A new perturbative approach to dense quark matter: thermal Loop Tree Duality

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Holography and Dense Matter

Paris, 18 September 2025

Main references:

1) Gorda, Kurkela, Paatelainen, Säppi, AV, PRL 127 (2021), 2103.05658
2) Gorda, Paatelainen, Seppänen, Säppi, PRL 131 (2023), 2307.08734
3) Navarrete, Paatelainen, Seppänen, PRD 110 (2024), 2403.02180
4) Kärkkäinen, Navarrete, Nurmela, Paatelainen, Seppänen, AV, PRL 135 (2025), 2501.17921

HELSINGIN YLIOPISTO
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$$\mathcal{L}_{\mathrm{QCD}} \quad = \quad \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \sum_f \bar{\psi}_f (\gamma_\mu D_\mu + m_f) \psi_f$$

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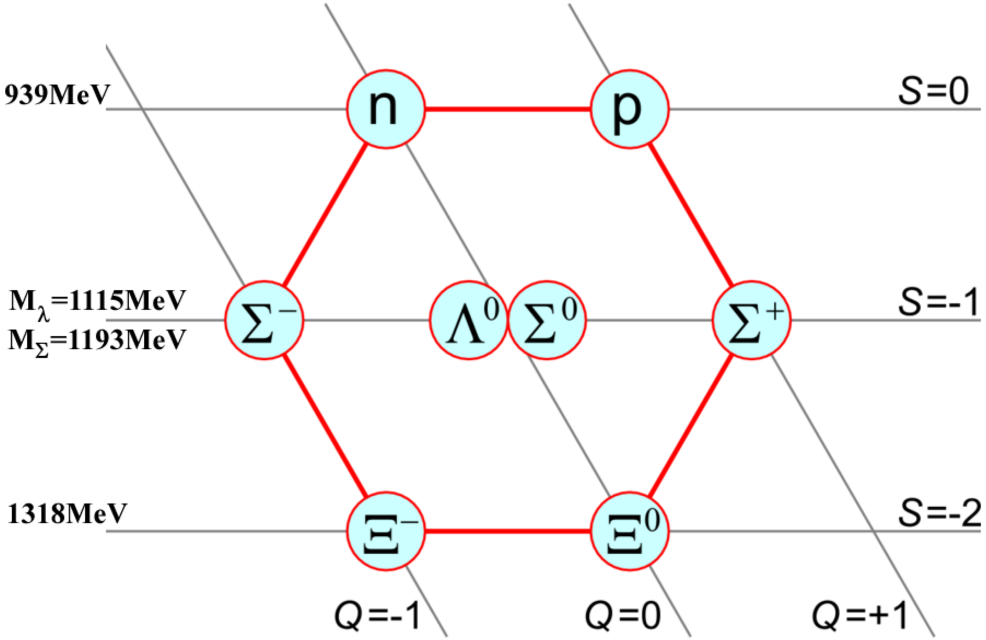

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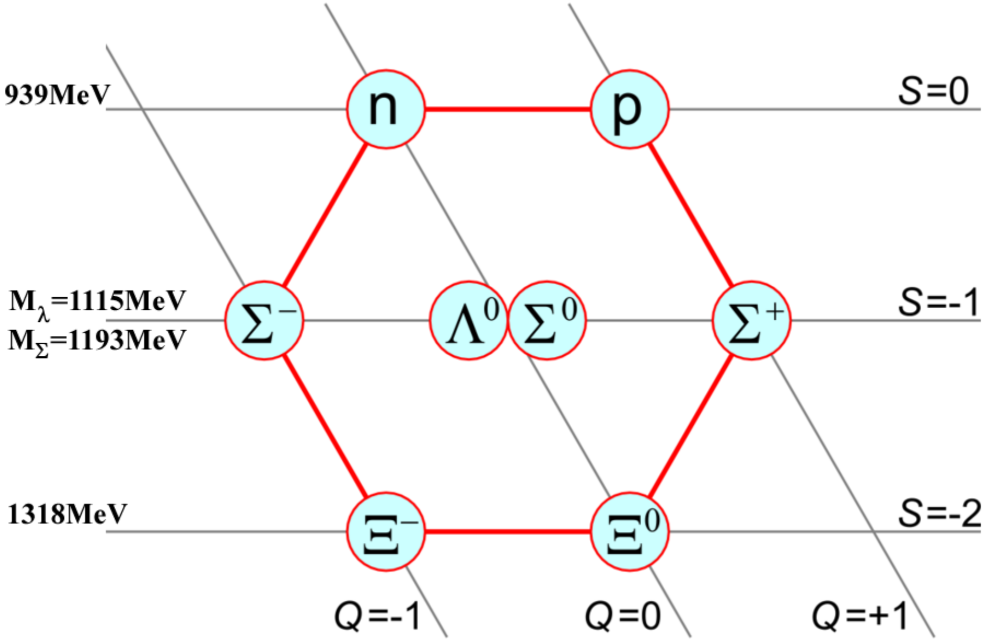
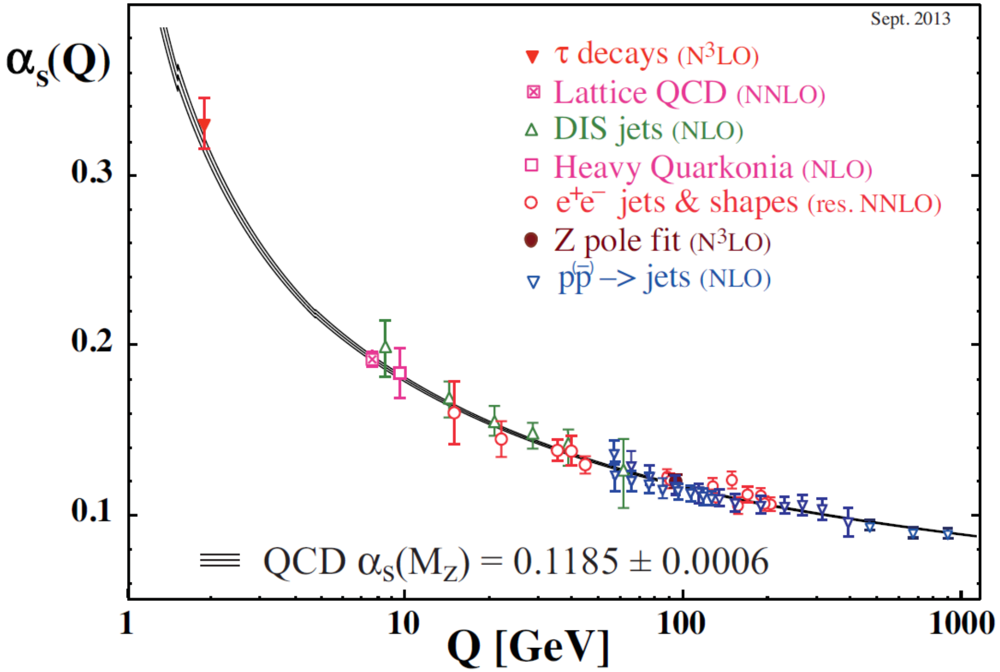
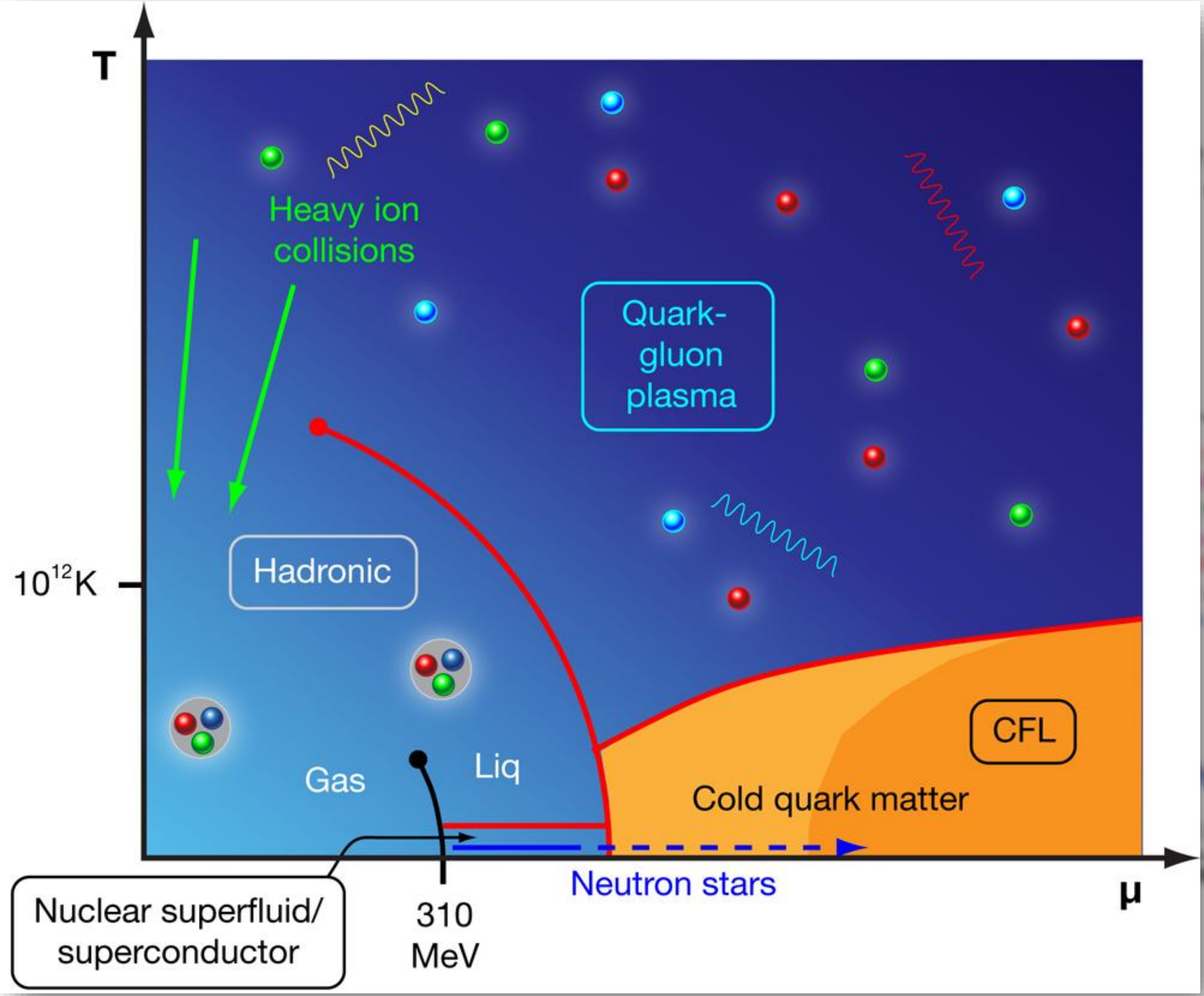
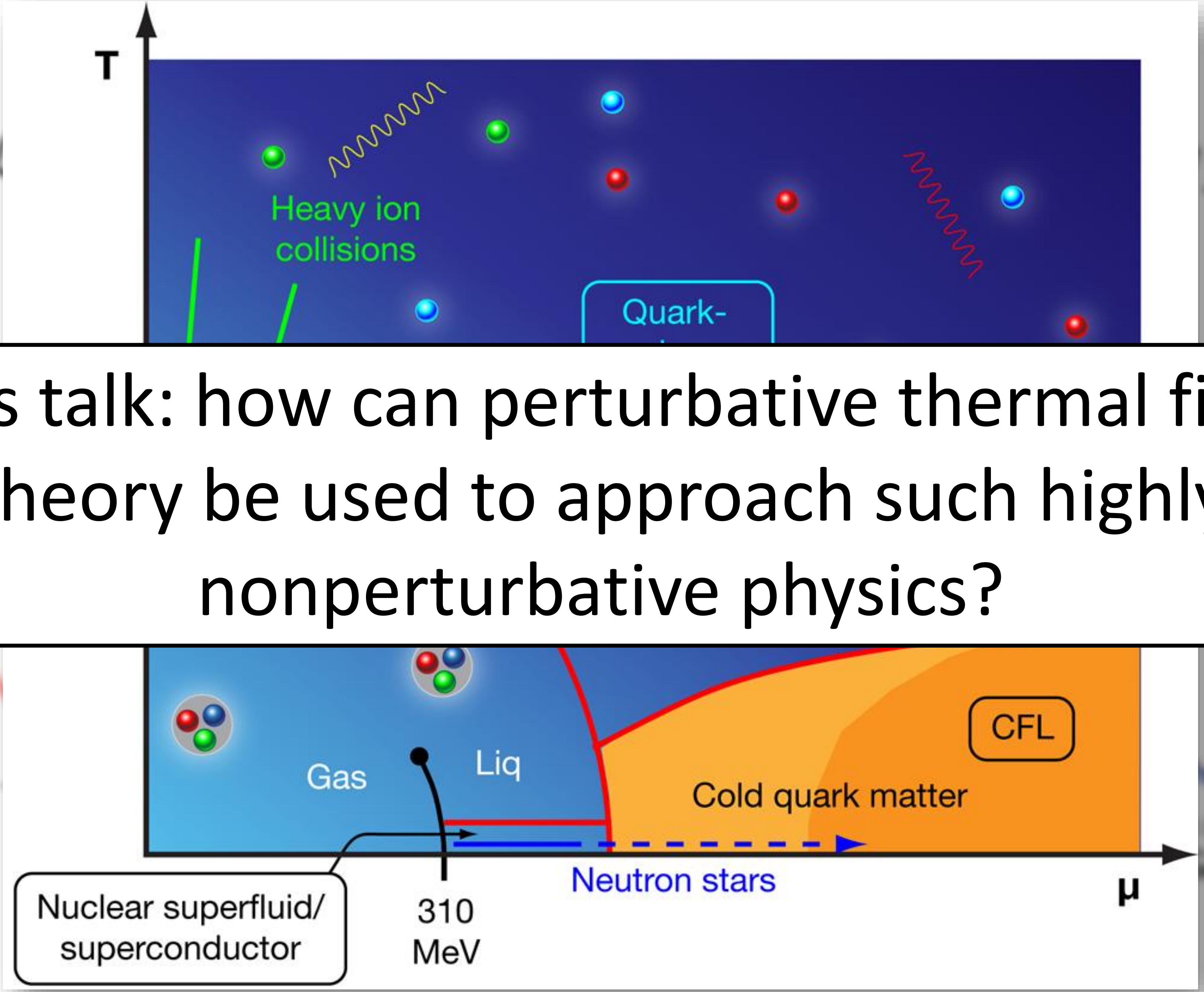





T
Heavy ion
collisions
Quark-
gluon
plasma
Hadronic
10^{12}K
Gas
Liq
CFL
Cold quark matter
Nuclear superfluid/
superconductor
310
MeV
Neutron stars
μ



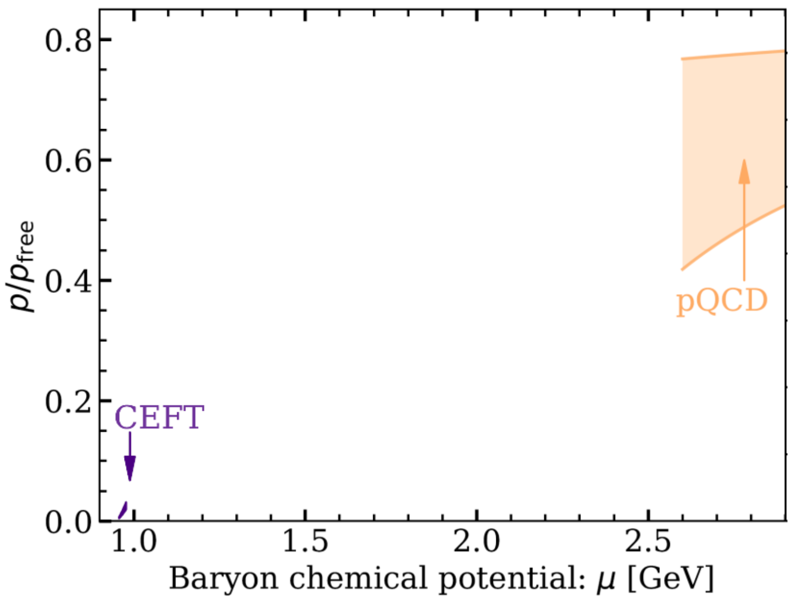
This talk: how can perturbative thermal field theory be used to approach such highly nonperturbative physics?

Rest of the talk:

I. Motivation: EoS inference and the high-density constraint
II. Basics of thermal field theory at high density
III. QM pressure at N3LO: general setup and soft contributions from HTL effective theory
IV. Hard sector & thermal Loop Tree Duality
V. Future directions

Motivation: EoS inference and the high-density constraint

Basic problem: Lack of first-principles tools for dense QCD matter



For the properties of NS matter – controlling stellar structure and composition – only low- and high-density limits under control via Chiral EFT & pQCD

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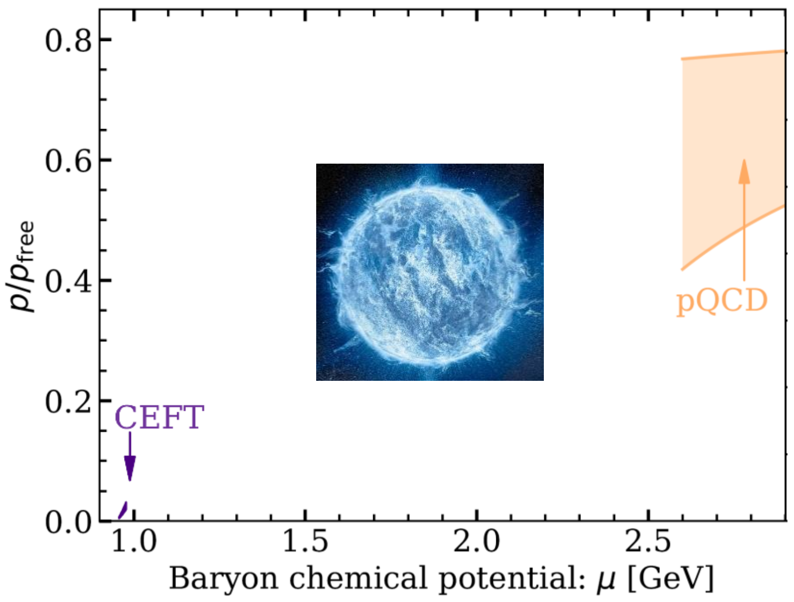

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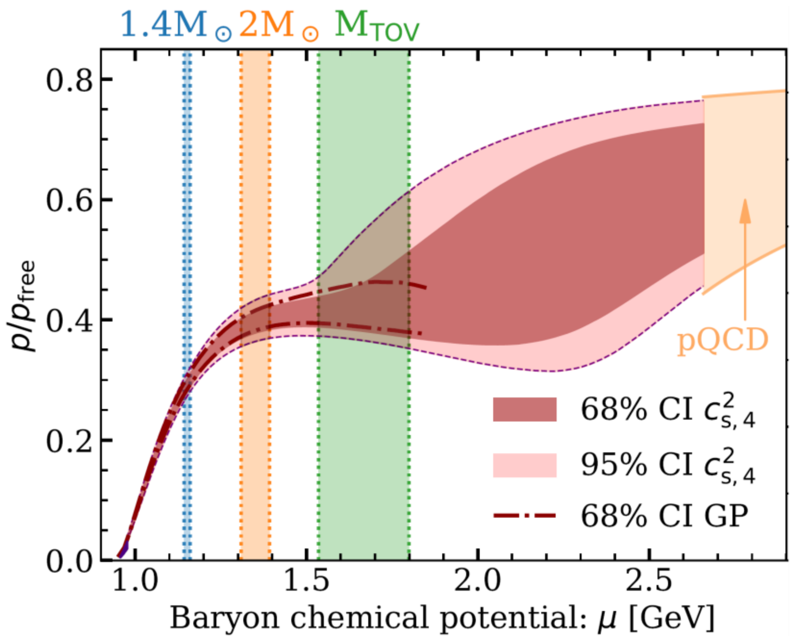

Annala et al. (incl. AV), PRL 120 (2018); Nature Phys. 16 (2020); PRX 12 (2022); Nature Comm. 14 (2023)

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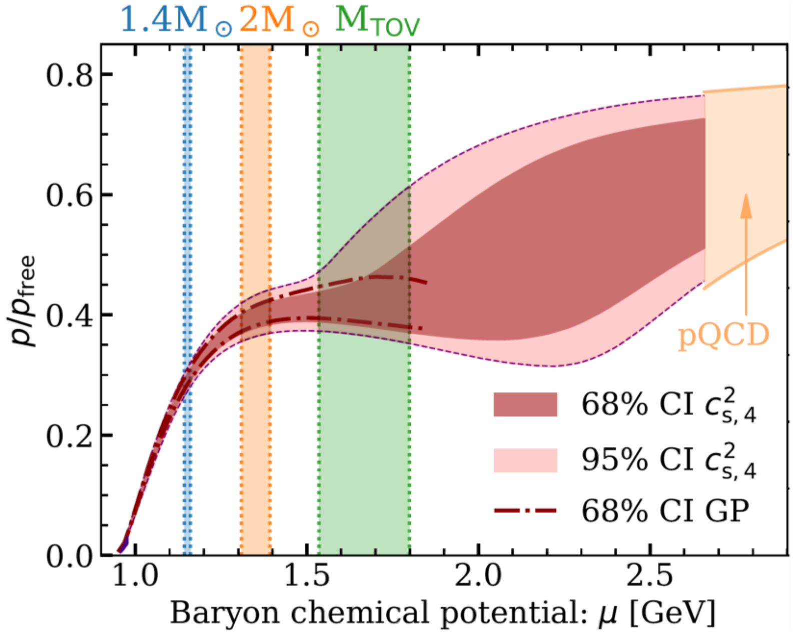
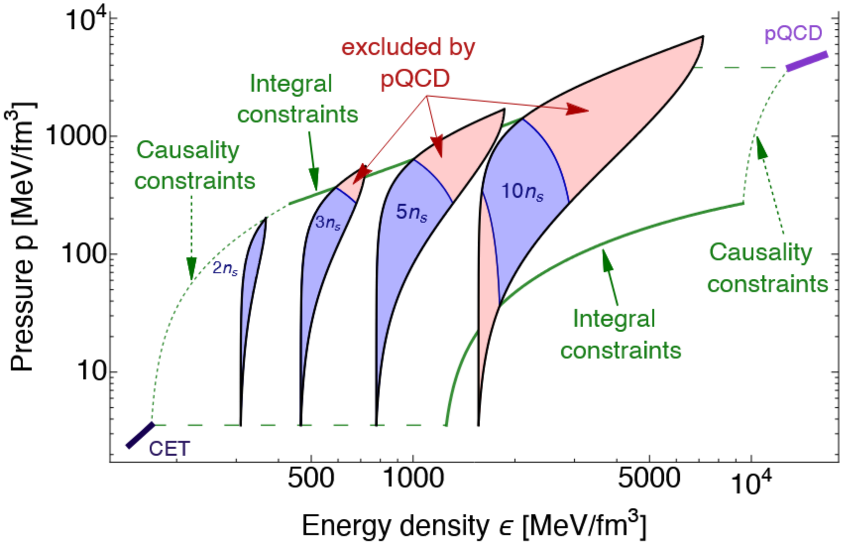


Annala et al. (incl. AV), Nature Comm. 14 (2023)



Komoltsev and Kurkela, PRL 128 (2021)

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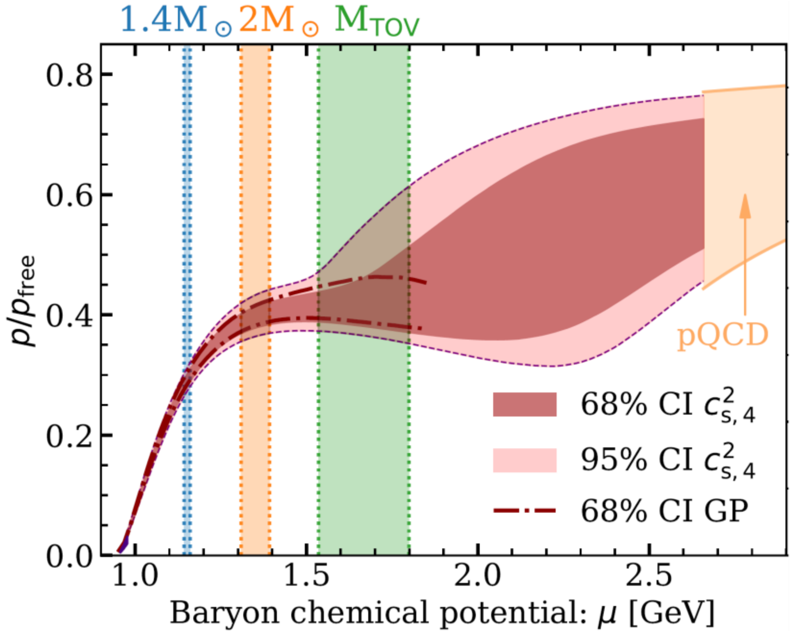

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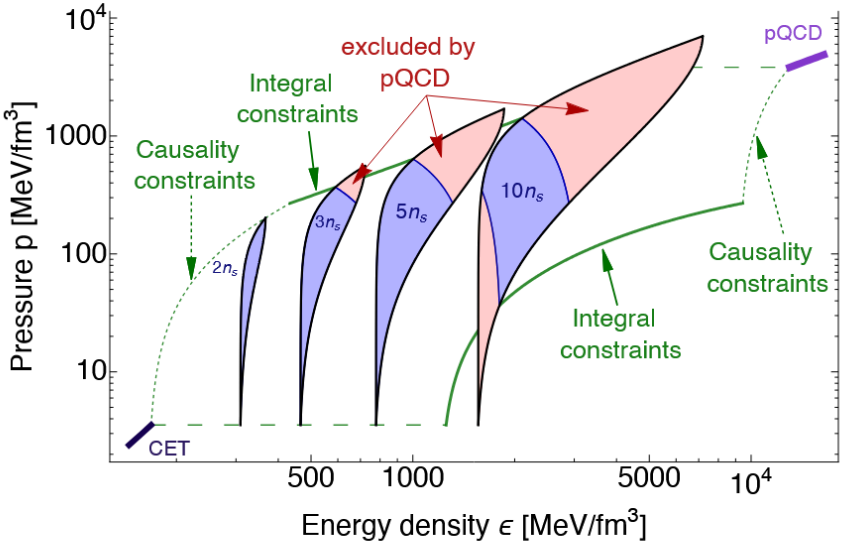

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Challenge: Is it possible to qualitatively improve the present situation without major observational breakthroughs?

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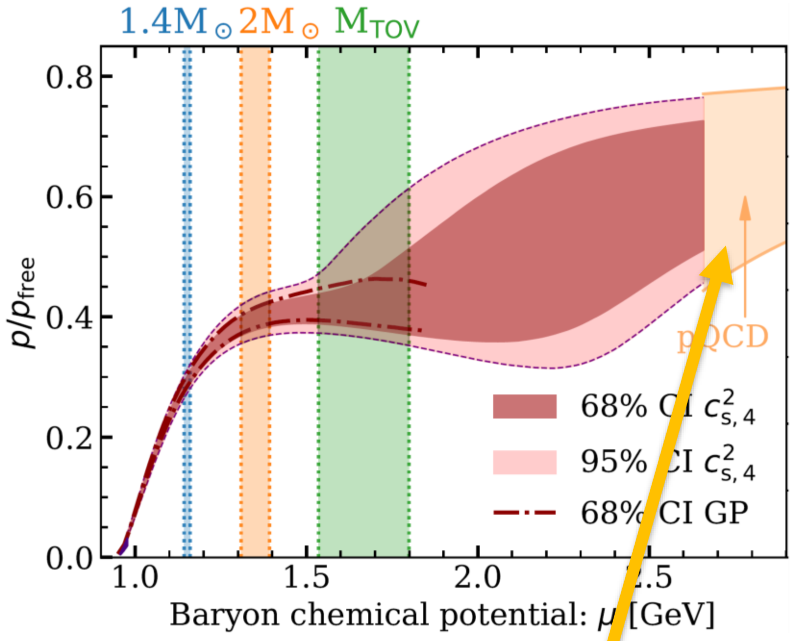

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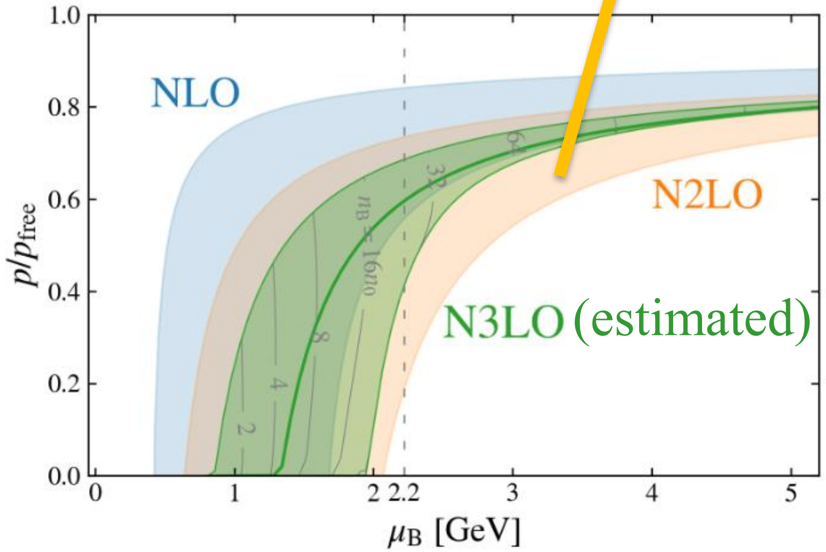

Gorda, Paatelainen, Säppi, Seppänen, PRL 131 (2023)

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Opportunity: Completion of N3LO QM pressure has potential for qualitative leap (no need for N4LO?)

- Easier said than done: determination of missing contribution an open problem since late 1970s!
- Recently, significant new hope from a major technical breakthrough (topic of this talk)

Basics of thermal field theory at high density

$$
\begin{aligned}
\Omega(T, \mu_u, \mu_d, \mu_s, m_s) &= -T \log \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\mathcal{D}A_\mu e^{-\int d^3x \int_0^{1/T} d\tau \mathcal{L}_{\mathrm{QCD}}}, \\
\mathcal{L}_{\mathrm{QCD}} &= \frac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} + \bar{\psi}_i(\gamma_\mu D_\mu + m_i - \mu_i \gamma_0)\psi_i
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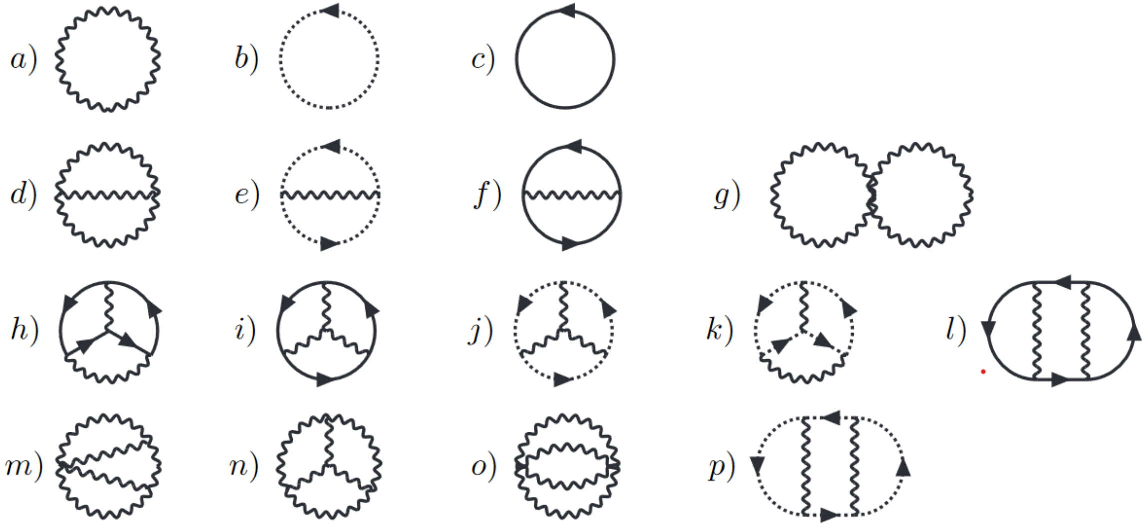

+ resummation of soft bosonic dof's, typically implemented via dimensionally reduced effective theories

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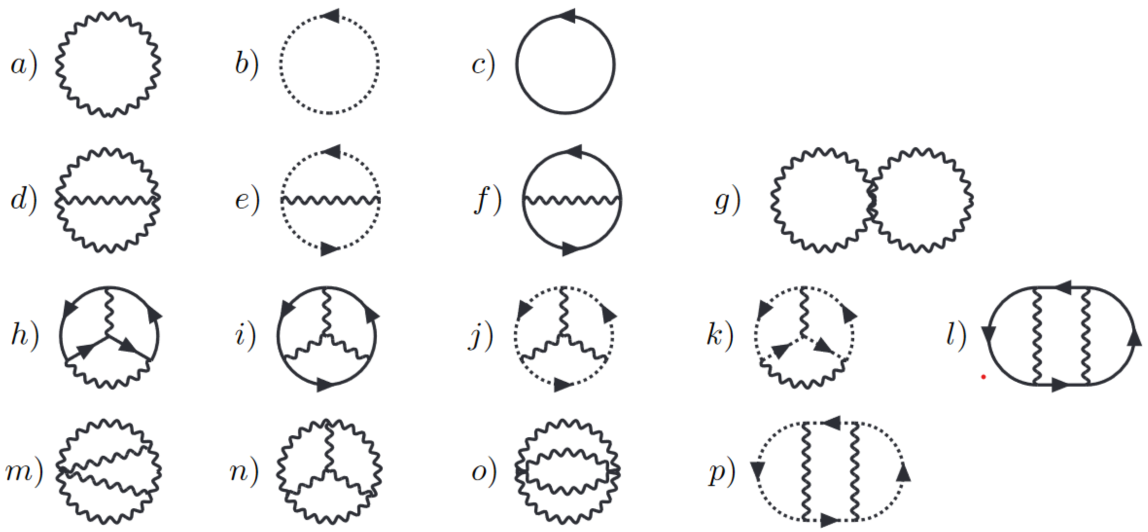

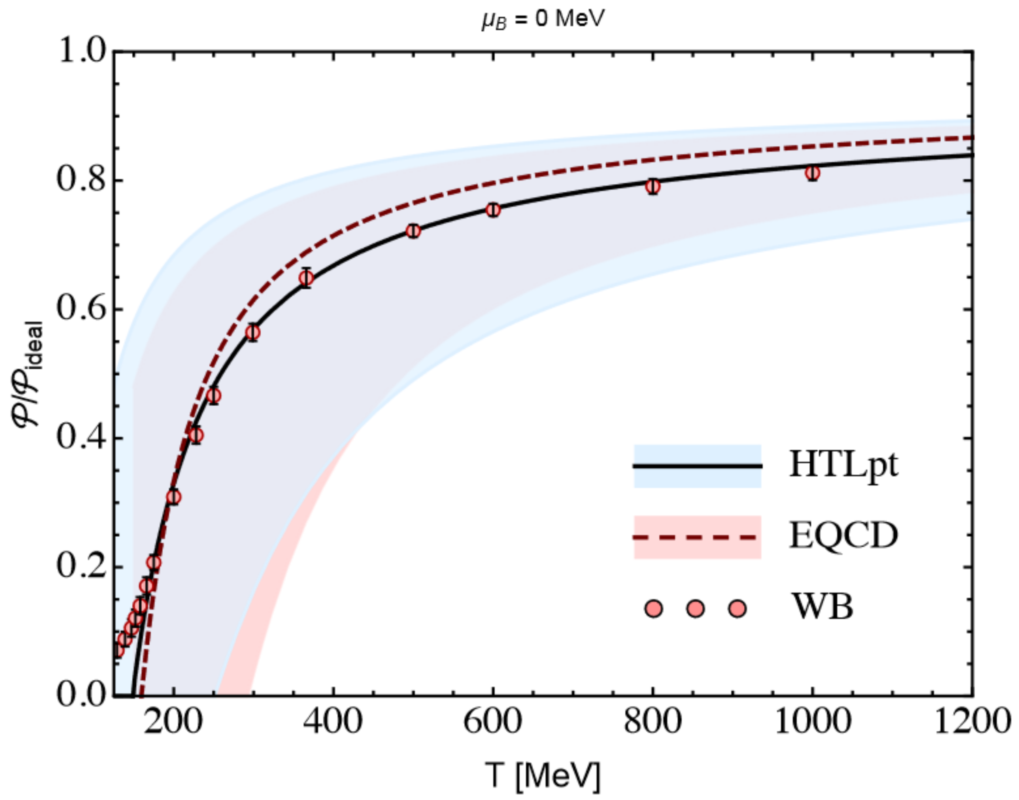

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Ghiglieri, Kurkela, Strickland, AV, Phys. Rept. 880 (2020)

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But what happens at high μ and $T = 0$?

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But what happens at high μ and $T = 0$?

1) Sum-integrals get replaced by four-dimensional continuous integrals, with fermionic $p_0 \rightarrow p_0 - i\mu$
   - Simplification from vanishing of diagrams with no fermion loops
   - In practical calculations, often advantageous to start from evaluation of temporal momentum integrals using the Residue theorem

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2) IR sensitive modes no longer 3d: all bosonic (Euclidean) four-momenta satisfying $|P| \lesssim m_E \sim g\mu_B$ need special treatment

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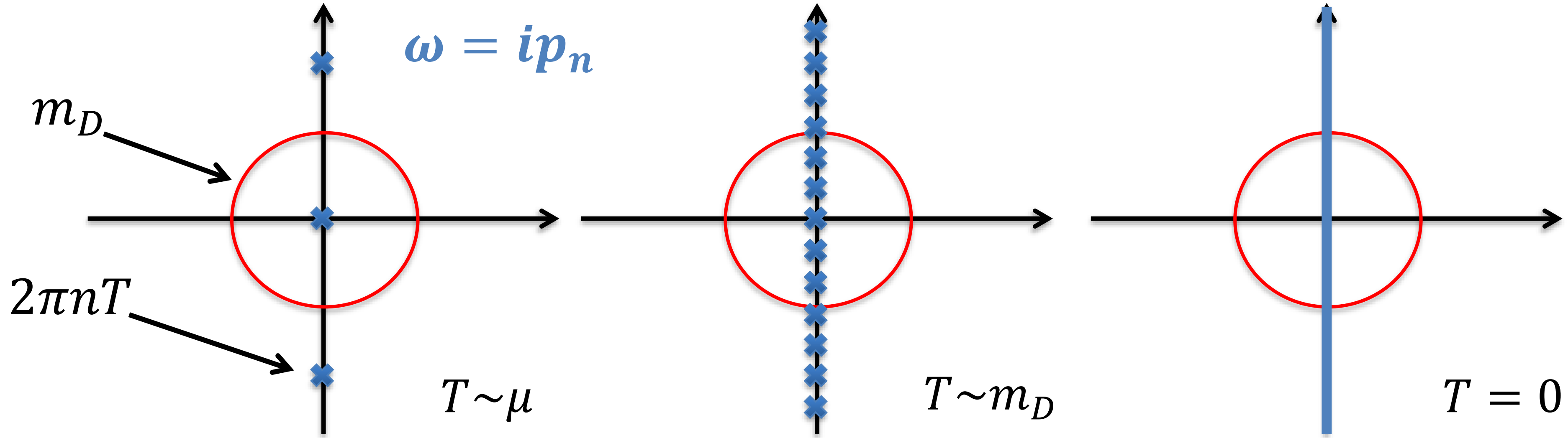

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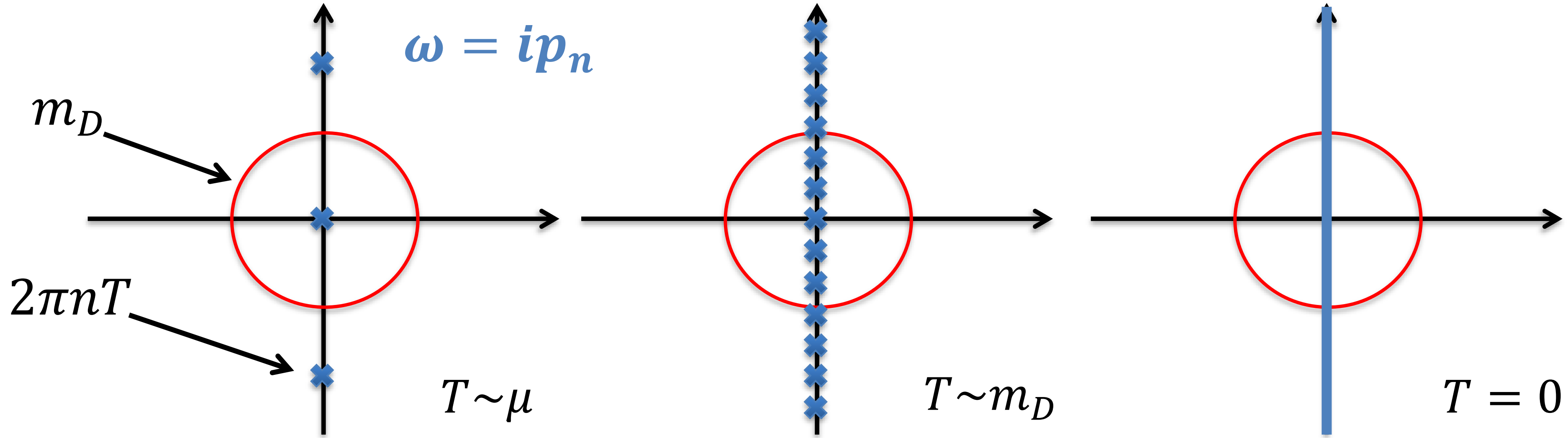

Correct soft EFT now the Hard Thermal Loop effective theory

QM pressure at N3LO: organization and soft contributions

Up to state-of-the-art $O(\alpha_s^3)$, three types of contributions to the pressure:

1) Hard modes (scale μ_B) and their interactions: naïve loop expansion up to and including four loops
2) Soft modes (scale $m_E \sim g\mu_B$) and their interactions: one- and two-loop graphs in HTL effective theory
3) Mixing of soft and hard modes

$$p = p_{\rm FD} + p_1^h \alpha_s + p_2^h \alpha_s^2 + p_3^h \alpha_s^3 + p_2^s \alpha_s^2 + p_3^s \alpha_s^3 + p_3^m \alpha_s^3$$

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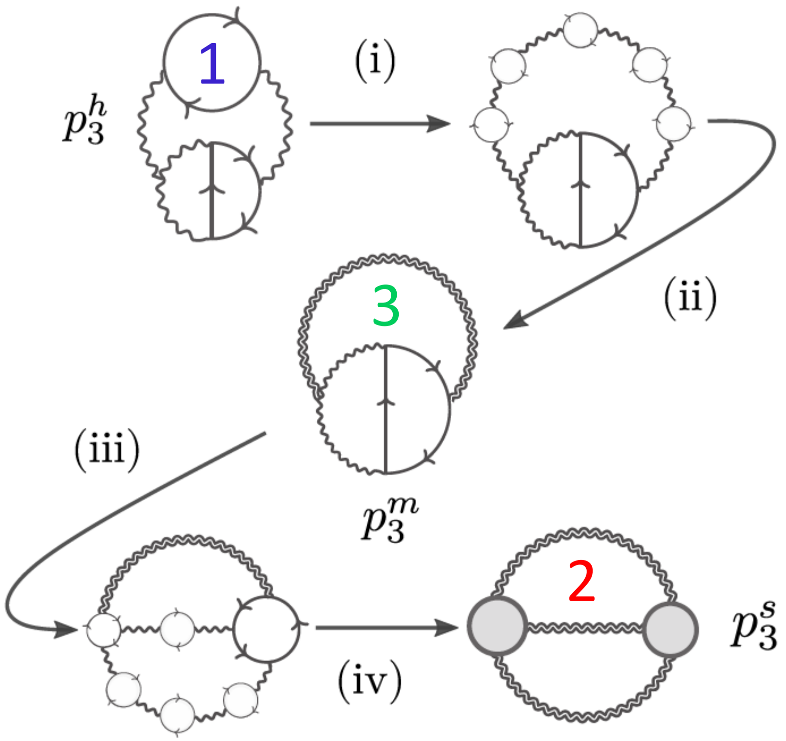

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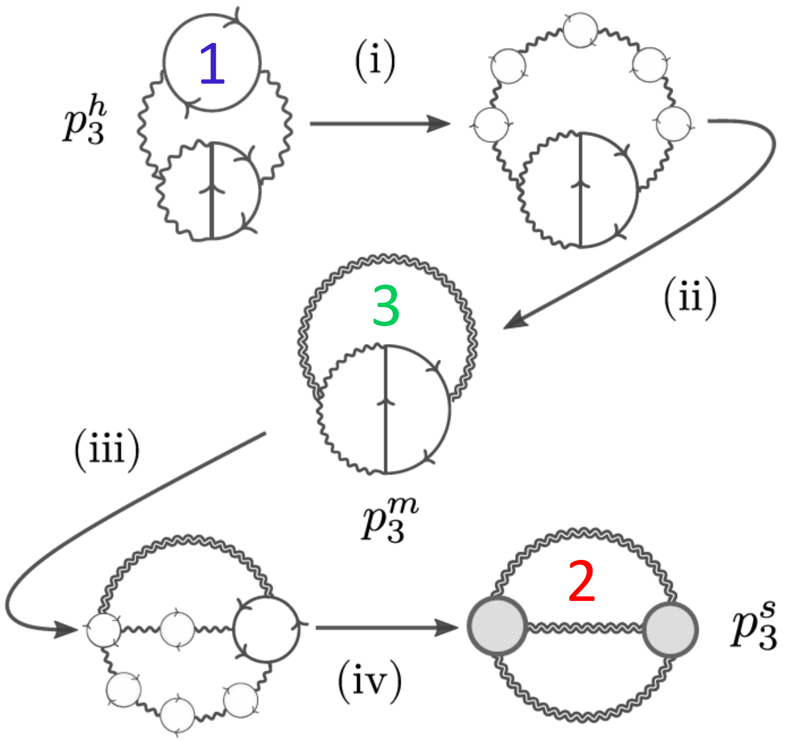

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Known since 1970's: Freedman, McLerran, PRD 16 (1977)

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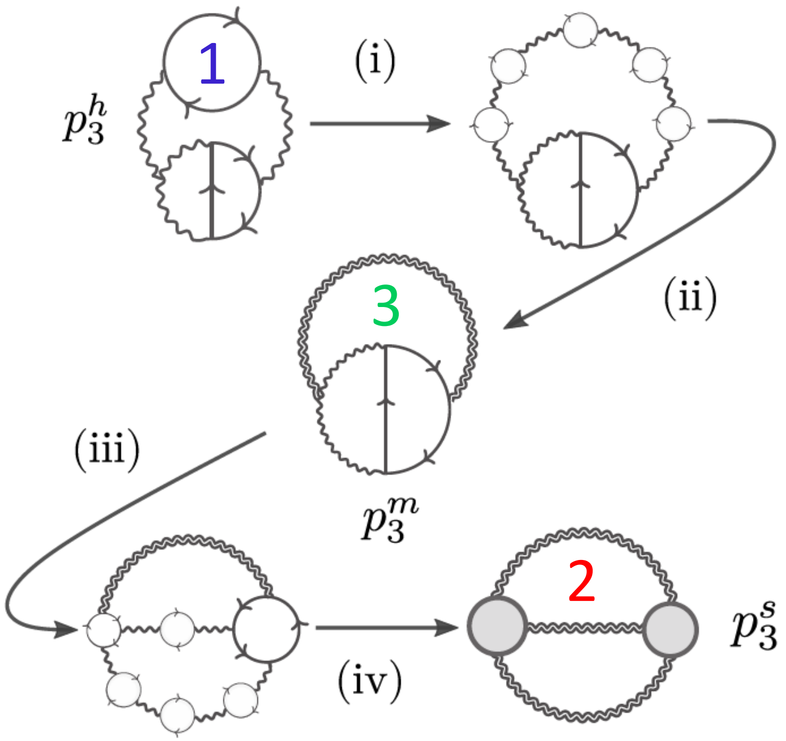

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Leading log from Gorda, Kurkela, Romatschke, Säppi, AV, PRL 121 (2018)

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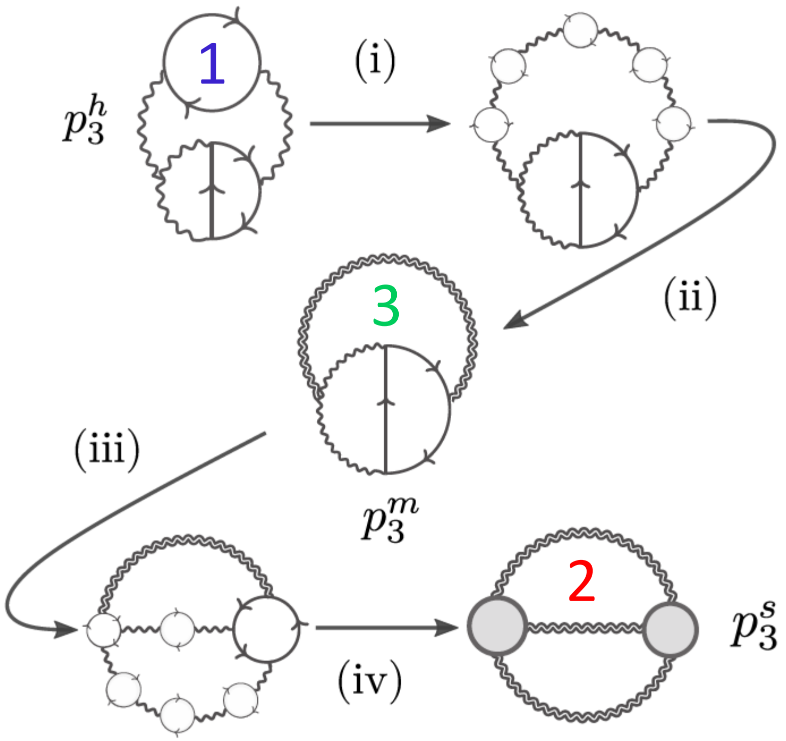

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Gorda, Kurkela, Paatelainen, Säppi, AV, PRL 127 (2021); Fernandez, Kneur, PRL 129 (2022)

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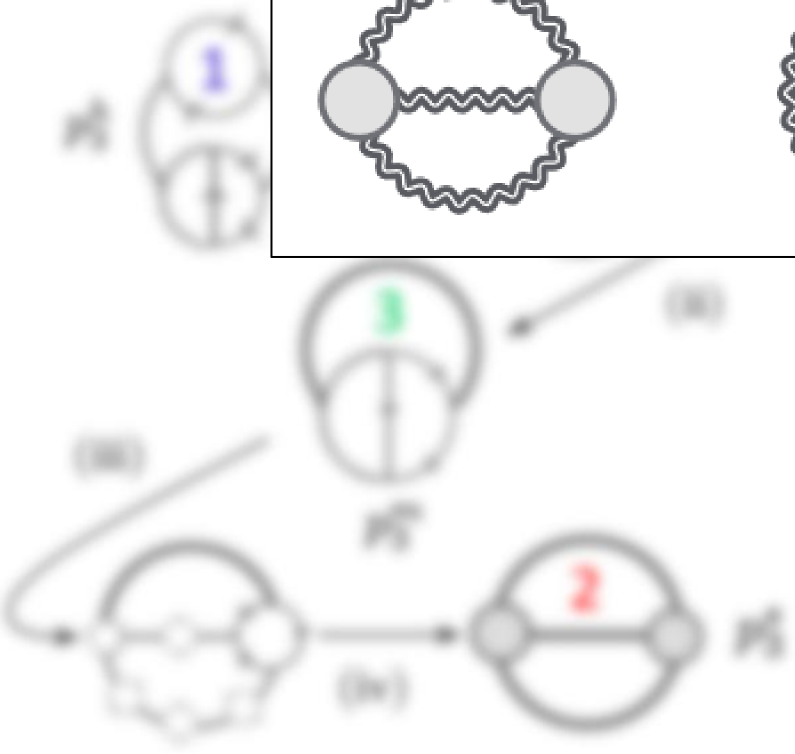

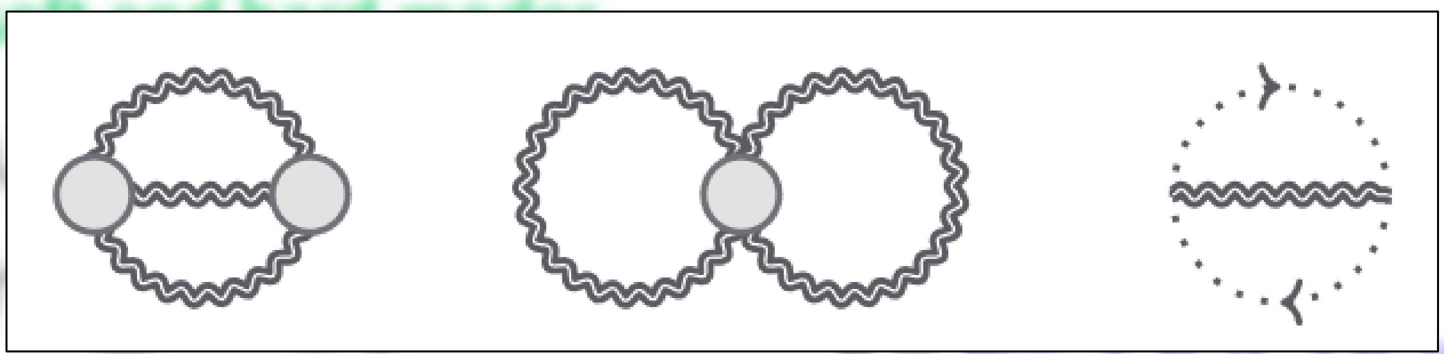

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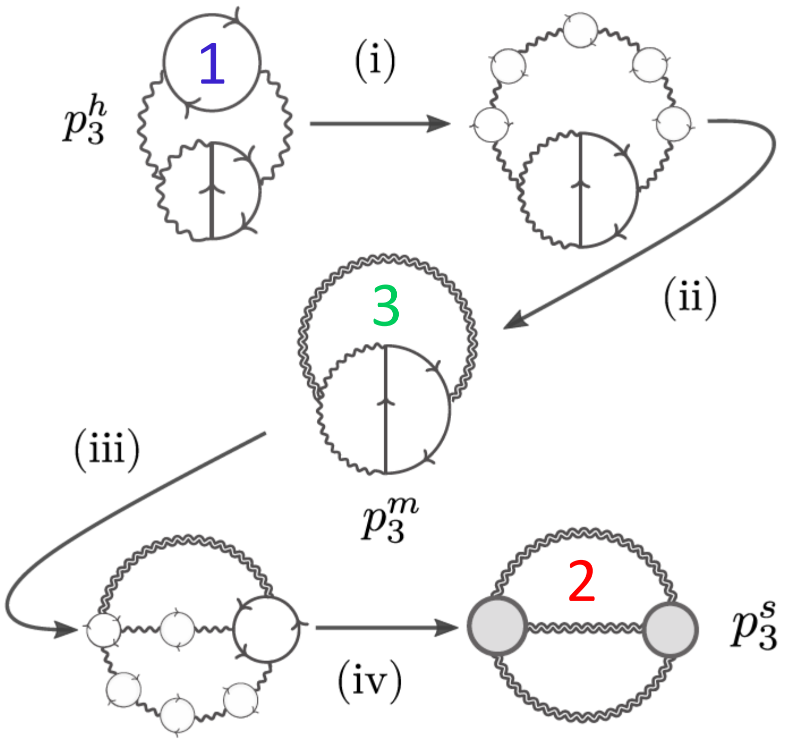

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QED: Gorda, Kurkela, Österman, Paatelainen, Säppi, Seppänen, Schicho, AV, PRD 107 (2023)
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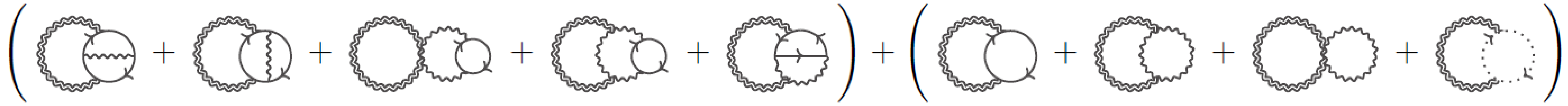
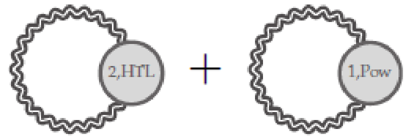
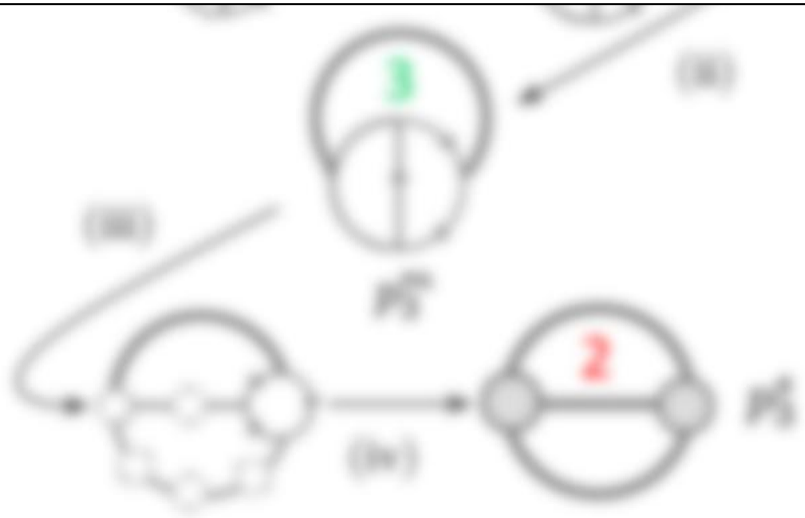




$$= -\frac{\alpha_s m_E^2 d_A}{8\pi}\int_K \mathrm{Tr}\left\{G_{\mathrm{LO}}(K)\left[\Pi^{2,\mathrm{HTL}}(K) + \frac{K^2}{m_E^2}\Pi^{1,\mathrm{Pow}}(K)\right]\right\}.$$



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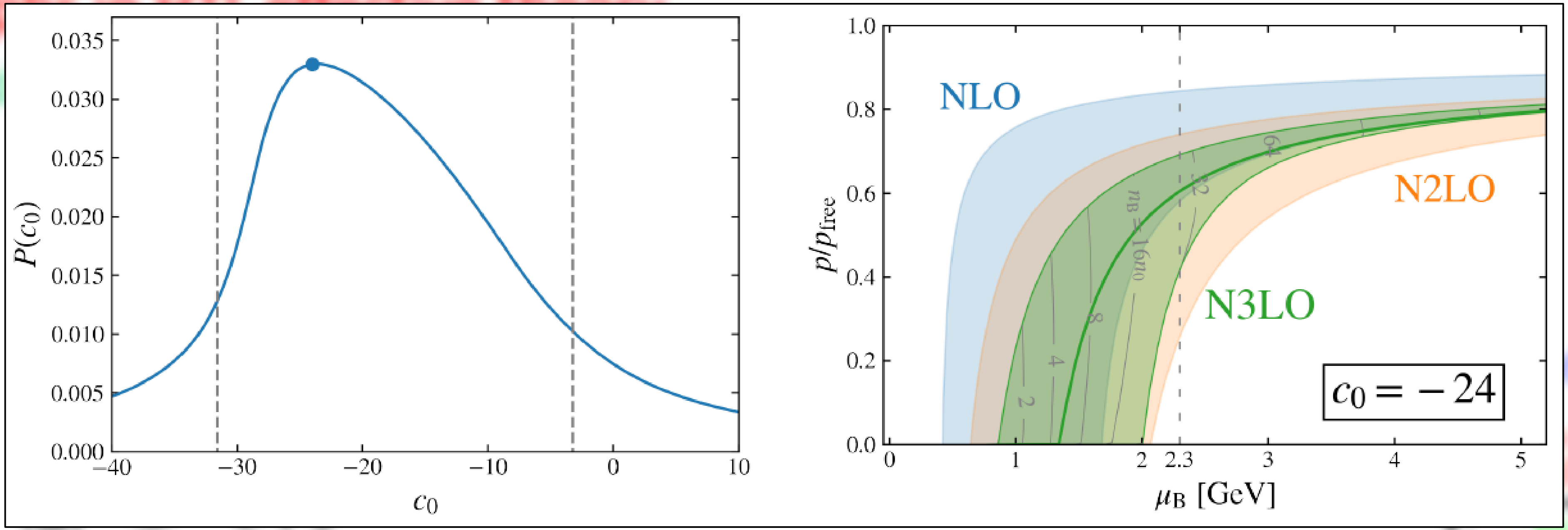
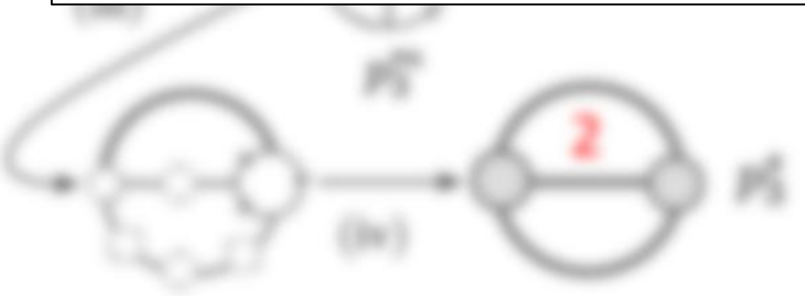
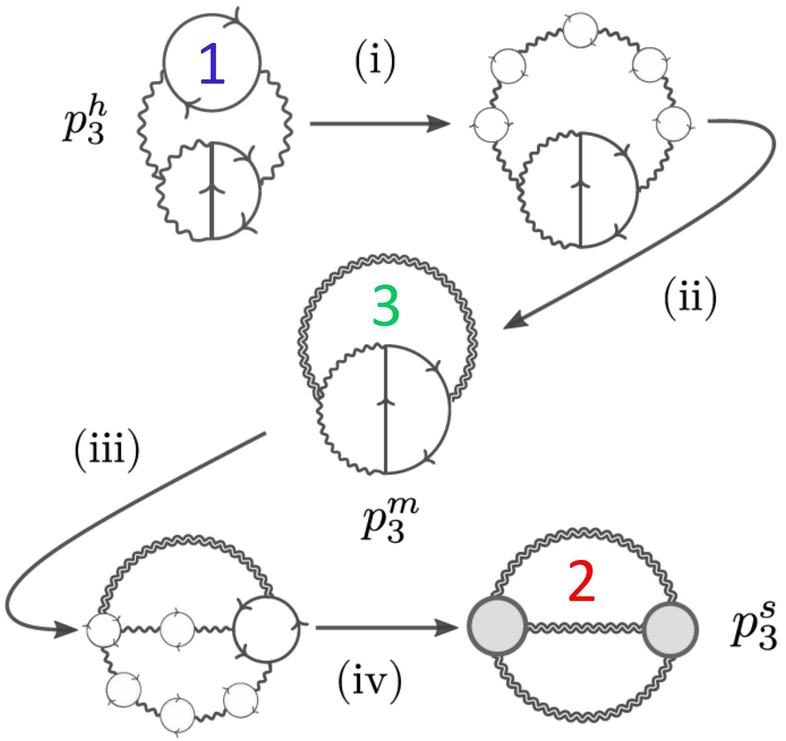




QED: Gorda, Kurkela, Österman, Paatelainen, Säppi, Seppänen, Schicho, AV, PRD 107 (2023)
QCD: Gorda, Paatelainen, Säppi, Seppänen, PRL 131 (2023)

Up to state-of-the-art $O(\alpha_s^3)$, three types of contributions to the pressure:

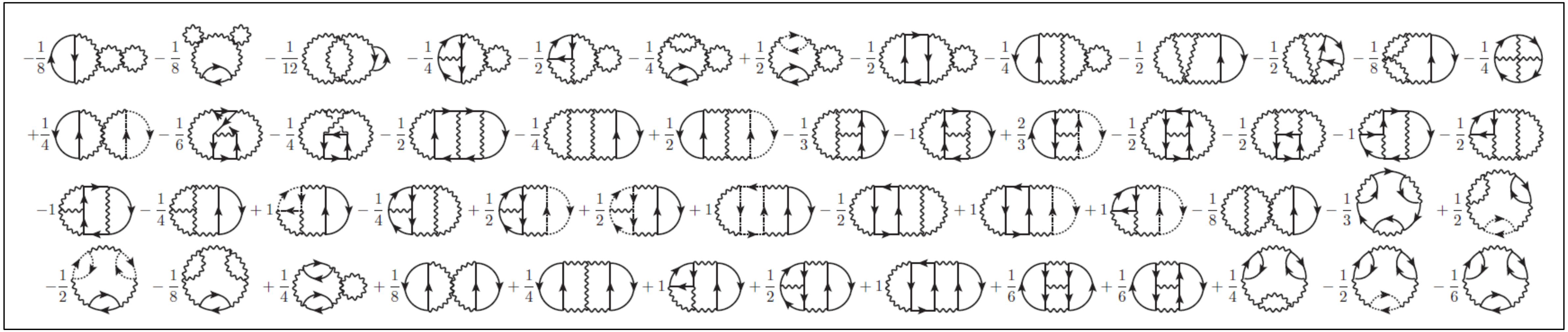
1) Hard modes (scale μ_B) and their interactions: naïve loop expansion up to and including four loops
2) Soft modes (scale $m_E \sim g\mu_B$) and their interactions: one- and two-loop graphs in HTL effective theory
3) Mixing of soft and hard modes

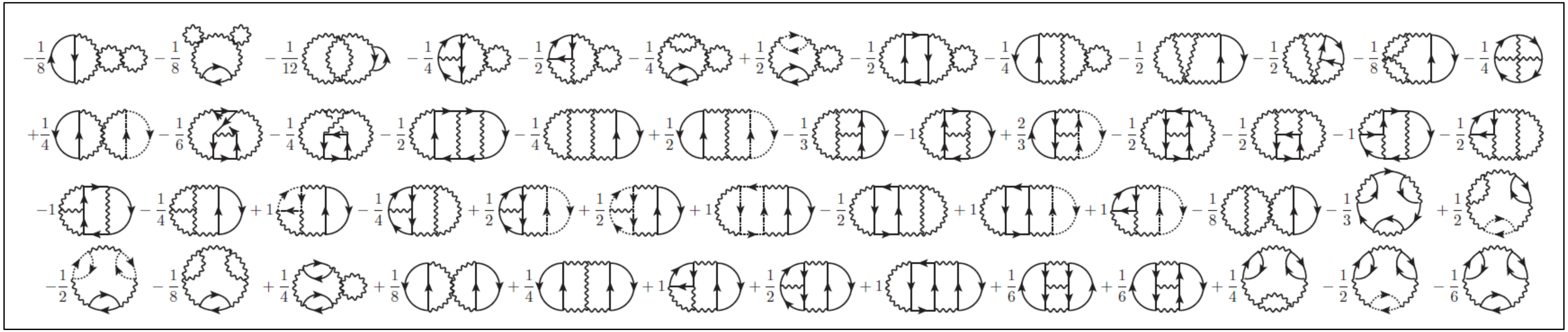


Last unknown part, topic of the remainder of this talk

$$p = p_{\rm FD} + p_1^h \alpha_s + p_2^h \alpha_s^2 \boxed{+ p_3^h \alpha_s^3} + p_2^s \alpha_s^2 + p_3^s \alpha_s^3 + p_3^m \alpha_s^3$$

Hard contributions at the N3LO: what, why, and how?





$$
\frac{p}{p_{\text{free}}} = 1 + \left(\frac{\alpha_s}{\pi}\right) a_{1,1} + N_f \left(\frac{\alpha_s}{\pi}\right)^2 \left[a_{2,1} \ln\left(N_f \frac{\alpha_s}{\pi}\right) + a_{2,2} \ln \frac{\bar{\Lambda}}{2\mu_q} + a_{2,3}\right]
$$

$$
+ N_f^2 \left(\frac{\alpha_s}{\pi}\right)^3 \left[a_{3,1} \ln^2 \left(N_f \frac{\alpha_s}{\pi}\right) + a_{3,2} \ln\left(N_f \frac{\alpha_s}{\pi}\right) + a_{3,3} \ln\left(N_f \frac{\alpha_s}{\pi}\right) \ln \frac{\bar{\Lambda}}{2\mu_q}\right.
$$

$$
\left. + a_{3,4} \ln^2 \frac{\bar{\Lambda}}{2\mu_q} + a_{3,5} \ln \frac{\bar{\Lambda}}{2\mu_q} + a_{3,6}\right] + O(\alpha_s^4)
$$

Hard contributions: Basic setup and removal of divergences

Construction of graphs in general ξ gauge & automated handling of all algebra

↙

Application of momentum shifts and other tricks to reduce # of integrals

↘

Recovery of gauge-parameter-independent result

↙

Demonstration of UV and IR finiteness of the pressure

↘

Semi-analytic integration w/ cutting rules and case-by-case tricks ☠ ☠ ☠

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

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Using QGraph, FORM, and a canonicalization procedure by Navarrete and Schröder, construct graphs in general covariant gauge & carry out color, Lorentz and Dirac algebras

$\Longrightarrow$ 156307 scalar integrals

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

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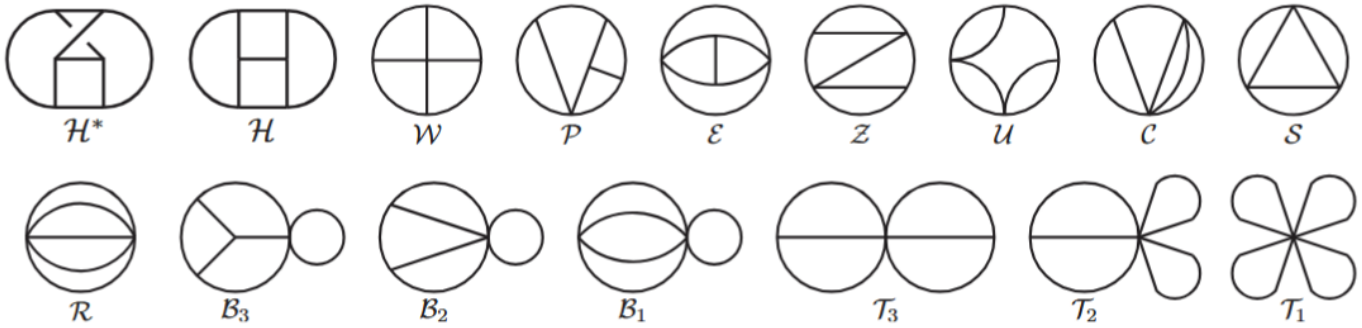
Using QGraph, FORM, and a canonicalization procedure by Navarrete and Schröder, construct graphs in general covariant gauge & carry out color, Lorentz and Dirac algebras $\Longrightarrow$ 156307 scalar integrals

Then systematically apply momentum shifts to find dramatic reduction to only 114 scalar masters with topologies



Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

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Semi-analytic integration w/ cutting rules and case-by-case tricks ☠ ☠ ☠

Important byproduct: explicit demonstration of the full gauge independence of the pressure through full cancellation of all ξ dependence in the result

# Ints.	N_f^3	$N_f^2 C_A$	$N_f^2 C_F$	$N_f C_A^2$	$N_f C_A C_F$	$N_f C_F^2$
ξ^0	132	2229	958	5975	2841	890
ξ^1	205	7428	2054	34554	11507	2209
ξ^2	173	9461	2452	72831	17340	2949
ξ^3	125	5507	1080	75344	10951	1300
ξ^4	-	2632	-	44618	3491	-
ξ^5	-	-	-	20036	-	-
ξ^0	18	50	48	65	55	45

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

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Semi-analytic integration w/ cutting rules and case-by-case tricks ☠ ☠ ☠

Expressing the pressure in terms of masters allows singling out all IR divergences into a small fraction of scalar integrals, typically featuring simple IR structures.

$\Longrightarrow$ IR pole extraction reduced to considering low-dimensional integrals over subdiagrams expanded in soft external momenta.

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

Hard contributions: Basic setup and removal of divergences

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Semi-analytic integration w/ cutting rules and case-by-case tricks ☠ ☠ ☠

Expressing the pressure in terms of masters allows singling out all IR divergences into a small fraction of scalar integrals, typically featuring simple IR structures.

$\Longrightarrow$ IR pole extraction reduced to considering low-dimensional integrals over subdiagrams expanded in soft external momenta.

A systematic analysis of these subdiagrams reveals three IR-divergent classes of graphs with semi-analytically computable divergences, leading to…

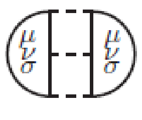
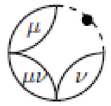
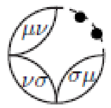
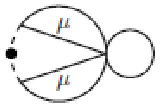
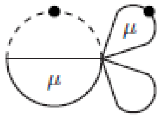
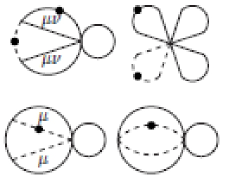
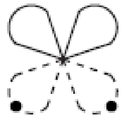
Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

Hard contributions: Basic setup and removal of divergences

- Construction of graphs in general ξ gauge & automated handling of all algebra
- Application of momentum shifts and other tricks to reduce # of integrals
- Recovery of gauge-parameter-independent result
- **Demonstration of UV and IR finiteness of the pressure**
- Semi-analytic integration w/ cutting rules and case-by-case tricks ☠ ☠ ☠

... a full cancellation of all $1/\epsilon_{\mathrm{IR}}$ poles against known divergences from soft & mixed contributions to the pressure!

∴ Remaining comp. should be amenable to numerics

Topology	$\tilde{p}^{\mathrm{h}}_{-2}$	$\tilde{p}^{\mathrm{h}}_{-1}$
	0	$-0.17590(60)C_A$
	$\frac{11}{6}C_A$	$\left(-\frac{11}{3}L - 3.22027\right)C_A$
	0	$\left(3 - \frac{\pi^2}{4}\right)(2C_F - C_A)$
	0	$\left(\frac{7\pi^2}{144} - \frac{5}{12} + \frac{2}{3}L\right)N_f$
	0	$\left(8 - \frac{2\pi^2}{3}\right)C_F$
	0	$-2C_F$
	0	$\left(-1 + \frac{\pi^2}{3}\right)C_F$
	0	$-\frac{1}{2}C_A$

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

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Challenge with masters: due to breaking of Lorentz symmetry by μ, most automated tools of vacuum QFT not applicable!

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

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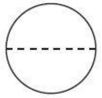
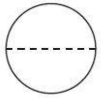
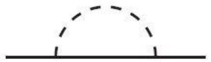
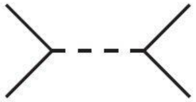
Challenge with masters: due to breaking of Lorentz symmetry by μ, most automated tools of vacuum QFT not applicable!

Traditional approach combines cutting rules and other tricks on case-by-case basis:

 $\rightarrow$  $- 2\int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\theta(\mu - E(\vec{p}))}{2E(\vec{p})}$ 

$+ \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{\theta(\mu - E(\vec{p}))}{2E(\vec{p})}\int \frac{d^3\vec{q}}{(2\pi)^3}\frac{\theta(\mu - E(\vec{q}))}{2E(\vec{q})}$ 

Severe oroblem: IR-safe integrals split to divergent parts, leaving no systematic way to apply numerics to the problem! ☹

Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921

Hard contributions: Renewed hope from Loop Tree Duality

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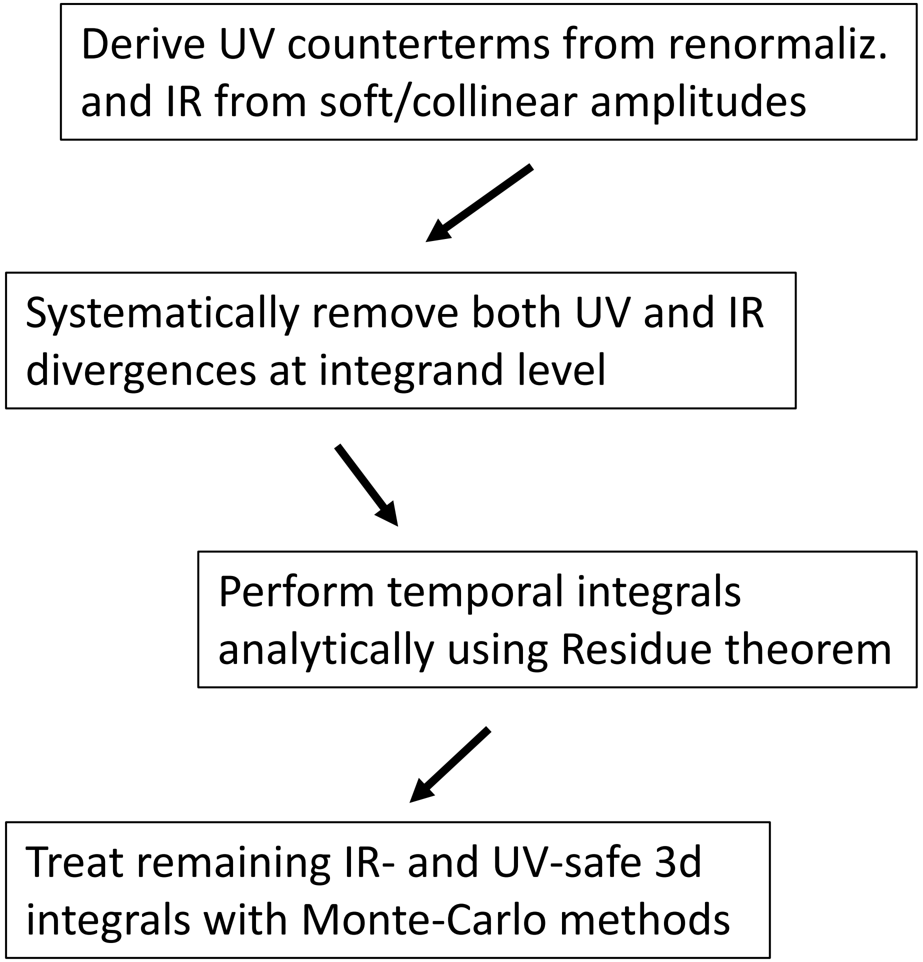

Established tool of vacuum QFT for efficient evaluation of scattering amplitudes:

1) Algorithmic removal of IR and UV divergences at integrand level via counterterms
2) Analytic handling of temporal mom. integrals
3) Numerical treatment of spatial integrals

Capatti, Hirschi, Kermanschach, and Ruijl, PRL 123 (2019), 1906.06138

Hard contributions: Renewed hope from Loop Tree Duality

Derive UV counterterms from renormaliz. and IR from soft/collinear amplitudes

↓

Systematically remove both UV and IR divergences at integrand level

↓

Perform temporal integrals analytically using Residue theorem

↓

Treat remaining IR- and UV-safe 3d integrals with Monte-Carlo methods

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1) Algorithmic removal of IR and UV divergences at integrand level via counterterms
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3) Numerical treatment of spatial integrals

$G = (G - G_{\mathrm{CT}}) + G_{\mathrm{CT}}$, where

- $G - G_{\mathrm{CT}}$ finite and computed numerically
- $G_{\mathrm{CT}} = G_{\mathrm{CT}}^{\mathrm{IR}} + G_{\mathrm{CT}}^{\mathrm{UV}}$ with
  - $G_{\mathrm{CT}}^{\mathrm{IR}}$ removes IR divergences
  - $G_{\mathrm{CT}}^{\mathrm{UV}}$ from renormalization (R theorem)

Capatti, Hirschi, Kermanschach, and Ruijl, PRL 123 (2019), 1906.06138

Hard contributions: Renewed hope from Loop Tree Duality

Important: with Lorentz invariance automatically broken by step 2, no obstacle to apply same procedure at nonzero μ or T

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Derive UV counterterms from renormaliz. and **IR counterterms from HTL theory**

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Navarrete, Paatelainen, Seppänen, PRD 110 (2024), 2403.02180

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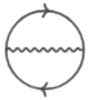
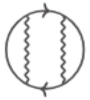
Navarrete, Paatelainen, Seppänen, PRD 110 (2024), 2403.02180
Kärkkäinen et al. (incl. AV), PRL 135 (2025), 2501.17921
Navarrete, Paatelainen, Seppänen, Tenkanen, 2507.07014

Hard contributions: Renewed hope from Loop Tree Duality

$T \neq 0, \mu = 0$:

Diagram	ε^{-1}	$\varepsilon^{0}_{\text{traditional}}$	$\varepsilon^{0}_{\text{tLTD}}$	$N\ [10^6]$	[μs]
$-\frac{1}{2}$ 	0	−0.208 333 333	−0.208 27(12)	30	5.7
$-\frac{1}{3}$ 	−0.002 375	−0.035 247 512	−0.035 30(6)	200	18.8
$-\frac{1}{4}$ 	−0.000 264	−0.004 098 706	−0.004 100 9(34)	200	16.3
$-\frac{1}{2}$ 	−0.002 111	−0.026 024 724	−0.026 00(4)	200	13.3

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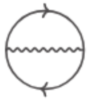
$-\frac{1}{2}$ 	0	−0.208 333 333	−0.208 27(12)	30	5.7
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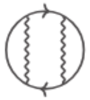
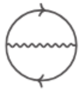
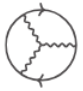
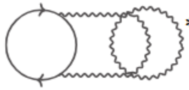
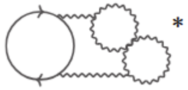
$-\frac{1}{2}$ 	−0.002 111	−0.026 024 724	−0.026 00(4)	200	13.3

$\mu \neq 0, T = 0$:

Diagram	$\varepsilon^{-2}\ [10^{-3}]$	$\varepsilon^{-1}\ [10^{-3}]$	$\varepsilon^{0}_{\text{traditional}}\ [10^{-3}]$	$\varepsilon^{0}_{\text{dLTD}}\ [10^{-3}]$	$N\ [10^6]$	[μs]
$-\frac{1}{2}$ 	0	0	−3.849 743	−3.8495(12)	30	5.8
$-\frac{1}{3}$ 	0	−0.219 409	−1.682 136	−1.682 10(23)	200	12.9
$-\frac{1}{12}$  *	0	0.001 563	0.023 186	0.023 18(6)	500	21.0
$-\frac{1}{8}$  *	0.014 068	0.173 504	1.174 161	1.174 23(10)	500	13.0

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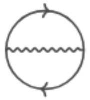
$-\frac{1}{2}$ 	0	−0.208 333 333	−0.208 27(12)	30	5.7
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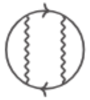
$-\frac{1}{2}$ 	−0.002 111	−0.026 024 724	−0.026 00(4)	200	13.3

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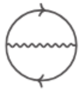
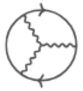
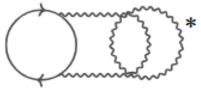
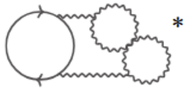
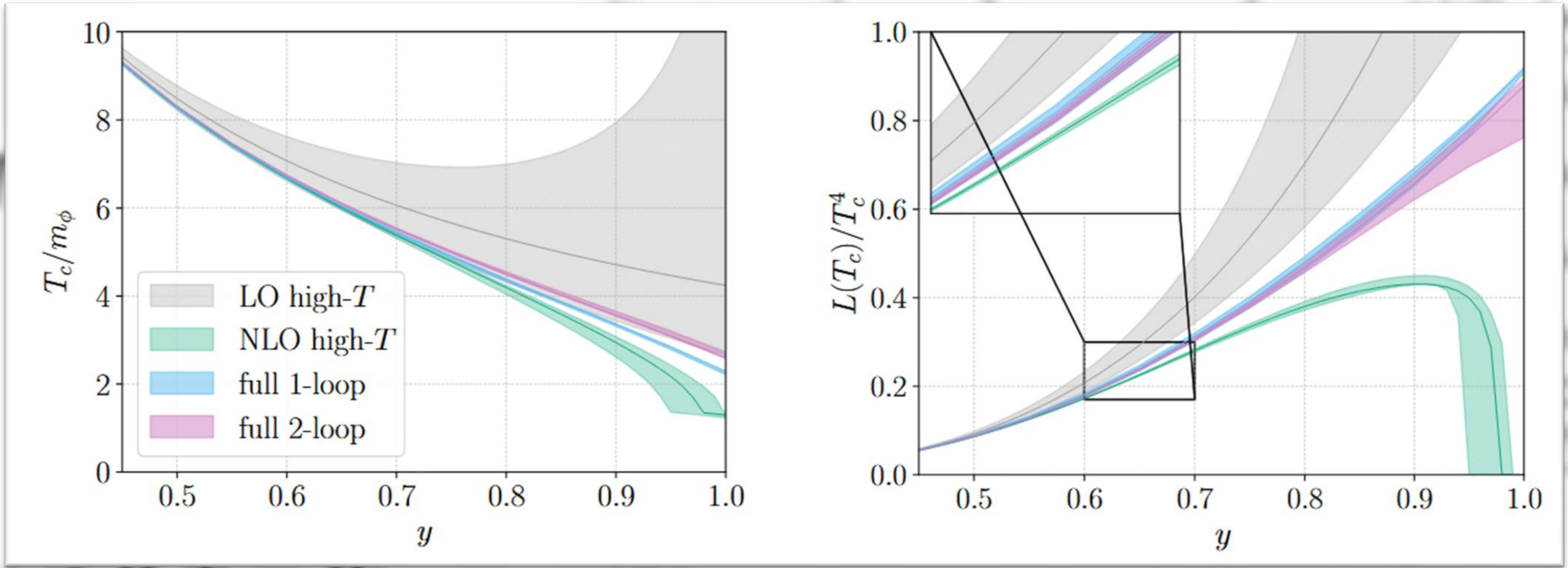
Diagram	$\varepsilon^{-2}\ [10^{-3}]$	$\varepsilon^{-1}\ [10^{-3}]$	$\varepsilon^0_{\text{traditional}}\ [10^{-3}]$	$\varepsilon^0_{\text{dLTD}}\ [10^{-3}]$	$N\ [10^6]$	[μs]
$-\frac{1}{2}$ 	0	0	−3.849 743	−3.8495(12)	30	5.8
$-\frac{1}{3}$ 	0	−0.219 409	−1.682 136	−1.682 10(23)	200	12.9
$-\frac{1}{12}$  *	0	0.001 563	0.023 186	0.023 18(6)	500	21.0
$-\frac{1}{8}$  *	0.014 068	0.173 504	1.174 161	1.174 23(10)	500	13.0

Coming up: $T \neq 0 \neq \mu$ (and $m_s \neq 0$) up to three loops; all 4-loopers at $\mu \neq 0$, $T = 0$; $O(\alpha_s^3)$ pure Yang-Mills pressure; $O(\alpha_s^3)$ quark matter pressure at all T; …

Hard contributions: Renewed hope from Loop Tree Duality



$\mu \neq 0, T = 0$:

Cosmological phase transitions without high-temperature expansions

Pablo Navarrete,[1,*] Risto Paatelainen,[1,†] Kaapo Seppänen,[1,‡] and Tuomas V. I. Tenkanen[1,§]

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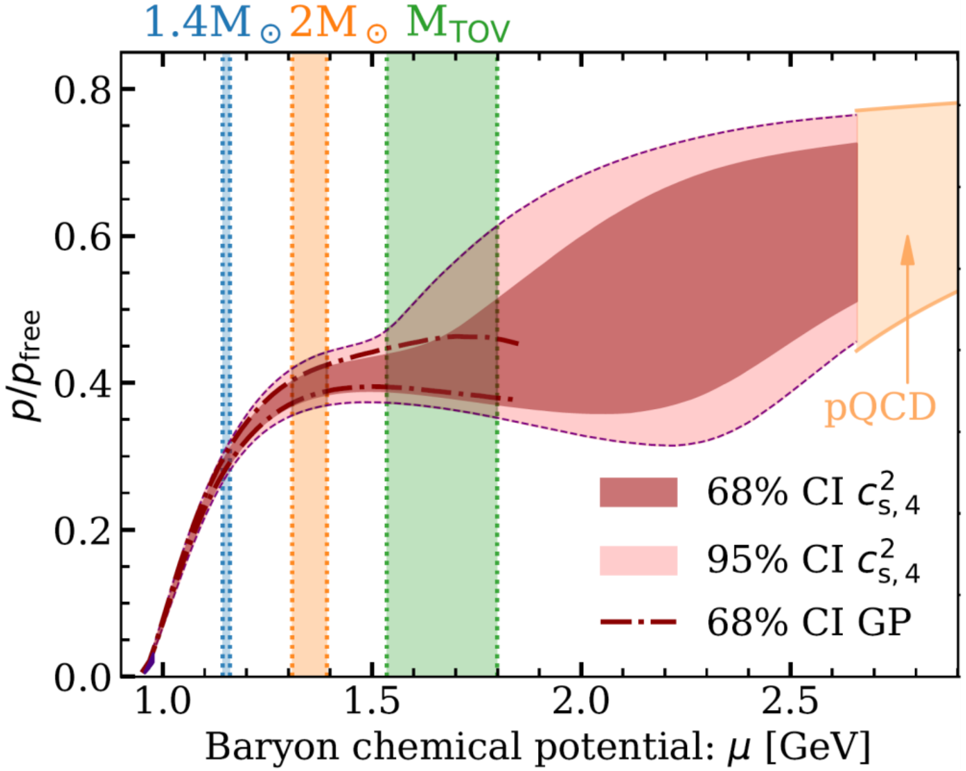
Coming up: ... at $\mu \neq 0$, $T = 0$; $O(\alpha_s^3)$ pure Yang-Mills pressure; $O(\alpha_s^3)$ quark matter pressure at all T; ...

Future directions

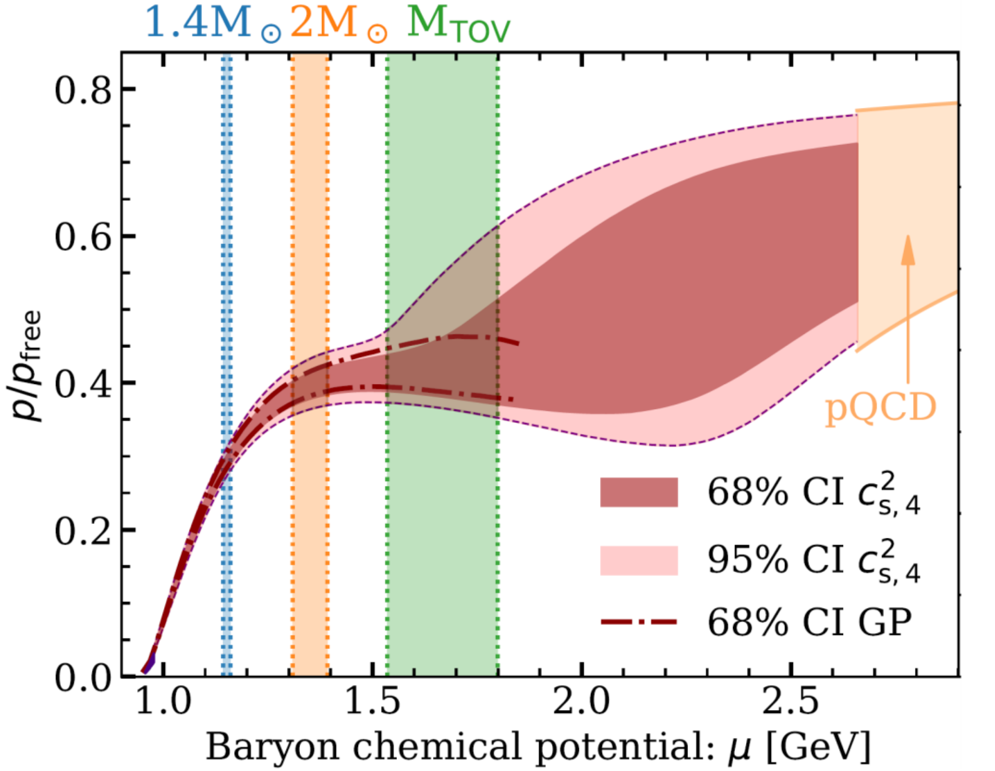
Inferring the EoS of NS-matter has become an active topic due to close connection to the phase of matter inside NS cores.

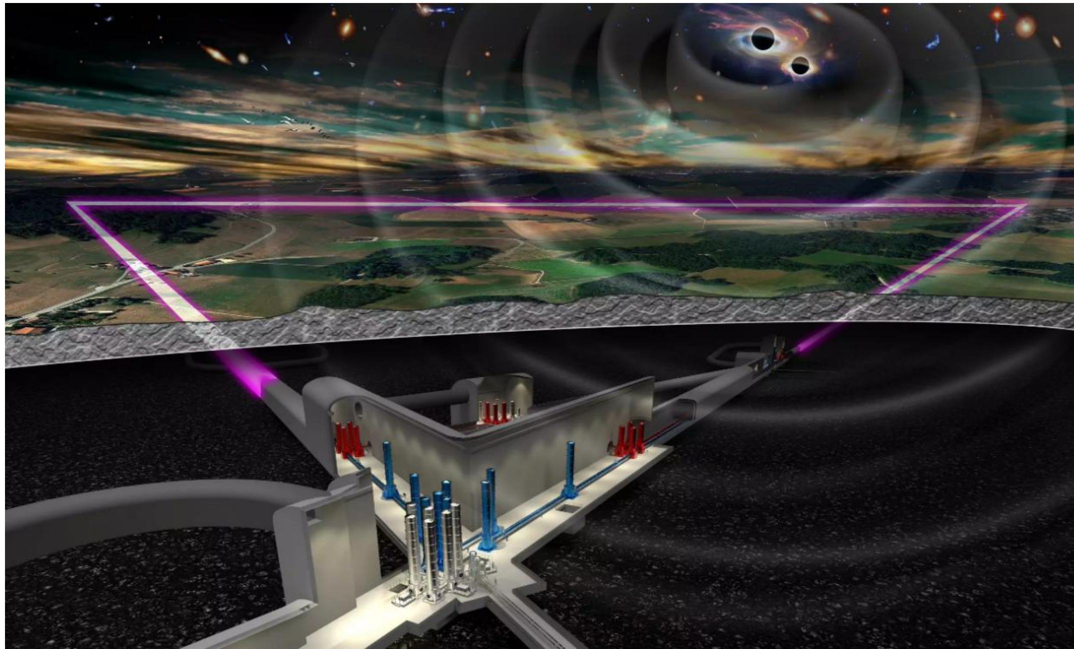


Inferring the EoS of NS-matter has become an active topic due to close connection to the phase of matter inside NS cores.

The near future holds great promise on both low-density and observational sides:

- NICER and future X-ray observatories (eXTP, etc.) will provide improved radii
- LVG (& later ET, CE) will vastly increase # of tidal-deformability measurements
- Rapid progress towards $2n_s$ in *ab-initio* Chiral Effective Theory calculations





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But due to a recent generalization of LTD to a thermal setting, we can expect to see equally dramatic improvements from the pQCD side!