L'aventure des cartes : an ongoing quest with Emmanuel and friends

Jérémie Bouttier

Emmanuel60, 15 May 2025

June 10th, 2005 on the garden side of this building

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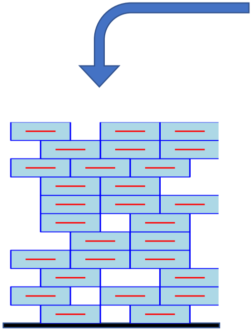
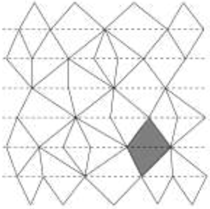
At the time, Emmanuel, Philippe and I had already written 9 papers together. But this was still the beginning of the adventure...

Back in time 6 years earlier (1999)





heaps

X. Viennot (1986)

Integrable 2D Lorentzian gravity and random walks

P. Di Francesco [a,1], E. Guitter [a], C. Kristjansen [b,2]

[a] CEA-Saclay, Service de Physique Théorique, F-91191, Gif sur Yvette Cedex, France
[b] The Niels Bohr Institute, Blegdamsvej 17, DK-2100 Copenhagen Ø, Denmark

Received 14 July 1999; accepted 11 October 1999

This paper by Emmanuel, Charlotte and Philippe marks the beginning of the application of combinatorial theory to 2D quantum gravity.

My first paper with Emmanuel and Philippe (2001-2002)

Still using the good old-fashioned matrix models to enumerate some decorated planar maps.
It involved interesting physics, and I still remember the morning where Emmanuel came up with his simple combinatorial interpretation for the 6-matrix-chain model.

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J. Phys. A: Math. Gen. **35** (2002) 3821–3854 PII: S0305-4470(02)33016-6

Critical and tricritical hard objects on bicolourable random lattices: exact solutions

J Bouttier, P Di Francesco and E Guitter

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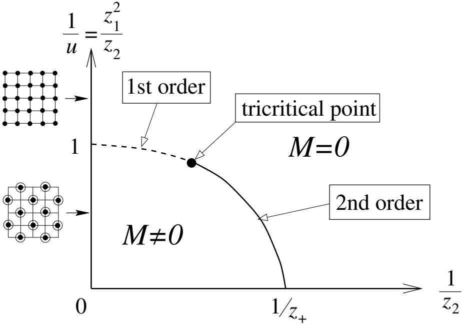

Our discovery of the bijective approach (2002)

One day, Emmanuel told us that he saw an interesting paper on the arXiv...

arXiv > math > arXiv:math/0205226

Mathematics > Combinatorics

[Submitted on 22 May 2002]

Random Planar Lattices and Integrated SuperBrownian Excursion

Philippe Chassaing, Gilles Schaeffer

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Random Planar Lattices and Integrated SuperBrownian Excursion

Philippe Chassaing, Gilles Schaeffer

It quickly inspired us a bijective proof of a 1997 result of Emmanuel, Philippe and Bertrand E.

Counting colored random triangulations

J. Bouttier, P. Di Francesco, E. Guitter

Service de Physique Théorique, CEA/DSM/SPhT, Unité de recherche associée au CNRS, CEA/Saclay, 91191 Gif sur Yvette Cedex, France

Received 26 June 2002; accepted 15 July 2002

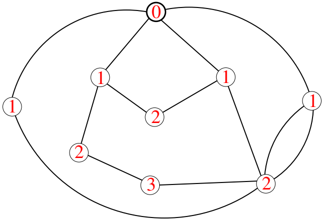
De: "Emmanuel Guitter" <guitter@spht>
À: "Gilles Schaeffer" <Gilles.Schaeffer@loria.fr>
Cc: philippe@spht.saclay.cea.fr, bouttier
Envoyé: Mardi 2 Juillet 2002 12:33:38
Objet: papier,...

Cher Gilles Schaeffer,

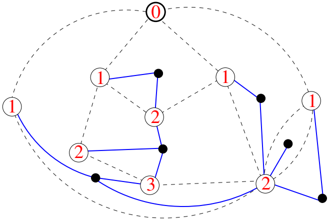
Merci infiniment de votre réponse.
> Je serais curieux de savoir par quel biais vous êtes arrivé à
> connaitre ma méthode.
En fait, nous avions remarqué dès 1997 en calculant la fonctionnelle génératrice des triangulations tricoloriées des vertex que la formule des U_i était celle de la fonctionnelle génératrice d'arbres tricoloriés avec une racine de couleur i. Nous avions

Several papers on bijections between planar maps and trees followed.

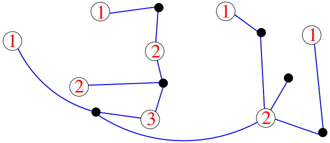
Our most successful one is Planar maps as labeled mobiles (2004) which introduces what is now known as “the” BDG bijection. It is Emmanuel’s (and my) most cited paper.



(a)



(b)



(c)

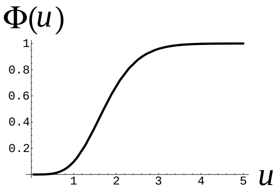
As already noted by Chassaing and Schaeffer, bijections are useful to study distances in random planar maps. In 2003 we came up with our first contribution on this topic:

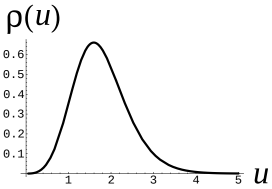
Geodesic distance in planar graphs

J. Bouttier, P. Di Francesco, E. Guitter

Service de Physique Théorique, CEA/DSM/SPhT, Unité de recherche associée au CNRS, CEA/Saclay, 91191 Gif-sur-Yvette cedex, France

Received 18 March 2003; accepted 23 April 2003





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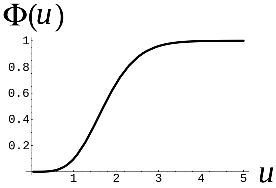

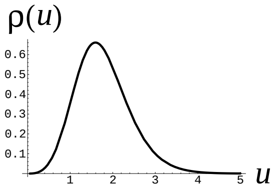

Using a bijection and an unexpected integrability phenomenon, we were able to prove a prediction of Ambjørn and Watabiki (1996) about the "two-point function" of pure 2D quantum gravity.

Our paper also contained a conjectured formula that Emmanuel and I were only able to prove 7 years later... Sofia Tarricone will talk tomorrow about this and another even more recent proof.

A few years and papers later...

Around 2007, Philippe moved to different topics, and we had the visit of Mark Bowick with whom Emmanuel and I wrote our only common paper not about maps: *Vacancy localization in the square dimer model*.

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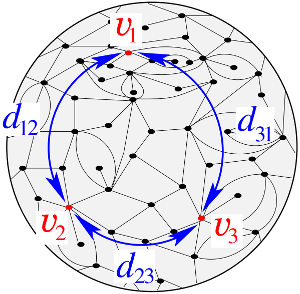
Also in 2007, a seminar on maps, first led by Grégory Miermont, started in Orsay. It gathered people from there, Saclay, and Palaiseau. Several of them are in this room today.

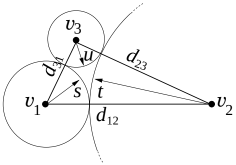
In the same year, Grégory wrote a paper introducing a new generalization of the Cori-Vauquelin-Schaeffer bijection. Grégory had probabilistic applications in mind, but Emmanuel and I were immediately puzzled about the possible applications to enumeration...

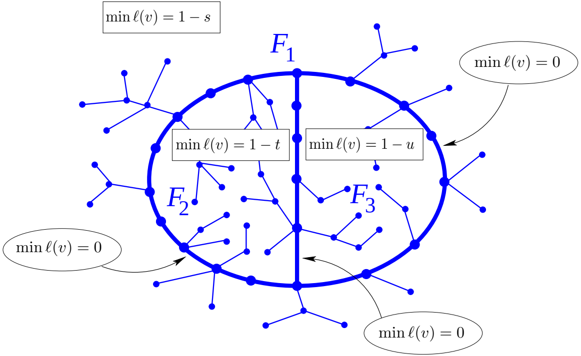
The three-point function of planar quadrangulations (2008)

Using Grégory's bijection and some own tricks, we found an exact formula for the generating function of planar quadrangulations with three marked vertices at prescribed distances:

$$F_{s,t,u} = \frac{[3]\left([s+1][t+1][u+1][s+t+u+3]\right)^2}{[1]^3[s+t+1][s+t+3][t+u+1][t+u+3][u+s+1][u+s+3]}, \quad [\ell] := \frac{(1-x^\ell)}{(1-x)}.$$







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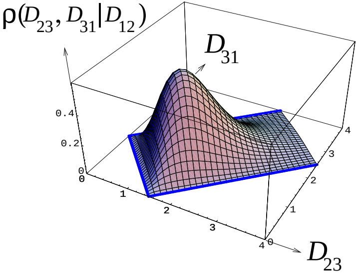

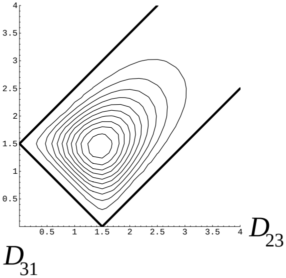

From distance statistics in random planar maps to continued fractions

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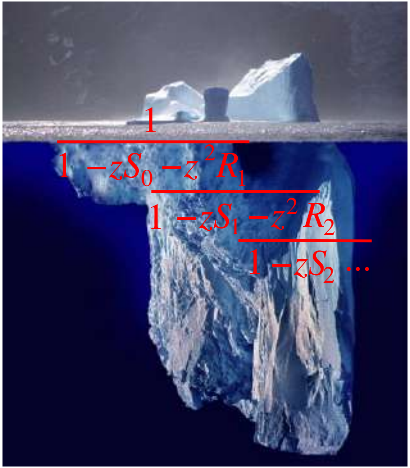
It allowed us in 2010 to prove the formula that we conjectured with Philippe back in 2003.



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Interestingly, this paper dared mentioning motivations from physics!

8. Some motivation from physics. In this section, we describe a motivation for the models discussed in this article that comes from the physics literature. In this discussion, we rely on a number of nonrigorous predictions and our only goal is to isolate some possible directions for future work. A useful reference is Appendix B in the survey by Duplantier [9] and the references therein.

As a starting point, we observe that models of random maps that are very similar to ours appear when studying annealed statistical physics models on random maps. These models are similar to more familiar models on regular lattices, such as percolation and Ising or Potts models, but they are defined on a random map that is chosen at the same time as the configuration of the model. To illustrate this, we will first deal with the so-called $O(N)$ model on a random planar quadrangulation. Let $\mathbf{q}$ be a rooted quadrangulation. A *loop configuration* on $\mathbf{q}$ is a collection $\mathcal{L} = \{c_1, \dots, c_k\}$, where $c_1, \dots, c_k$ are cycles, that is, paths on $\mathbf{q}$ starting and ending at the same point and never visiting the same vertex twice. It is further required that the paths c_i do not intersect. We set

$$\#\mathcal{L} = k \quad \text{and} \quad \lg(\mathcal{L}) = \sum_{i=1}^{k} \lg(c_i),$$

where $\lg(c_i)$ is the number of edges in the path c_i; see Figure 3 for an example.

Let $N \geq 0$ be fixed. The annealed $O(N)$ measure is the σ-finite measure over the set of all pairs $(\mathbf{q}, \mathcal{L})$, where $\mathbf{q}$ is a rooted quadrangulation and $\mathcal{L}$ is a loop configuration on $\mathbf{q}$, defined by

$$W_{O(N)}(\mathbf{q}, \mathcal{L}) = e^{-\beta \# F(\mathbf{q})} x^{\lg(\mathcal{L})} N^{\#\mathcal{L}},$$

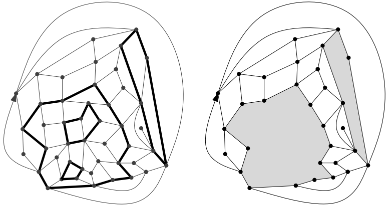

FIG. 3. *An $O(N)$ configuration on a rooted quadrangulation, with 4 cycles of total length 30, and the external gasket associated with this configuration, with shaded holes of degrees 6 and 14.*

The $O(n)$ loop model on random planar maps (2010-2012)

We were intrigued by the "gasket" construction of Jean-François and Grégory. We discovered a variant of their idea, where enumeration seemed tractable. As the computations were quite intricate, we were happy to be joined by Gaëtan Borot, who was a PhD student in Saclay at the time.

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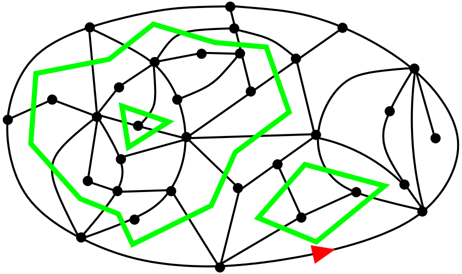


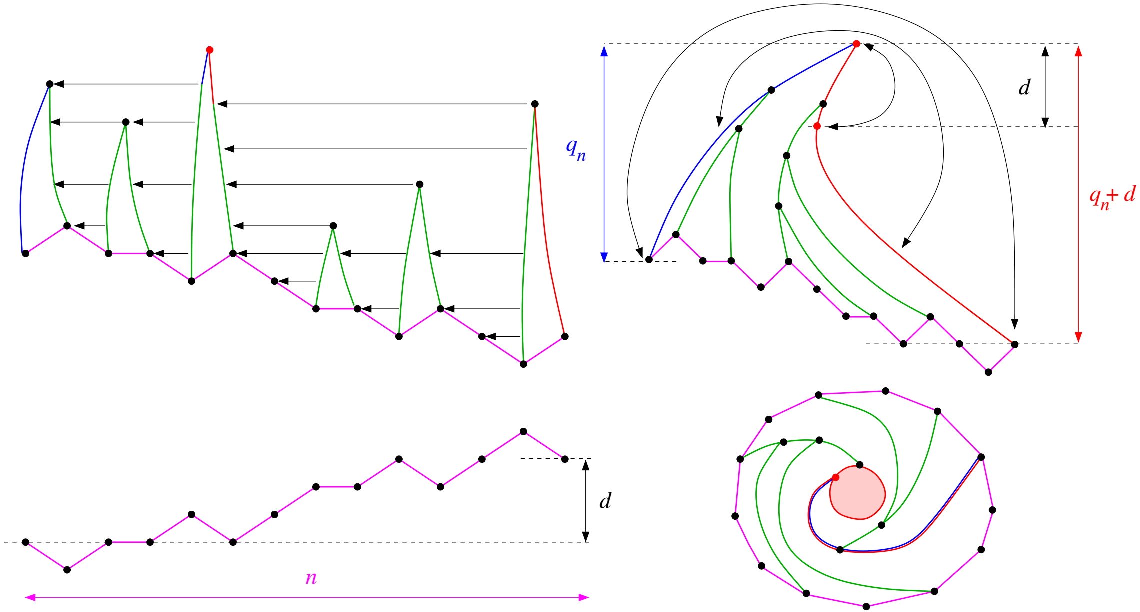
We wrote together three papers on the gasket decomposition of planar maps decorated with loops, thereby confirming the connection between the stable map and loop models.

Bijections again (2014)

With Éric Fusy, we wrote a paper introducing a common generalization of the BDG and of the Miermont-Ambjørn-Budd bijections.

Éric was also working at the time with Olivier Bernardi on a unified approach to bijections between planar maps and trees. Trying to understand their results, we developed the formalism of slice decomposition.

In a nutshell, the idea is to cut recursively maps along leftmost geodesics. Slices are our basic building blocks, which admit a recursive decomposition.

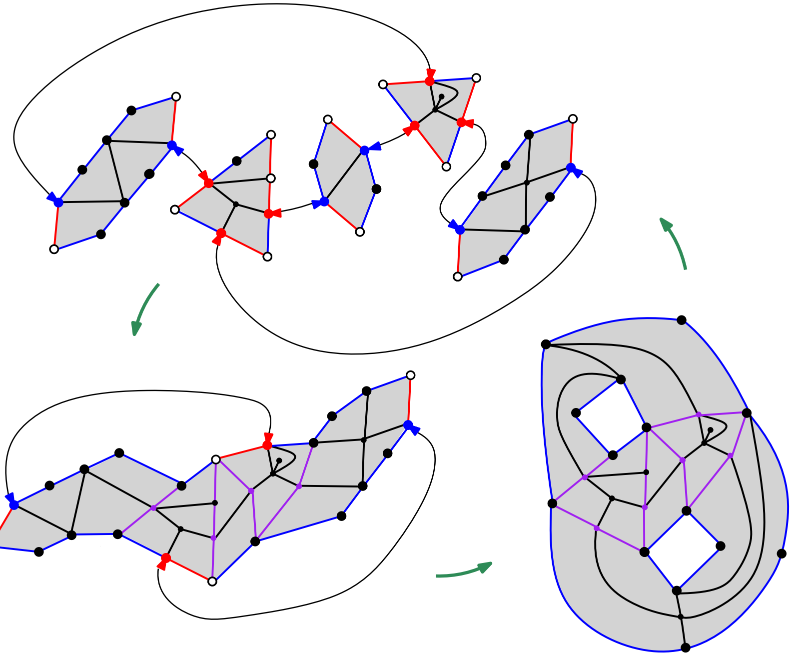

q_n
d
$q_n + d$
d
n

Restarting our collaboration (2020-2025+)

In 2015 I moved to Lyon and my collaboration with Emmanuel was paused for a while.

In 2019, I passed my *habilitation*, which led me to look back at my past contributions and think about possible new directions to explore. One of them was trying to extend the slice decomposition to maps with arbitary topologies. The first unsolved case was that of pairs of pants.

I first talked about this problem to Grégory, my "neighbor" in Lyon. Then, Emmanuel visited to give a seminar, and here was our new collaboration which kept us busy during the 2020 lockdowns.



Our collaboration nowadays

Emmanuel, Grégory and I have written three papers together and we still have projects going on.

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L'aventure des cartes continue!

Thank you Emmanuel, looking forward to still many years of collaboration!