QCD sum rules: basis and applications.

Carlos Albarrán Calvo

September 15, 2025

Introduction

- QCD sum rules were first invented by Shifmann, Vainstein and Zhakarov in the 80's.
- The thecnique's aim is to obtain analitic expressions for different hadronic parameters.
- They also have a series of advantages when compared to other methods.
- There are two types of sum rules in QCD...

SVZ sum rules

These are specially useful when trying to obtain decay constants of hadrons, as it is our case.

The starting point is a correlation function:

$$\Pi_{\mu\nu}(q) = i \int d^4x e^{iq\cdot x} \langle 0 | T\{j_\mu(x) j_\nu(0)\} | 0 \rangle = (q_\mu q_\nu - q^2 g_{\mu\nu}) \Pi(q^2) \; , \quad (1)$$

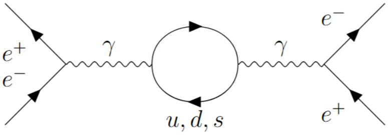

Figure 1: Feynmann diagram for the SVZ sum rule

We begin by considering we are in the high-energy regime ($Q^2 = -q^2 \gg \Lambda_{QCD}$) and using the optical theorem:

$$2Im\Pi_{\mu\nu}(q) = \sum_n \langle 0|j_\mu|n\rangle\langle n|j_\nu|0\rangle d\tau^n (2\pi)^4 \delta^{(4)}(q - p_n) \ . \qquad (2)$$

where n are all possible intermediate states. Let's only take into account a vectorial meson, like the pion.

One can use $\langle V(q)|j_\nu|0\rangle = f_V m_V \epsilon_\mu^{(\lambda)*}$. Adding up the polarizations, the imaginary part yields:

$$\frac{1}{\pi} Im\Pi(q^2) = f_V^2 \delta(q^2 - m_v^2) + \rho^h(q^2)\theta(q - s_0^h) \ . \qquad (3)$$

One now is to use a dispersion relation linking $q^2 > 0$ and $q^2 < 0$ regimes, as well as to obtain the whole correlation function and not only its imaginary part.

We do this by integrating in the complex plane, yielding:

$$\Pi(q^2) = \frac{q^2 f_V^2}{m_V^2(m_V^2 - q^2)} + q^2 \int_{s_0^h}^{\infty} ds \frac{\rho^h(s)}{s(s - q^2)} \; - \Pi(0) \; . \tag{4}$$

where we substract its first Taylor series term as it is UV divergent.

Finally, one has to perform a Borel transform, which aims to exponentially supresses the continuous contributions.

$$\mathcal{B}_{M^2}[\Pi(q^2)] = \lim_{-q^2, n \longrightarrow \infty} \frac{(-q^2)^{n+1}}{n!} \left(\frac{d}{dq^2} \right)^n \Pi(q^2) \ , \tag{5}$$

$$\Pi(M^2) = f_V^2 e^{-m_V^2/M^2} + \int_{s_h^0}^{\infty} ds \ \rho^h(s) e^{-s/M^2} \ . \tag{6}$$

The first step is to explicitly calculate the correlator. For that purpose, we make use of the OPE for the two currents inside the T-product. The local operators are the following:

$$\begin{aligned} \mathcal{O}_3 &= \bar{\psi}\psi \\ \mathcal{O}_4 &= G^a_{\mu\nu} G^{a\mu\nu} \\ \mathcal{O}_5 &= \bar{\psi}\sigma_{\mu\nu}\frac{\lambda^a}{2} G^{a\mu\nu}\psi \\ \mathcal{O}_6^{\psi} &= (\bar{\psi}\Gamma_r\psi)(\bar{\psi}\Gamma_s\psi) \\ \mathcal{O}_6^{G} &= f^{abc} G^a_{\mu\nu} G^{b\nu}_{\sigma} G^{c\sigma\mu} \end{aligned} \tag{7}$$

Adding up the perturvative part and all the contributions from the condensates we get to the complete expression of the perturbative part after applying the corresponding Borel transform:

$$\Pi(q^2) = \frac{1}{4\pi^2}\left(1+\frac{\alpha_s(M)}{\pi}\right)\int_0^\infty ds\, e^{-s/M^2} + \frac{2m\langle\bar{\psi}\psi\rangle}{M^2} + \frac{\langle\frac{\alpha_s}{\pi}G_a^{\mu\nu}G_{a\mu\nu}\rangle}{12M^2} - \frac{112\pi}{81}\frac{\alpha_s\langle\bar{\psi}\psi\rangle^2}{M^4}\ . \quad (8)$$

The next step is then to match both sides of the correlation function, yielding:

$$f_V^2\, e^{-m_V^2/M^2} + \int_{s_h}^\infty ds\, \rho_h(s)\, e^{-s/M^2} = \frac{1}{4\pi^2}\left(1+\frac{\alpha_s(M)}{\pi}\right)\int_0^\infty ds\, e^{-s/M^2} + \frac{2m\langle\bar{\psi}\psi\rangle}{M^2} + \frac{\langle\frac{\alpha_s}{\pi}G_a^{\mu\nu}G_{a\mu\nu}\rangle}{12M^2} - \frac{112\pi}{81}\frac{\alpha_s\langle\bar{\psi}\psi\rangle^2}{M^4}\ . \quad (9)$$

Quark-hadron duality

As the last step for finally obtaining the final form of the sum rule, one has to perform a useful as well as necessary aproximation. One must notice that for $q^2 \rightarrow \infty$, the correlation function can be approximated by its perturbative part:

$$\int_{s_0^h}^{\infty} ds \rho^h(s) e^{-s/M^2} \simeq \frac{1}{\pi} \int_{s_0}^{\infty} ds Im\Pi^{(pert)}(s) e^{-s/M^2}. \tag{10}$$

This allows us to eliminate the continuous part on the hadronic part. The final sum rule reads:

$$f_V^2 \, e^{-m_V^2/M^2} = \frac{1}{4\pi^2} \left(1 + \frac{\alpha_s(M)}{\pi}\right) \int_0^{s_0^h} ds \, e^{-s/M^2} + \frac{2m\langle\bar{\psi}\psi\rangle}{M^2} + \frac{\langle \frac{\alpha_s}{\pi} G_a^{\mu\nu} G_{a\mu\nu}\rangle}{12M^2} - \frac{112\pi}{81} \frac{\alpha_s \langle\bar{\psi}\psi\rangle^2}{M^4} \, . \tag{11}$$

Extra. Perturbative part calculation (I)

- Starting from eq. 1 we contract the operators using Wick's theoremand matching color indices we arrive at:

$$\langle 0|T\{j_\mu(x)j_\nu(0)\}|0\rangle = \delta_{ij}\,Tr\left[\gamma_\mu S_0(x,0)\gamma_\nu S_0(0,x)\right]\ . \tag{12}$$

- Write explicitly the propagators and integrate in d^4x. Then use the delta function to get rid of the integral in one of the momentums.
- Take traces in dimension D and integrate using dimensional regularization.

$$\begin{aligned} Tr\left[\gamma_\mu(\not p - m)\gamma_\nu((\not p - \not q) - m)\right] =& D(2p_\mu q_\nu - g_{\mu\nu}p^2) \\ & - D(p_\mu q_\nu + p_\nu q_\mu - g_{\mu\nu}p\cdot q) \\ & + Dm^2 g_{\mu\nu}\ . \end{aligned} \tag{13}$$

Extra. Perturbative part calculation (II)

Finally, we arrive at:

$$q^2\Pi(q^2) = -\frac{12i}{D-1}\int_0^1 dv \int \frac{d^D p}{(2\pi)^D}\frac{(2-D)(p^2-q^2v(1-v))+Dm^2}{(p^2+q^2v(1-v)-m^2)^2} \,. \tag{14}$$

One will need to integrate this expression and write it in the form of a dispersion integral to arrive to the desired result.

Extra. Perturbative part calculation (III)

Doing so, it yields:

$$\Pi^{(0)}(q^2) = \frac{q^2}{\pi} \int ds \frac{Im\Pi^{(0)}(s)}{s(s-q^2)} \tag{15}$$

where

$$Im\Pi^{(0)}(s) = \frac{1}{8\pi} v(3-v^2)\theta(s-4m^2) \ , \tag{16}$$

and $v = \sqrt{1-4m^2/s}$. The perturvative part can also be computed up to $\mathcal{O}(\alpha_s)$:

$$Im\Pi^{(pert)}(s) = Im\Pi^{(0)}(s) \left[1 + \alpha_s C_F \left(\frac{\pi}{2v} - \frac{v+3}{4} \left(\frac{\pi}{2} - \frac{3}{4\pi} \right) \right) \right] \ . \tag{17}$$

Extra. Quark condensate calculation (I)

For the quark condensate we have:

$$\Pi^{(\bar{\psi}\psi)}_{\mu\nu}(q) = i\int d^4x \; e^{iq\cdot x}\langle 0|\{\bar{\psi}^i(0)\gamma_\mu S^{ij}(x,0)\gamma_\nu \psi^j(0) + \bar{\psi}^j(0)\gamma_\nu S^{ij}(0,x)\gamma_\mu \psi^i(0)\}|0\rangle \; , \tag{18}$$

where we can expand the spinors as

$$\begin{aligned} \psi(x) &= \psi(0) + x^\rho \overrightarrow{D}_\rho \psi(0) + \dots \; , \\ \bar{\psi}(x) &= \bar{\psi}(0) + \bar{\psi}(0)\overleftarrow{D}_\rho x^\rho + \dots \; . \end{aligned} \tag{19}$$

Finally one arrives at:

$$\Pi^{(\bar{\psi}\psi)}(q) = \frac{2m}{q^4}\langle\bar{\psi}\psi\rangle \; . \tag{20}$$