\longrightarrow$ the shape is sufficient to describe all GR parameter space

Degeneracy beyond GR

Beyond-GR modifications are degenerate with GR parameters variation:

Parameters	M	a	ℓ	$y_{\mathcal{O}}$	R_0	R_1	R_2	Residuals
Kerr	1	0.2		0	10.0	0.394	0.370	1.88×10^{-5}
SV	1.32	0.1	0.82	0	10.0	0.402	0.380	1.53×10^{-5}

⟶ cannot discriminate up to 10^{-2} between Kerr and Simpson-Visser

⟶ Independent measurements of mass and spin required to test General Relativity

Conclusion

- Our work: consider the unique generalization of Kerr that preserves its Killing tower symmetry
- This yields an analytical computation of the critical curve for many different deformations Δ_r and Δ_y
- Degeneracy is expected between modified gravity effects and inner properties of the black hole
- Prospectives:
    - use the simple mathematical properties of geodesic motion to get analytical results for finite n subrings
    - find modified theories of gravity with solutions belonging to the Kerr off-shell family

Thank you for your attention!