Pairing beyond BCS

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in collaboration with:

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Outline

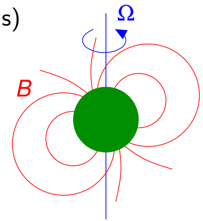
- Introduction: pairing in neutron stars
- Screening of the pairing interaction ($V_{\text{low-}k}$+Skyrme)
- HFB+BMBPT approach with low-momentum interactions
- Results for a s-wave contact interaction
- Conclusions and outlook

References:

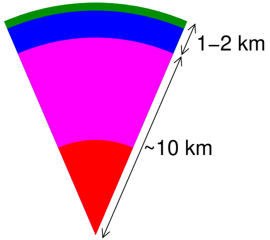
Screening with $V_{\text{low-}k}$: S. Ramanan & MU, PRC 98, 024314 (2018)
Screening with RPA: MU & S. Ramanan, PRC 101, 035803 (2020)
Pairing in neutron matter: S. Ramanan & MU, EPJ ST 230, 567 (2021)
EoS with $V_{\text{low-}k}$: V. Palaniappan, S. Ramanan & MU, PRC 107, 025804 (2023)
Induced 3BF: V. Palaniappan, S. Ramanan & MU, PRC 111, 035803 (2025)

Basic properties of neutron stars

- Produced in core-collapse supernova explosions
- Very compact: $M \sim 1 - 2M_\odot$ $(2 - 4 \times 10^{30}$ kg) in a radius of $R \sim 10$ km
  $\rightarrow$ $\rho >$ nuclear saturation density
- Rapid rotation (periods range from seconds to milliseconds)
- Strong magnetic field B
  typically 10^{12} G, in "magnetars" up to 10^{14} G
- B not aligned with the rotation axis leads to periodic e.m. emission (pulsar) and slows down the rotation



- A neutron star has a complex inner structure:



outer crust: Coulomb lattice of neutron rich nuclei in a degenerate electron gas

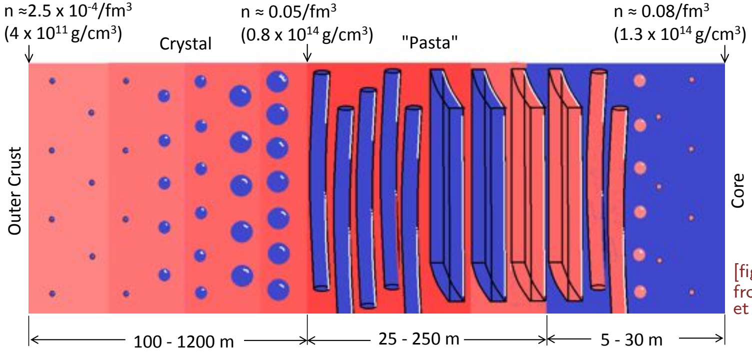
inner crust: unbound neutrons form a neutron gas between the nuclei (clusters)

outer core: homogeneous $n, p, e^-, (\mu^-)$ matter

inner core: densities up to a few times ρ_0, new degrees of freedom: hyperons? quark matter?

Structure of the inner crust

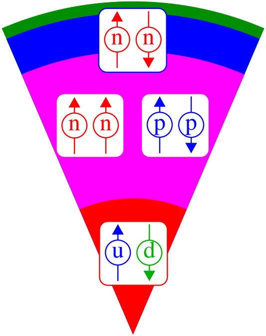
- Presence of a gas of unbound neutrons between the nuclei (clusters) + almost uniform degenerate electron gas to ensure global charge neutrality
- BCC crystal and "pasta phases": rods ("spaghetti"), slabs ("lasagne")



[figure adapted from W. Newton et al. (2011)]

Superfluidity in neutron stars

- Typical temperature of a neutron star: $T \sim 10^6 - 10^9$ K $\sim 0.1 - 100$ keV
- Compared to nuclear energy scales, this is very low!
- BCS gap equation ($T = 0$): $\Delta_p = -\sum_{p'} V_{p,p'} \frac{\Delta_{p'}}{2E_{p'}}$ $\quad E_{p'} = \sqrt{(\epsilon_{p'} - \mu)^2 + \Delta_{p'}^2}$
- Different types of superfluidity may exist in neutron stars:



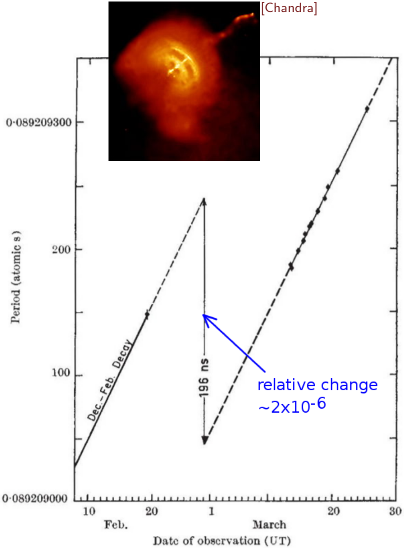
inner crust:
neutron pairing in s wave (pairs with total spin $S = 0$),
$T_c \sim 1$ MeV $\quad \rightarrow$ subject of this talk

outer core:
neutron pairing in p wave (pairs with total spin $S = 1$)
proton pairing in s wave

quark core (speculative):
"color superconductivity", $T_c \sim 10$ MeV
[e.g. Alford et al. RMP (2008)]

Pulsar glitches

- Rotation of a neutron star: very regular, period increases slowly with time
- Glitch = sudden speed-up of the rotation, followed by a slow relaxation
- First glitch observed 1969 in the Vela pulsar, since then 520 glitches in 180 different pulsars [R.N. Manchester (2017)]
- Widely accepted explanation by Manchester and Itoh (1975): pinning of quantized vortices to the clusters in the inner crust
- While the normal part of the star is slowing down (Ω_n), the superfluid neutrons are spinning at constant frequency (Ω_s)
- When $\Omega_s - \Omega_n$ becomes too large, the vortices get unpinned and the superfluid transfers angular momentum to the normal part



[Chandra]

[Radhakrishnan and Manchester, Nature 222 (1969)]

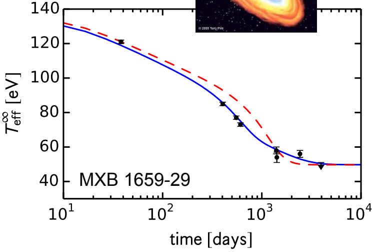
Cooling

- One day after the supernova, T has already dropped from $\sim 10^{11}$ to $\sim 10^{9}$ K
- For about 10^5 years, ν emission (from the core) is the dominant cooling mechanism
- For older stars, cooling is dominated by photon emission
- Cooper pairing affects cooling through:
  - $\nu\bar{\nu}$ emission via the PBF (pair breaking and formation) mechanism,
  - strongly reduced specific heat

Special case: accreting neutron stars

- Neutron star with a companion star
- Matter falling on the neutron star heats the surface
- Deep crustal heating: nuclear reactions in deeper layers of the crust
- X-ray outbursts take a few weeks or months (or even years), then cooling during a couple of years of quiescence
- Particularly sensitive to Cooper pairing in the neutron-star crust

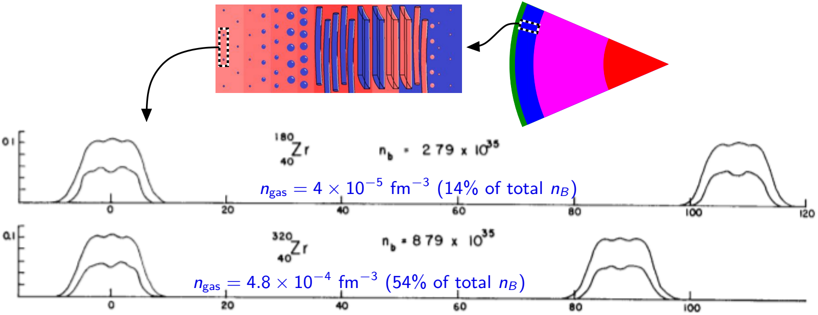
[figure by T. Piro]



[Deibel et al., ApJ 839 (2017)]

Relevant densities of "dilute" neutron matter

- Upper layers of the inner crust (close to neutron-drip density $\sim 2.5 \times 10^{-4}$ fm^{-3})



[Negele and Vautherin, NPA 207 (1973); similar results by Baldo et al., PRC 76 (2007)]

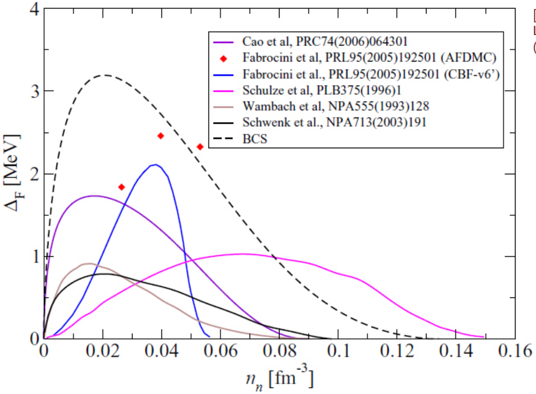
- In spite of its "low" density (still $\rho \gtrsim 10^{11}$ g/cm^3), the neutron gas is relevant because it occupies a much larger volume than the clusters

- Deeper in the crust: n_{gas} increases up to $\sim n_0/2 = 0.08$ fm^{-3}

$\rightarrow$ Relevant range: $n \sim 4 \times 10^{-5} \ldots 0.08$ fm^{-3}, $k_F = (3\pi^2 n)^{1/3} \sim 0.1 \ldots 1.3$ fm^{-1}

Pairing in neutron matter: results in the literature (2008)

- Concentrate on *s*-wave pairing (*p*-wave pairing expected at higher densities)

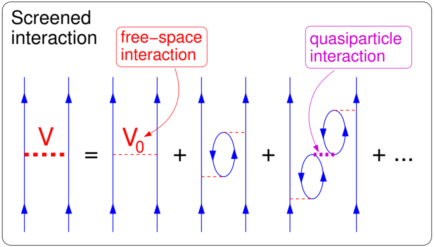


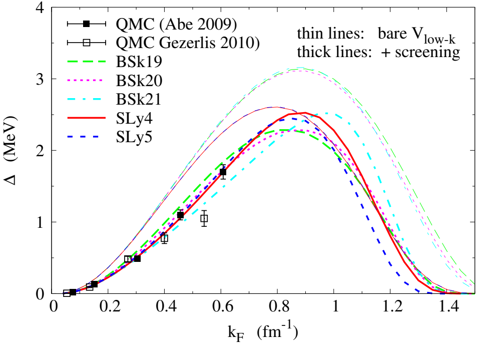
[Chamel & Haensel, Liv. Rev. Relativity (2008)]

- Gap first increases with density (because of increasing density of states) but then it decreases (because of the finite range of the interaction)
- Large corrections beyond BCS, but no consensus

Recent progress at low densities

- Screening calculation with low-momentum interaction $V_{\text{low-}k}$ for the pairing and Skyrme functionals for m^* and the RPA [M.U. and S.Ramanan, PRC (2020), EPJ ST (2021)]
- Zoom on low density: $k_F \propto n^{1/3}$
- Necessary to scale the cutoff with k_F ($\Lambda = 2.5k_F$) to recover the GMB result $\Delta/\Delta_{\text{BCS}} \to 0.45$ for $k_F a \to 0$
- $\Delta/\Delta_{\text{BCS}} \approx 0.6$ at relevant low densities, in good agreement with QMC calculations
- But inner crust involves densities up to $n \simeq 0.08\,\text{fm}^{-3}$ ($k_F \simeq 1.3\,\text{fm}^{-1}$) where large uncertainties persist: m^*, quasiparticle interaction (Landau parameters), 3-body force, ...





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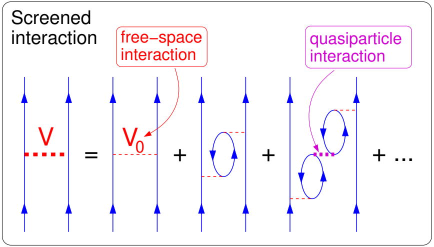
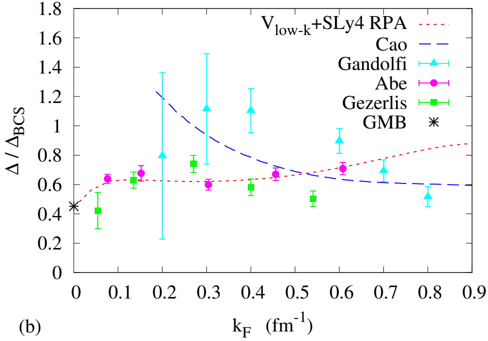




(b)

Superfluid neutron matter within HFB + BMBPT

- Goal: eliminate uncertainties due to different Skyrme functionals
- Hard core of "realistic" nucleon-nucleon potentials requires resummations, nuclei are not bound in Hartree-Fock(-Bogoliubov, HFB) approximation
- Soft interactions ($V_{\text{low-}k}$, SRG) much better suited for perturbative methods
- HFB with low-momentum interactions and perturbative corrections can give good results for open-shell nuclei [e.g., Tichai et al. 2019]
- Hamiltonian: $H = \mathcal{E}_{\text{HFB}} + \sum_{k\sigma} E_k \beta^\dagger_{k\sigma}\beta_{k\sigma} + {:}V{:}$

  $\beta_{k\sigma} = u_k a_{k\sigma} + (-1)^{\frac{1}{2}-\sigma} v_k a^\dagger_{-k-\sigma}$ = quasiparticle (qp) operator

  ${:}V{:}$ = interaction normal-ordered w.r.t. the HFB ground state
- Bogoliubov Many-Body perturbation theory (BMBPT):

  treat ${:}V{:}$ as a perturbation

  $\rightarrow$ ground state is expanded as $|\text{HFB}\rangle$ + 2qp, 4qp, ...excitations.
- Nambu-Gor'kov diagrammatic formalism [see e.g. Schrieffer's book (1964)]
  2×2 propagators combining normal and anomalous Green's functions

Low-momentum interactions

$V_{\text{low}-k}$ interactions [Bogner et al. NPA 784 (2007)]

- Matrix elements $V_{kk'} \to 0$ for k or $k' > \Lambda$
- Scattering phase shifts $\delta(k)$ for $k < \Lambda$ are independent of Λ

Similarity renormalization group (SRG) [Bogner et al. PRC 75 (2007)]

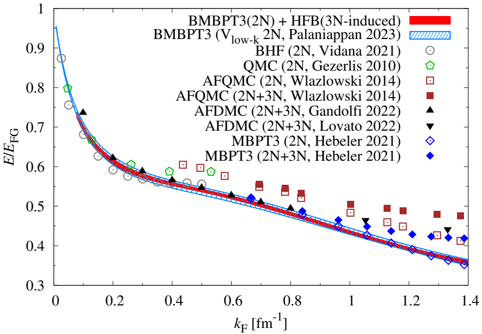
- Unitary transformation: phase shifts unchanged at all energies
- For $k, k' \gg \Lambda$: only diagonal matrix elements survive $\longrightarrow$ decoupling
- One can compute "induced" 3- (and higher-) body forces that are generated by the transformation when decreasing the cutoff

The cutoff Λ is in both cases unphysical

- Final results should be cutoff independent
(at least in the range of Λ where the BMBPT converges)
- Varying Λ allows one to estimate the magnitude of missing contributions

EoS of dilute neutron matter within BMBPT

- E/N in units of $E_{FG}/N = \frac{3}{5}E_F$
- Our most recent calculation:
  - 3rd order BMBPT
  - chiral N4LO 2-body force (2BF) softened with SRG
  - induced 3BF included
- To get right asymptotics at low density, scale the SRG cutoff λ with k_F
- error band: residual cutoff dependence for $1.3 \leq \lambda/k_F \leq 2.5$



[Palaniappan et al. PRC 111 (2025)]

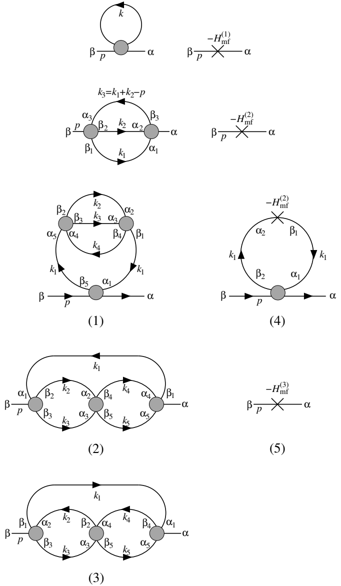
- Even if the bare 3BF is negligible at low density, the SRG induced 3BF is necessary at $\lambda \lesssim 2.5k_F$ to reduce cutoff dependence
- To be done: BMBPT corrections to the pairing gap

BMBPT diagrams for gap Δ and mean field U

- In Nambu-Gor'kov formalism:

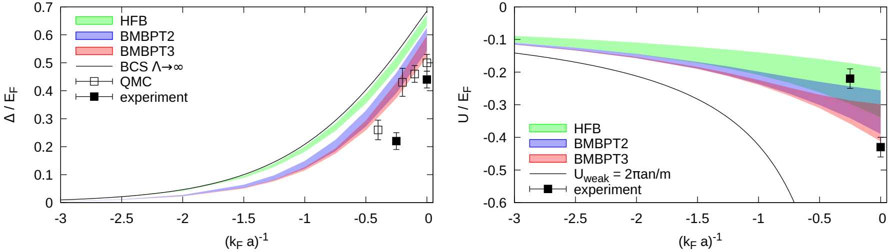
  $$2 \times 2 \text{ self-energy } \Sigma = \begin{pmatrix} U & \Delta \\ \Delta & -U \end{pmatrix}$$

- BMBPT corrections can be computed diagrammatically
- Start from a self-consistently corrected gap instead of the HFB gap (counterterms $\times$)
- Screening with one bubble appears at 2nd order
- At 3rd order, one can identify corrections from:
  - quasiparticle interaction in the screening (first step towards RPA resummation)
  - 2nd-order effective mass (m^*)
  - 2nd-order qp strength (Z factor)
  - vertex corrections



BMBPT3 results for Δ and U [S. Ramanan & MU, in preparation]

- Numerical implementation so far only for *s*-wave contact interaction corresponding to the situation in experiments with ultracold atoms
- $1/(k_F a)$ characterizes the interaction strength (a = scattering length)
- Vary cutoff in the range $1.5k_F \leq \Lambda \leq 2.5k_F$: cutoff dependence as indicator for missing contributions (induced 3-body force, higher orders of BMBPT)



QMC: [Carlson& Reddy PRL (2005), Gezerlis& Carlson PRC (2008)];
exp: [Schirotzek et al. PRL (2008)]; GMB: [Gor'kov & Melik-Barkhudarov JETP (1961)]

- Weak coupling: $\Delta \to (4e)^{-1/3}\Delta_{\text{BCS}} \approx 0.45\Delta_{\text{BCS}}$, $\qquad U \to \frac{4\pi a}{m} n_\sigma$

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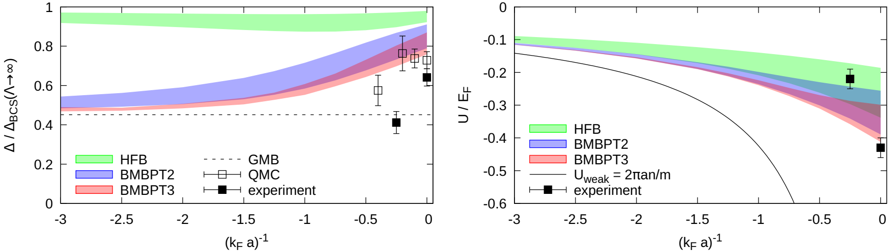

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Conclusions

- Superfluidity has important observational consequences in neutron stars, but difficult to pin down the values of Δ or T_c from observations
- At low densities: screening reduces the s-wave gap by a factor of ~ 0.6 compared to BCS
- Large uncertainties at the highest densities of the inner crust
- HFB+BMBPT with low-momentum interactions as in nuclear structure
- Diagrammatic calculation of the gap includes not only screening but also other effects
- First results for a contact interaction indicate convergence
- Experiments with ultracold atoms can serve as a test of the method

Outlook

- Diagrammatic calculation of the gap with neutron-neutron interaction
- Include 3BF (induced and bare) in the calculation of the gap
- Inner crust of neutron stars $\neq$ infinite uniform matter
- Outer core: role of protons? p-wave pairing?