Exercises - WG3 - WISPS and Compact Object

You may not have time to complete all of these exercises, so consider picking a subset which interests you most. You are encouraged you to work in groups of $\sim$ 2 to 4 to aid collaboration.

1. Superradaint Scatttering off a Rotating Cylinder.

[*You may use a computer algebra package such as Mathematica in this exercise.*]

Consider the problem discussed in the lectures, which describes a rotating conducting cylinder, for which the governing equation can be written as

$$\Box\phi - \sigma(\Omega\partial_\varphi - \partial_t)\phi = 0, \tag{1}$$

where σ is constant for cylindrical radial distances $r < R$ and zero for $r > R$.

(a) Assume an infinitely tall cylinder, such that you can choose a z-independent separable solution of the form $\phi = \phi(t, r, \varphi) = \psi(r)e^{im\varphi}e^{i\omega t}$, where m is an integer. Obtain a radial equation for $\psi(r)$.

(b) Solve this radial equation to obtain an expression for $\psi(r)$ in terms of known functions. You will need to impose sensible boundary conditions, e.g., regularity at the origin, continuity across $r = R$, etc.

(c) By using the asymptotic forms of these known functions as $r \to \infty$, express the exterior solutions at $r \to \infty$ as a sum of outgoing and ingoing waves, with amplitudes $A_{\rm out}$ and $A_{\rm in}$ respectively.
[*hint: it may be useful to express the asymptotic solutions in terms of Hankel functions*]

(d) Finally, derive an expression for $Z = 1 - |A_{\rm out}|^2/|A_{\rm in}|^2$ in the small $\omega R \ll 1$ limit to linear order in σ.

2. Resonant Axion-Photon Conversion in an Inhomogeneous 3D Medium

Consider the Boltzmann-like equation we will see in lectures:

$$\partial_k\mathcal{H} \cdot \partial_x f_\gamma - \partial_x\mathcal{H} \cdot \partial_k f_\gamma = g_{a\gamma\gamma}^2 \left|k \cdot F_{\rm ext} \cdot \varepsilon\right|^2 2\pi\delta\left(E_\gamma(\mathbf{k}, x)^2 - E_\phi(\mathbf{k})^2\right) f_\phi\,. \tag{2}$$

Here $k = k_\mu$ and $x = x^\mu$ are the 4-momentum and spacetime position of photons respectively, which have a phase-space density $f = f_\gamma(k, x)$. Note ε is the 4-polarisation vector of the photon and f_ϕ gives the phase-space density of axions.

(a) Solve this equation by a method of characteristics by using characterstic curves $x = x(\lambda)$ and $k = k(\lambda)$ which give the characteristic curves of photons corresponding to the operator appearing on the left-hand side of Eq. (2). What is the interpretation of the charactersitic equations for $x'(\lambda)$ and $k'(\lambda)$, have you seen such equations before? You should find that you end up with an equation for the form

$$\frac{df(k(\lambda), x(\lambda))}{d\lambda} = \cdots \tag{3}$$

(b) By integrating this equation and assuming a stationary background, derive the following expression for the conversion probability $P_{a\gamma} = f_\gamma(k_c, x_c)/f_\phi(k_c, x_c)$ where (k_c, x_c) are the points where the resonance occurs, i.e. where the argument of the delta-function above vanishes.

$$P_{a\gamma} = \frac{\pi g_{a\gamma\gamma}^2 \left|k \cdot F_{\text{ext}} \cdot \varepsilon\right|^2}{E_\gamma \partial_{k_0} \mathcal{H} \left|\mathbf{v}_p \cdot \nabla_{\mathbf{x}} E_\gamma(\mathbf{k}, \mathbf{x})\right|}, \tag{4}$$

where $\mathbf{v}_p = \mathbf{k}/k_0$ is the phase-velocity. You may find the chain-rule result $\partial_{\mathbf{x}} \mathcal{H}/\partial_{k_0} \mathcal{H} = \nabla_{\mathbf{x}} E_\gamma$ helpful.

3. Radio telescope sensitivity to axion dark matter

In general, the Hamiltonian $\mathcal{H}$ for a photon in a magnetised plasma can be quite complicated. Instead we'll do something simpler and assume an weakly magnetised plasma where

$$\mathcal{H} = k_\mu k^\mu + \omega_p^2, \tag{5}$$

where ω_p is the plasma mass. We will model the plasma around a neuton star as spherical toy model:

$$\omega_p^2 = \frac{4\pi \alpha n_e}{m_e}, \qquad n_e = \frac{2\Omega B}{e}, \qquad B = B_s \left(\frac{R}{r}\right)^3, \tag{6}$$

where R is the neutron star radius, $\Omega = 2\pi/P$ is the angular frequency with which the neutron star is rotating, and P is its corresponding period. B_s is the surface magnetic field. Let's assume a stationary setup, in which case, by integrating Eq. (9) over phase space, we arrive at[1]

$$\int \mathrm{d}^3\mathbf{k} \int \mathrm{d}\mathbf{A} \cdot \mathbf{v}_g \, \omega f_\gamma = \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}\mathbf{\Sigma}_{\mathbf{k}} \cdot \mathbf{v}_p \omega P_{a\gamma} f_\phi \equiv \mathcal{P} \tag{7}$$

[1] You can show this in the bonus exercise of this problem sheet.

where $\mathcal{P}$ is the power (i.e. energy per unit time) produced by axions converting into photons. Hence by deriving an expression for the right-hand side of (7), you will be able to derive an expression for the total power $\mathcal{P}$ emitted by resonantly produced photons. You may estimate the flux density (that is, power, per unit area, per unit freqeuncy) arriving on earth as

$$\mathcal{S} = \frac{\mathcal{P}}{4\pi d^2}\frac{1}{\Delta f}, \tag{8}$$

where d is the distance to source, and Δf is the bandwidth of the signal, which for an axion line signal you can take to be $\Delta f = v_0^2\, m_a$.

(a) Argue that for the toy model above, the critical surface is spherical, with radius r_c, and drive an expression of r_c in terms of m_a and other quantities.

(b) By evaluating the expression in the right-hand side of Eq. (7) using the model described above, obtain, and evaluate an integral for the total power $\mathcal{P}$ in this model. You may take the axion density to be $f_a(\mathbf{x}, \mathbf{k}) = v_a \rho_{\rm DM}^{r_c}/m_a \, \frac{\delta(|\mathbf{k}| - \omega_c)}{4\pi k^2}$ where $\omega_c = \sqrt{m_a^2 + k_c^2}$ and $k_c = m_a v_a$. Where $\rho_{\rm DM}^{r_c} = \rho_{\rm DM}^{\infty} \frac{2}{\sqrt{\pi}} \frac{1}{v_0} \sqrt{\frac{2GM_{\rm NS}}{r_c}}$.

$\Big[$*hint: it may be useful to express the conversion probability in the following form* $P_{a\gamma} = \frac{\pi}{2} \frac{g_{a\gamma\gamma}^2 \left|\mathbf{B}_{\rm ext} \cdot \boldsymbol{\varepsilon}\right|^2}{|\boldsymbol{v}_p \cdot \nabla_{\boldsymbol{x}} E_\gamma(\boldsymbol{k}, \boldsymbol{x})|}$. $\Big]$

(c) Finally, let's consider a pulsar PSR J2144-3933 which is $d = 180\,$parsec from Earth. You can assume a stellar radius of $R = 10\,$km, $v_a \simeq \sqrt{\frac{GM}{r_c}}$, $B_s = 2 \times 10^{12}\,$Gauss, and $\rho_{\rm DM}^{\infty} \simeq 0.45 \times$ GeV cm^{-3}, $P = 8.5$s and $v_0 \sim 200$ km/sec. Using these numbers and the results above, derive an expression for $\mathcal{S}$ for a generic axion mass.

(d) Finally, the minimal detectable signal is

$$S_{\rm min} = {\rm SNR}_{\rm min} \frac{\rm SEFD}{\sqrt{n_{\rm pol} \Delta f t_{\rm obs}}}.$$

By taking a typical system equivalent flux density of a telescope to be SEFD = 2 Jy, an observing time of 100 hours and an SNR of 3, derive the value of $g_{a\gamma\gamma}$ to which you would be sensitive for an axion mass of $m_a = \mu eV$. You may also wish to find a generic expression for fiducial values of different parameters of the form $g_{a\gamma\gamma} \sim (\mu eV/m_a)^{\rm index} (B_a/10^{14})^{\rm index} \cdots$ etc to get some feel for the scaling.

4. Bonus Question - Continuity Equation

This question allows you to derive the result (7) from first principles. By integrating Eq. (2) over a *finite* spatial 3-volume $\mathcal{V}$, whose bounding surface has area element $\mathrm{d}A$, and a region in 4-momentum space $\int \mathrm{d}^4 k$ running over all momenta. Derive the following equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\int \mathrm{d}\mathcal{V} \int \mathrm{d}^3\mathbf{k}\, \omega f_\gamma + \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}\mathbf{A}\cdot\mathbf{v}_g\, \omega f_\gamma + \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}\mathcal{V}\, \partial_t E_\gamma f_\gamma = \int \mathrm{d}\mathcal{V}\, Q, \tag{9}$$

You can use the chain-rule definitions, $\partial_{\mathbf{k}}\mathcal{H}/\partial_{k_0}\mathcal{H} = \mathbf{v}_g$ and $\partial_{\mathbf{x}}\mathcal{H}/\partial_{k_0}\mathcal{H} = \nabla_{\mathbf{x}} E_\gamma$. Where

$$Q = \int \mathrm{d}^3\mathbf{k}\, \omega g_{a\gamma\gamma}^2 (k \cdot \tilde{F}_{\mathrm{ext}} \cdot \varepsilon^c)^2 2\pi\delta\left(E_\gamma(\mathbf{k}, x)^2 - E_\phi(\mathbf{k})^2\right) \frac{1}{\partial_{k_0}\mathcal{H}} f_\phi. \tag{10}$$

Next, show that

$$\int \mathrm{d}\mathcal{V}\, Q = \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}\Sigma_{\mathbf{k}}\, \omega\, \frac{\pi g_{a\gamma\gamma}^2 \left|k \cdot \tilde{F}_{\mathrm{ext}} \cdot \varepsilon\right|^2}{E_\gamma \partial_{k_0}\mathcal{H}\, |\nabla_{\mathbf{x}} E_\gamma|} f_\phi \equiv \int \mathrm{d}^3\mathbf{k} \int \mathrm{d}\mathbf{\Sigma}_{\mathbf{k}}\cdot\mathbf{v}_p \omega P_{a\gamma} f_\phi \tag{11}$$

You can make use of the identity:

$$\int \mathrm{d}^n\mathbf{x}\, \delta(G(\mathbf{x})) = \int_{G^{-1}(0)} \mathrm{d}\Sigma\, \frac{1}{|\nabla G|}, \tag{12}$$

where $\mathrm{d}\Sigma$ is the area element on the surface defined by $G(\mathbf{x}) = 0$ and where $\Sigma_{\mathbf{k}}$ is the spatial surface on which $E_\gamma(\mathbf{k}, \mathbf{x}) = E_\phi(\mathbf{k})$. What's the interpretation of $\Sigma_{\mathbf{k}}$?

(b) What is the interpretation of Eq. (9) ? Staring at the connection between Eqs. (4) and (11).

[*hint : you can cheat at look at question 3 for a clue on interpretation.*]