Multiple mountains on a pulsar: implications for gravitational waves and the spin-down rate

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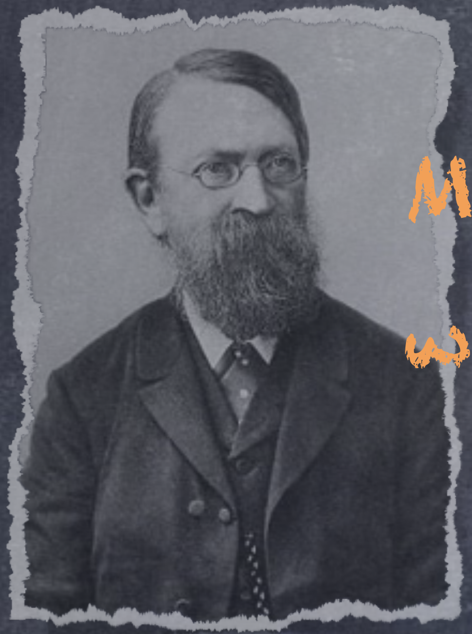
Jordan-Fierz-Brans-Dicke theory

Newton believed that inertial forces such as centrifugal forces must arise from acceleration with respect to "absolute space".

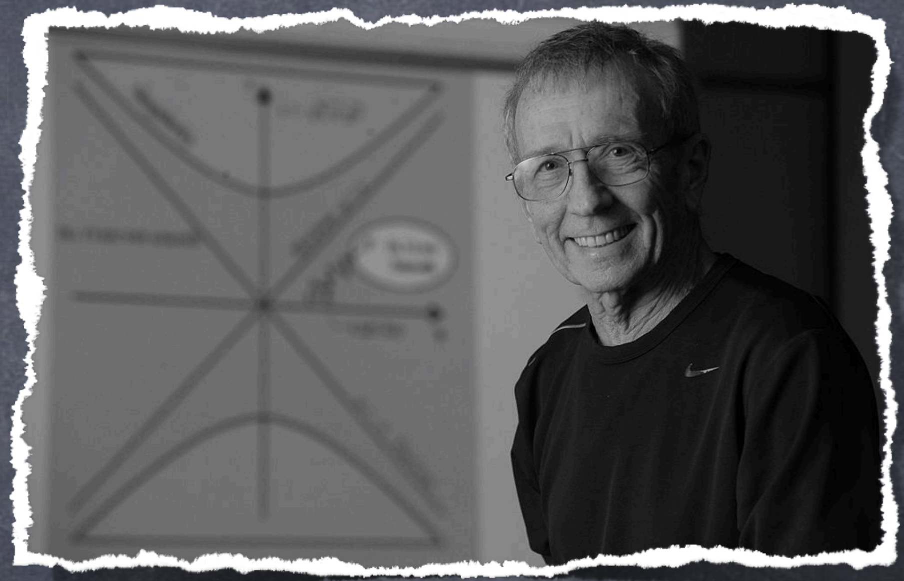


Mach argued that they were more likely caused by acceleration with respect to the mass of the celestial bodies.

In Brans-Dicke (BD) theory:

- G is not a constant
- G is determined by the totality of the matter in the universe through an auxiliary field equation
- This theory still has general coordinate invariance but it has an additional degree of freedom (ϕ)

$$G \rightarrow G(x) = \frac{1}{\phi(x)}$$



Carl H Brans

Robert H Dicke

The action in BD theory is given by

$$S = S_g[g_{\mu\nu}, \phi] + S_m[\psi_m, g_{\mu\nu}]$$

where

$$S_g \equiv \frac{1}{16\pi}\int d^4x\sqrt{-g}\left[\phi R - \frac{\omega}{\phi}\nabla^\mu\phi\,\nabla_\mu\phi\right] \qquad S_m \equiv \int d^4x\sqrt{-g}\,\mathscr{L}_m\left[\psi_m, g_{\mu\nu}\right]$$

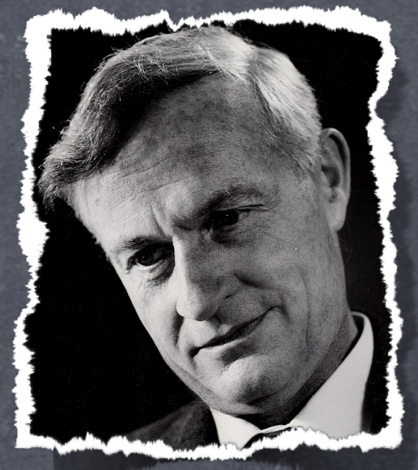
Pascual Jordan

Equations obtained after varying the action are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi}{\phi}T_{\mu\nu} + \frac{\omega}{\phi^2}\left[\partial_\mu\phi\partial_\nu\phi - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi\right] + \frac{1}{\phi}\left[\nabla_\mu\partial_\nu\phi - g_{\mu\nu}g^{\alpha\beta}\nabla_\alpha\partial_\beta\phi\right]$$

$$R - \frac{\omega}{\phi^2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \frac{2\omega}{\phi}g^{\mu\nu}\nabla_\mu\partial_\nu\phi = 0$$

ϕ = scalar field
ω = parameter
ψ_m = matter field
$\mathscr{L}_m$ =



Markus Fierz

$$T_{\mu\nu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta S_m}{\delta g^{\mu\nu}}$$

In GR, the field equation is

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \frac{8\pi G}{c^4}T_{\mu\nu}$$

We linearize the field equations and obtain

$$h_{\mu\nu} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & h_+(t) + h_S(t) & h_\times(t) & 0 \\ 0 & h_\times(t) & -h_+(t) + h_S(t) & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $\left(h_S \equiv -\frac{\delta\phi}{\phi_0} \right)$

$$\Box h_{\mu\nu} = -16\pi G \left(\frac{3+2\omega}{4+2\omega} \right) \left[T_{\mu\nu} - \eta_{\mu\nu} \frac{1+\omega}{3+2\omega} T^\lambda_\lambda \right]$$

solution

$$\begin{cases} h_+(t) = \dfrac{G}{rc^4}(1-\zeta)(\ddot{Q}^{xx}_W(t') - \ddot{Q}^{yy}_W(t')) \\ h_\times(t) = \dfrac{2G}{rc^4}(1-\zeta)\ddot{Q}^{xy}_W(t') \end{cases}$$

$$\Box(\delta\phi) = \frac{8\pi}{3+2\omega} T^\lambda_\lambda$$

solution

$$h_S(t) = \frac{2G}{rc^2}\zeta \left[M(t') + \frac{1}{c}\dot{D}^z_W(t') - \frac{1}{2c^2}\ddot{Q}^{zz}_W(t') \right]$$

t' is the retarded time

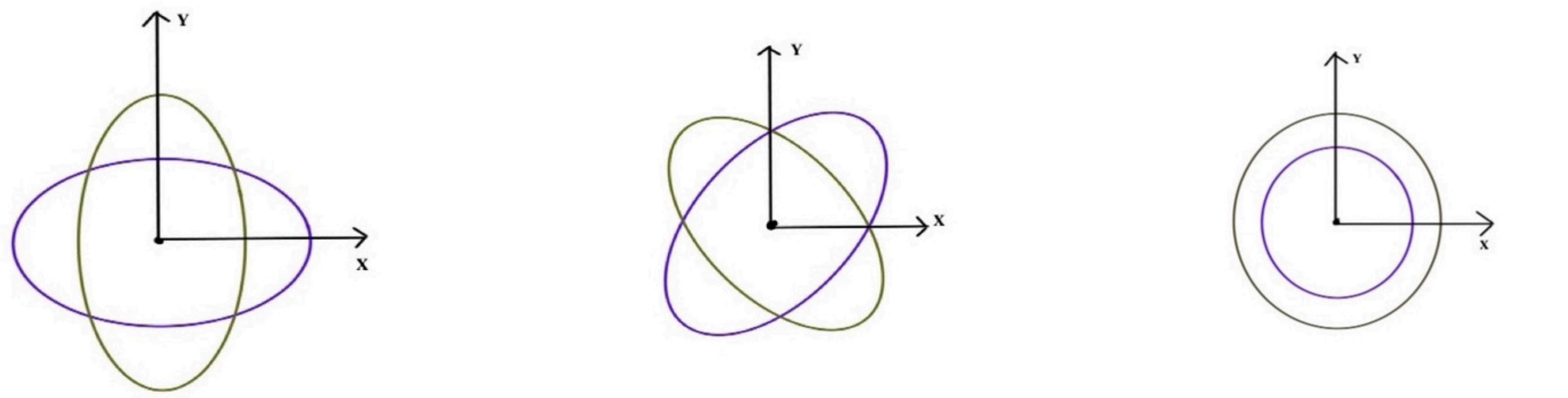
r is the distance of the source

$\ddot{Q}^{ij}_W$ is the second time derivative of the mass quadrupole moment in the wave frame

$\dot{D}^i_W$ is the first time derivative of the mass dipole moment in the wave frame

M is the mass monople moment

$$\zeta \equiv \frac{1}{2\omega + 4}$$



Plus polarization (h_+) Cross polarization $(h_\times)$ Scalar polarization (h_S)

h_+ & $h_\times$:

- Tensor polarizations in GR
- Dominated by time-varying quadrupole moment
- 2-fold symmetric
- Ellipses preserve the area

h_S :

- Scalar polarization in BD
- Dominated by time-varying dipole moment
- ∞-fold symmetric

Multiple mountains on a pulsar

Let the mass of the k^{th} mountain be m_k and its coordinates are

$$x_k = a\sin\theta_k\cos\phi_k$$
$$y_k = a\sin\theta_k\sin\phi_k$$
$$z_k = a\cos\theta_k$$

a is radius of the star

$\theta_k \in [0,\pi]$

$\phi_k \in [0,2\pi)$

Dipole moment and quadrupole moment in the source frame are given by:

$$D_s^i = \int \rho_k x^i dxdydz$$

$$Q_s^{ij} = \int \rho_k \left[x^i x^j - \frac{1}{3}r^2\delta_{ij}\right] dxdydz$$

where $\rho_k = m_k\delta(x - x_k)\delta(y - y_k)\delta(z - z_k)$

We obtain,

$$Q_s^{ij} = m_k a^2 \begin{bmatrix} \sum_{k=1}^{N} (\sin^2\theta_k \cos^2\phi_k - \frac{1}{3}) & \sum_{k=1}^{N} \frac{1}{2}\sin^2\theta_k \sin 2\phi_k & \sum_{k=1}^{N} \frac{1}{2}\sin 2\theta_k \cos\phi_k \\ \sum_{k=1}^{N} \frac{1}{2}\sin^2\theta_k \sin 2\phi_k & \sum_{k=1}^{N} (\sin^2\theta_k \sin^2\phi_k - \frac{1}{3}) & \sum_{k=1}^{N} \frac{1}{2}\sin 2\theta_k \sin\phi_k \\ \sum_{k=1}^{N} \frac{1}{2}\sin 2\theta_k \cos\phi_k & \sum_{k=1}^{N} \frac{1}{2}\sin 2\theta_k \sin\phi_k & \sum_{k=1}^{N} (\cos^2\theta_k - \frac{1}{3}) \end{bmatrix}$$

$$D_s = \begin{bmatrix} \sum_{k=1}^{N} m_k x_k \\ \sum_{k=1}^{N} m_k y_k \\ \sum_{k=1}^{N} m_k z_k \end{bmatrix}$$

We move to the wave frame using following transformations

$$D_W(t) = S \cdot R(t) \cdot D_s$$

$$Q_W(t) = S \cdot R(t) \cdot Q_s \cdot R(t)^T \cdot S^T$$

where

$$R(t) \equiv \begin{bmatrix} \cos\omega t & -\sin\omega t & 0 \\ \sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$S \equiv \begin{bmatrix} \cos\iota & 0 & -\sin\iota \\ 0 & 1 & 0 \\ \sin\iota & 0 & \cos\iota \end{bmatrix}$$

ω is the angular frequency of the star and ι is the inclination angle.
This allows us to write the three polarisation states as:

$$h_S(t) \approx -\frac{2G}{rc^3}\zeta\omega a \sin\iota \sum_{k=1}^{N} m_k \sin\theta_k \sin\left(\omega t' + \phi_k\right)$$

Scalar polarization (8)

Tensor polarizations

$$h_+(t) = -\frac{2G(1-\zeta)}{rc^4}\omega^2 a^2 \left(1+\cos^2\iota\right)\sum_{k=1}^{N} m_k \cos\left(2\omega t' + 2\phi_k\right)\sin^2\theta_k$$

$$+\frac{G(1-\zeta)}{2rc^4}\omega^2 a^2 \sin 2\iota \sum_{k=1}^{N} m_k \cos\left(\omega t' + \phi_k\right)\sin 2\theta_k,$$

$$h_\times(t) = -\frac{4G(1-\zeta)}{rc^4}\omega^2 a^2 \cos\iota \sum_{k=1}^{N} m_k \sin\left(2\omega t' + 2\phi_k\right)\sin^2\theta_k$$

$$+\frac{G(1-\zeta)}{rc^4}\omega^2 a^2 \sin\iota \sum_{k=1}^{N} m_k \sin\left(\omega t' + \phi_k\right)\sin 2\theta_k,$$

This is known as dual harmonic model

Power radiated

$$\frac{dP_{\text{grav}}}{dA} = \frac{c^3}{16\pi G(1-\zeta)} < \dot{h}_+^2(t) + \dot{h}_\times^2(t) + \left(\frac{1-\zeta}{\zeta}\right)\dot{h}_s^2(t) > \equiv \frac{dP^{(T)}}{dA} + \frac{dP^{(S)}}{dA}$$

where $dA = r^2 \int_{\rho=0}^{2\pi} \int_{\iota=0}^{\pi} \sin\iota d\iota d\rho$

After integration, we get

$$P^{(T)} = \frac{32}{5}\frac{G(1-\zeta)}{c^5}a^4\omega^6\left[\sum_{k=1}^{N}\sum_{j=1}^{N} m_k m_j \sin^2\theta_k \sin^2\theta_j \cos(2\phi_k - 2\phi_j)\right]$$

$$+\frac{1}{10}\frac{G(1-\zeta)}{c^5}a^4\omega^6\left[\sum_{k=1}^{N}\sum_{j=1}^{N} m_k m_j \sin 2\theta_k \sin 2\theta_j \cos(\phi_k - \phi_j)\right]$$

$$P^{(S)} = \frac{1}{3}\frac{G\zeta}{c^3}\omega^4 a^2 \sum_{k=1}^{N}\sum_{j=1}^{N} m_k m_j \sin\theta_k \sin\theta_j \cos(\phi_k - \phi_j)$$

(10)

Specific Examples of Mountain distribution

Two mountains

$$P^{(T)} = \frac{32}{5}\frac{G(1-\zeta)}{c^5}a^4\omega^6\left[m_1^2\sin^4\theta_1 + m_2^2\sin^4\theta_2 + 2m_1m_2\sin^2\theta_1\sin^2\theta_2\cos 2(\phi_1-\phi_2)\right]$$
$$+\frac{1}{10}\frac{G(1-\zeta)}{c^5}a^4\omega^6\left[m_1^2\sin^2 2\theta_1 + m_2^2\sin^2 2\theta_2 + 2m_1m_2\sin 2\theta_1\sin 2\theta_2\cos(\phi_1-\phi_2)\right]$$

$$P^{(S)} = \frac{1}{3}\frac{G\zeta}{c^3}a^2\omega^4\left[m_1^2\sin^2\theta_1 + m_2^2\sin^2\theta_2 + 2m_1m_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2)\right]$$

Let us assume that both mountains lie on the same latitude $(\theta_1 = \theta_2 \equiv \theta)$ and $|\phi_1 - \phi_2| = \frac{\pi}{2} \;\; or \frac{3\pi}{2}$

$$P^{(T)} = \frac{32}{5}\frac{G(1-\zeta)}{c^5}a^4\omega^6\sin^4\theta\left[m_1-m_2\right]^2 + \frac{1}{10}\frac{G(1-\zeta)}{c^5}a^4\omega^6\sin^2 2\theta\left[m_1^2+m_2^2\right]$$
$$P^{(S)} = \frac{1}{3}\frac{G\zeta}{c^3}a^2\omega^4\sin^2\theta\left[m_1^2+m_2^2\right]$$

If we assume $m_1 = m_2$ and $\theta = \frac{\pi}{2}$, power emitted in tensor waves is zero and only scalar waves carry away energy from the system.

Let us take the mass of the i^{th} mountain to be $m_i \equiv \epsilon_i M$, M being the mass of the pulsar. We can write GW strain as:

$$h_0^q = \frac{16\pi^2 G(1-\zeta)}{c^4}\frac{Ma^2 f_0^2}{r}\epsilon^q$$

where

$$\epsilon^q \equiv \left[\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1^2\epsilon_2^2\cos(2\phi_1 - 2\phi_2)\right]^{\frac{1}{2}}$$

$$h_0^d = \frac{4\pi G}{c^3}\zeta\frac{Maf_0}{r}\epsilon^d$$

where

$$\epsilon^d \equiv \left[\epsilon_1^2 + \epsilon_2^2 + 2\epsilon_1^2\epsilon_2^2\cos(\phi_1 - \phi_2)\right]^{\frac{1}{2}}$$

The strain h_0 is measured by detectors. So, we can use these strains to constrain the BD parameter using the relation:

$$\zeta = \frac{1}{1 + \frac{c}{4\pi a f_0}\frac{h_0^q \epsilon^d}{h_0^d \epsilon^q}}$$

Four mountains

$$P^{(T)} = \frac{32}{5}\frac{G(1-\zeta)}{c^5}a^4\omega^6\big[m_1^2\sin^4\theta_1 + m_2^2\sin^4\theta_2 + m_3^2\sin^4\theta_3 + m_4^2\sin^4\theta_4 + 2m_1m_2\sin^2\theta_1\sin^2\theta_2\cos 2(\phi_1-\phi_2) + 2m_1m_3\sin^2\theta_1\sin^2\theta_3\cos 2(\phi_1-\phi_3)$$
$$+2m_1m_4\sin^2\theta_1\sin^2\theta_4\cos 2(\phi_1-\phi_4) + 2m_2m_3\sin^2\theta_2\sin^2\theta_3\cos 2(\phi_2-\phi_3) + 2m_2m_4\sin^2\theta_2\sin^2\theta_4\cos 2(\phi_2-\phi_4) + 2m_3m_4\sin^2\theta_3\sin^2\theta_4\cos 2(\phi_3-\phi_4)\big]$$
$$+\frac{1}{10}\frac{G(1-\zeta)}{c^5}a^4\omega^6\big[m_1^2\sin^2 2\theta_1 + m_2^2\sin^2 2\theta_2 + m_3^2\sin^2 2\theta_3 + m_4^2\sin^2 2\theta_4 + 2m_1m_2\sin 2\theta_1\sin 2\theta_2\cos(\phi_1-\phi_2) + 2m_1m_3\sin 2\theta_1\sin 2\theta_3\cos(\phi_1-\phi_3)$$
$$+2m_1m_4\sin 2\theta_1\sin 2\theta_4\cos(\phi_1-\phi_4) + 2m_2m_3\sin 2\theta_2\sin 2\theta_3\cos(\phi_2-\phi_3) + 2m_2m_4\sin 2\theta_2\sin 2\theta_4\cos(\phi_2-\phi_4) + 2m_3m_4\sin 2\theta_3\sin 2\theta_4\cos(\phi_3-\phi_4)\big]$$

$$P^{(S)} = \frac{1}{3}\frac{G\zeta}{c^3}a^2\omega^4\big[m_1^2\sin^2\theta_1 + m_2^2\sin^2\theta_2 + m_3^2\sin^2\theta_3 + m_4^2\sin^2\theta_4 + 2m_1m_2\sin\theta_1\sin\theta_2\cos(\phi_1-\phi_2) + 2m_1m_3\sin\theta_1\sin\theta_3\cos(\phi_1-\phi_3)$$
$$+2m_1m_4\sin\theta_1\sin\theta_4\cos(\phi_1-\phi_4) + 2m_2m_3\sin\theta_2\sin\theta_3\cos(\phi_2-\phi_3) + 2m_2m_4\sin\theta_2\sin\theta_4\cos(\phi_2-\phi_4) + 2m_3m_4\sin\theta_3\sin\theta_4\cos(\phi_3-\phi_4)\big].$$

Let us assume that both mountains lie on the same latitude (θ) and
$|\phi_1-\phi_2| = \frac{\pi}{2}, |\phi_1-\phi_4| = \frac{\pi}{2}, |\phi_2-\phi_3| = \frac{\pi}{2}, |\phi_3-\phi_4| = \frac{\pi}{2}, |\phi_1-\phi_3| = \pi, |\phi_2-\phi_4| = \pi$, we get

$$P^{(T)} = \frac{32}{5}\frac{G(1-\zeta)}{c^5}a^4\omega^6\sin^4\theta\,\big[m_1-m_2+m_3-m_4\big]^2 + \frac{1}{10}\frac{G(1-\zeta)}{c^5}a^4\omega^6\sin^2 2\theta\,\big[(m_1-m_3)^2+(m_2-m_4)^2\big]$$
$$P^{(S)} = \frac{1}{3}\frac{G\zeta}{c^3}a^2\omega^4\sin^2\theta\,\big[(m_1-m_3)^2+(m_2-m_4)^2\big]$$

No tensor or scalar waves if all mountains of same mass!!!

Summary

1. We study the effect of multiple mountains on a pulsar on GW emission and spin-down rate.
2. We consider Brans-Dicke theory which has three polarization states.
3. Brans-Dicke theory reduces to general relativity by choosing $\zeta = 0$.
4. We observe that multiple irregularities on the surface can counterbalance each other and hence reduce the GW emission.

Thank you