Exercise 2

Let's consider a Boltzmann-like equation

$$\partial_k \mathcal{H} \cdot \partial_x f_\gamma - \partial_x \mathcal{H} \cdot \partial_k f_\gamma = g_{a\gamma\gamma}^2 |k \cdot F_{ext} \cdot \varepsilon|^2 2\pi \delta\left(E_\gamma^2(k,x) - E_\phi^2(k)\right) f_\phi$$

where $\partial_k = \frac{\partial}{\partial k^\mu}$ $\quad \partial_x = \frac{\partial}{\partial x^\mu}$ $\quad f_\gamma = f_\gamma(k^\mu, x^\mu)$.

In this expression the parametric equation for the photon worldline in phase-space is given by $(k^\mu(\lambda), x^\mu(\lambda))$.

a) Let's consider the l.h.s. of the Boltzmann-like equation. Along the photon worldline, classical Hamilton's equations apply:

$$\frac{\partial \mathcal{H}}{\partial k^\mu} = \frac{\partial x^\mu}{\partial \lambda} \qquad \frac{\partial \mathcal{H}}{\partial x^\mu} = -\frac{\partial k^\mu}{\partial \lambda}$$

~~es~~ Then, the l.h.s. becomes

$$\frac{\partial x^\mu}{\partial \lambda} \partial_x f_\gamma + \frac{\partial k^\mu}{\partial \lambda} \partial_k f_\gamma = \frac{df(k^\mu(\lambda), x^\mu(\lambda))}{d\lambda}$$

and we end up with:

$$\frac{df(k^\mu(\lambda), x^\mu(\lambda))}{dk} = g_{a\gamma\gamma}^2 |k \cdot F_{ext} \cdot \varepsilon|^2 2\pi \delta\left(E_\gamma^2(\bar{k}, \bar{x}) - E_\phi^2(R)\right) f_\phi$$

b) Let's denote with (k_c, x_c) the point of the phase space where the resonance occurs, i.e. $E_\gamma^2(\bar{k}_c, \bar{x}_c) - E_\phi^2(\bar{k}_c) = 0$. In this expression we have defined $\bar{k}_c = \bar{k}(\lambda_c)$ and $\bar{x}_c = \bar{x}(\lambda_c)$.

Let me recall the properties of the delta function:

$$\delta(f(x)) = \sum_i \frac{\delta(x-x_i)}{|f'(x_i)|}, \quad \text{where } f'(x_i) = \frac{df}{dx}\Big|_{x=x_i} \text{ and } f(x_i)=0$$

In our case, at the resonance $E_\gamma^2(k(\lambda_c), x(\lambda_c)) - E_\phi^2(k(\lambda_c)) = 0$

$$\Rightarrow \frac{d}{d\lambda}\left(E_\gamma^2(\lambda) - E_\phi^2(\lambda)\right)\Big|_{\lambda=\lambda_c} = 2E_\gamma\left(E_\gamma' - E_\phi'\right)$$

where all the quantities are evaluated at the resonance.

Then by integrating the equation we have:

$$f_\gamma(k_c, x_c) = g_{\phi\gamma\gamma}^2 \pi |k \cdot F_{ext} \cdot \varepsilon|^2 \frac{1}{E_\gamma |E_\gamma' - E_\phi'|} f_\phi(k_c, x_c)$$

Now, under the assumption of a stationary background, the photon energy must be conserved along its worldline $\Rightarrow E_\gamma' = 0$.

Moreover, since $E_\phi^2 = \bar{k}(\lambda)\cdot\bar{k}(\lambda) + m_\phi^2$

$$\Rightarrow 2E_\phi E_\phi' = 2\bar{k}(\lambda)\cdot\bar{k}'(\lambda) \Rightarrow E_\phi' = \frac{\bar{k}(\lambda)\cdot\bar{k}'(\lambda)}{E_\phi} = -\frac{\bar{k}\cdot\partial_{\bar{x}}\mathcal{H}}{E_\phi} =$$

$$= -\bar{v}_p \cdot \partial_{\bar{x}}\mathcal{H}$$

Now, from the chain rule $\partial_{\bar{x}}\mathcal{H} = \partial_{k_0}\mathcal{H}\, \nabla_{\bar{x}} E_\gamma$. So in the end we have:

$$P_{\phi\gamma} = \frac{f_\gamma(k_c, x_c)}{f_\phi(k_c, x_c)} = \frac{g_{\phi\gamma\gamma}^2 \pi |k \cdot F_{ext} \cdot \varepsilon|^2}{E_\gamma\, \partial_{k_0}\mathcal{H}\, |\bar{v}_p \cdot \nabla_{\bar{x}} E_\gamma|}$$