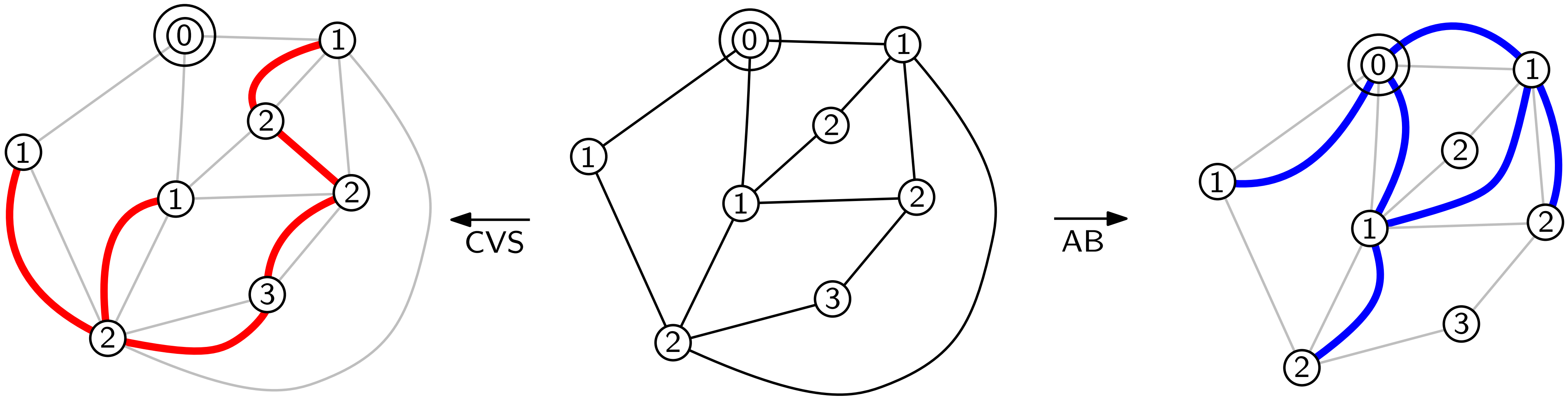
Schnyder orientations for d-irreducible maps

Éric Fusy (CNRS/LIGM, Université Gustave Eiffel)
Joint work with Olivier Bernardi and Shizhe Liang

L'esprit des cartes, une conférence en l'honneur d'Emmanuel Guitter
IPhT, CEA, May 15,16 2025

Ambjørn-Budd bijection



Journées Cartes 2013 at IPhT

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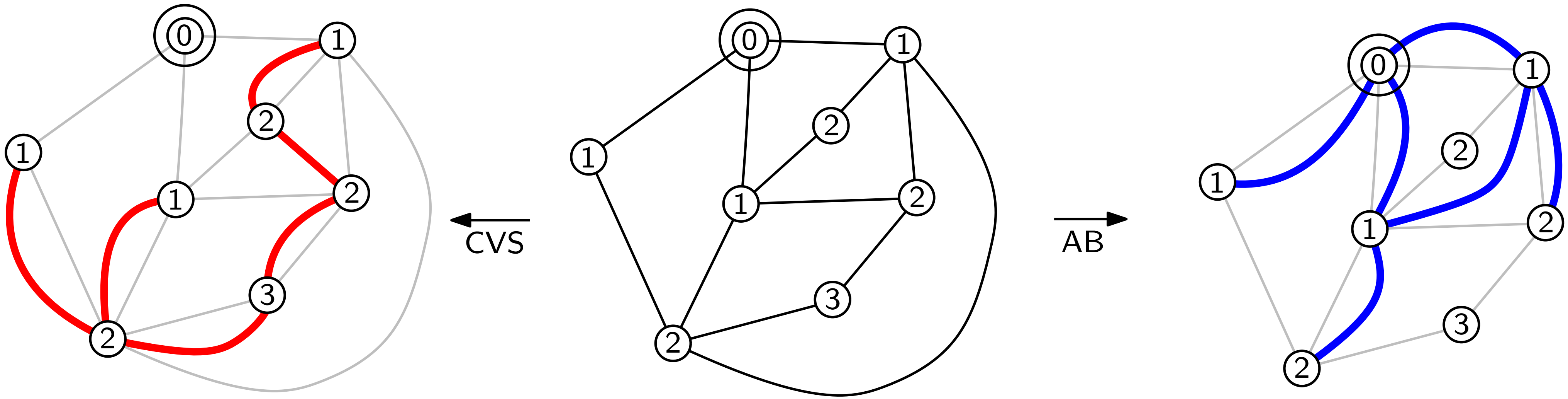
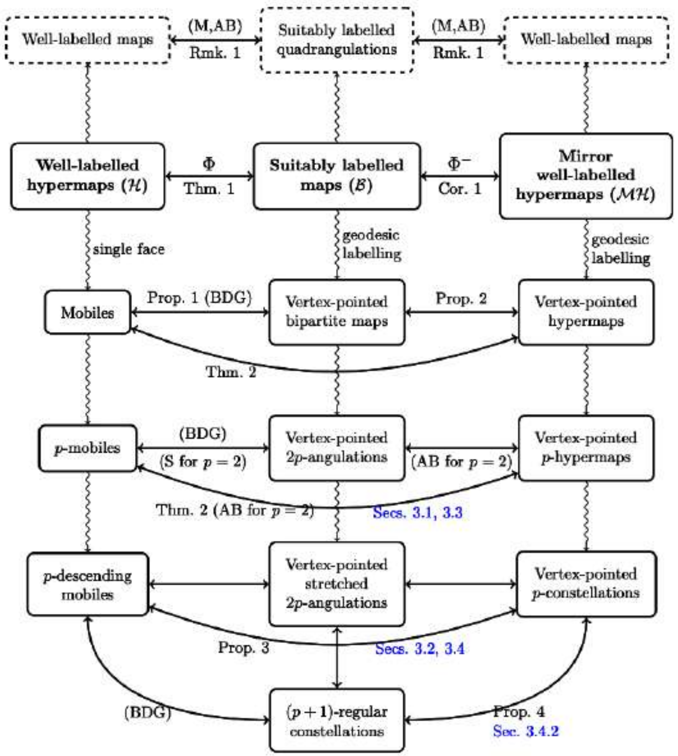


extension to the BDG setting



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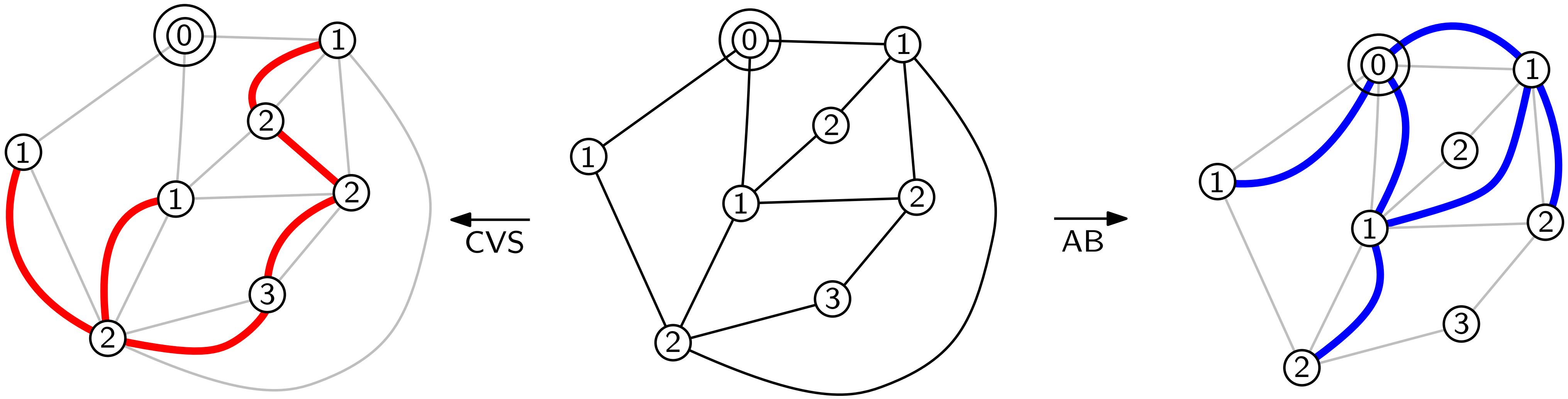

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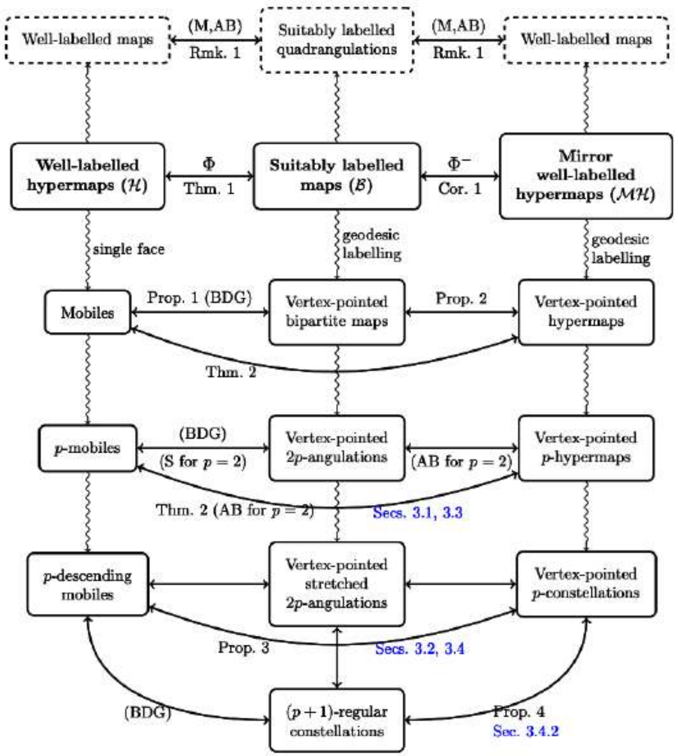

two bivariate extensions of 2-point function:

- odd labels (faces via Tutte's bijection)
- local-max labels (faces via AB)

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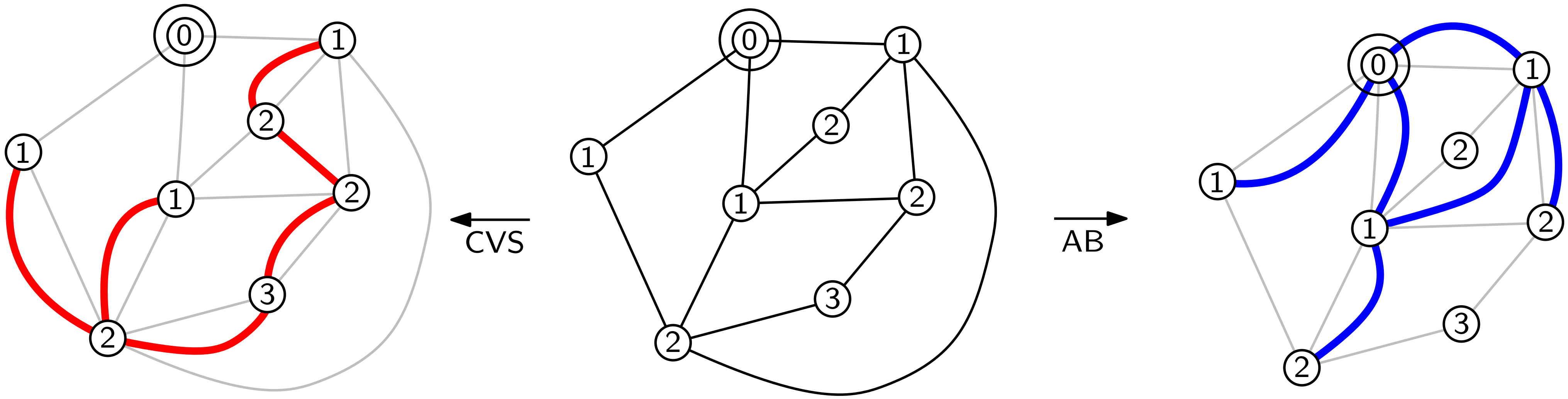

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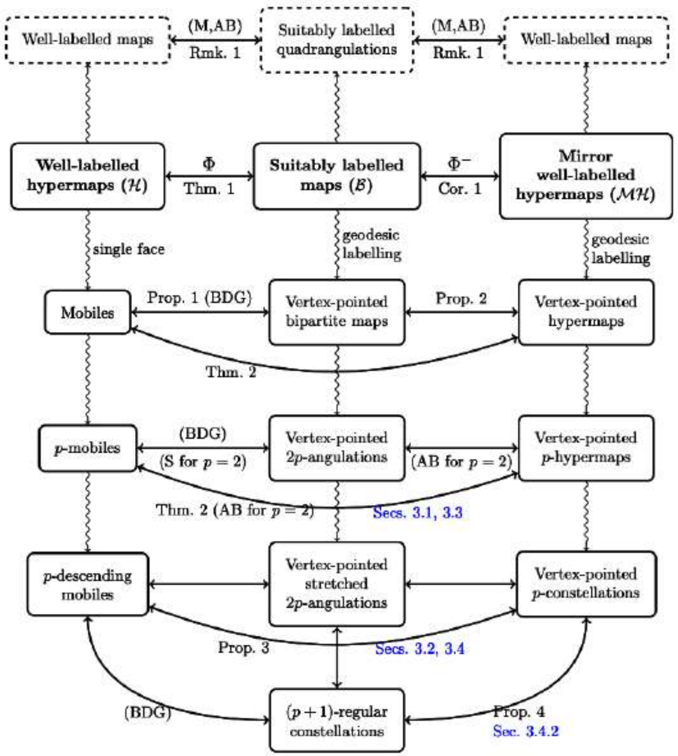

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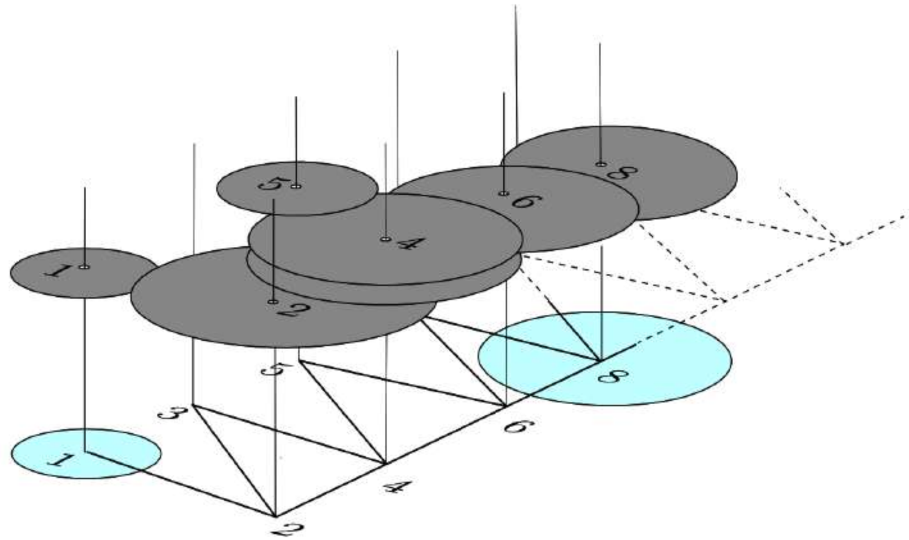
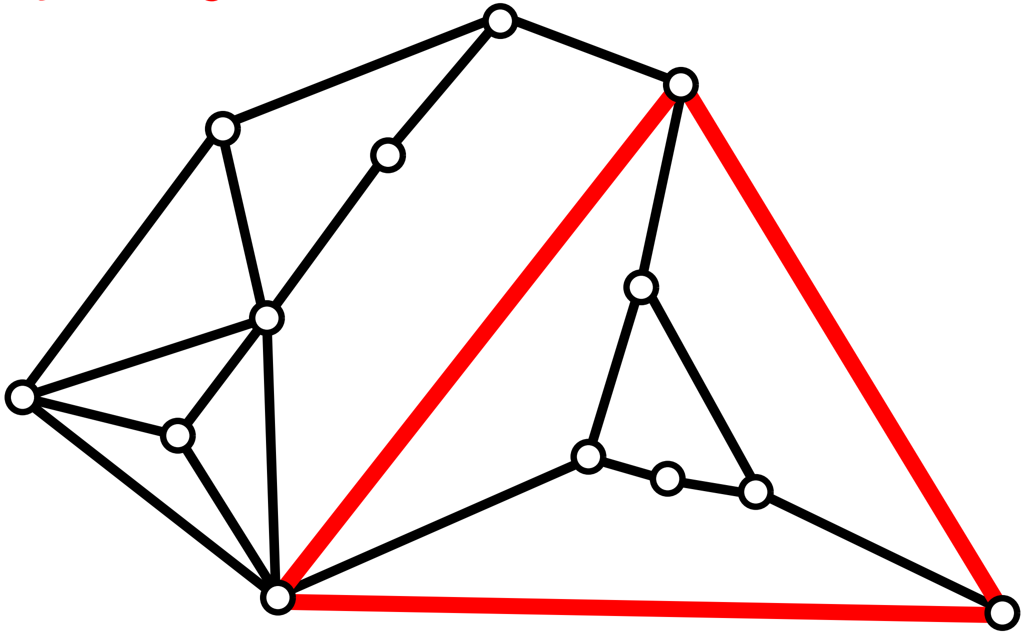


Girth parameters on maps

girth = length of a shortest cycle

girth = 3



Rk: If $girth = d$ then all faces have **degree at least** d

Girth parameters on maps

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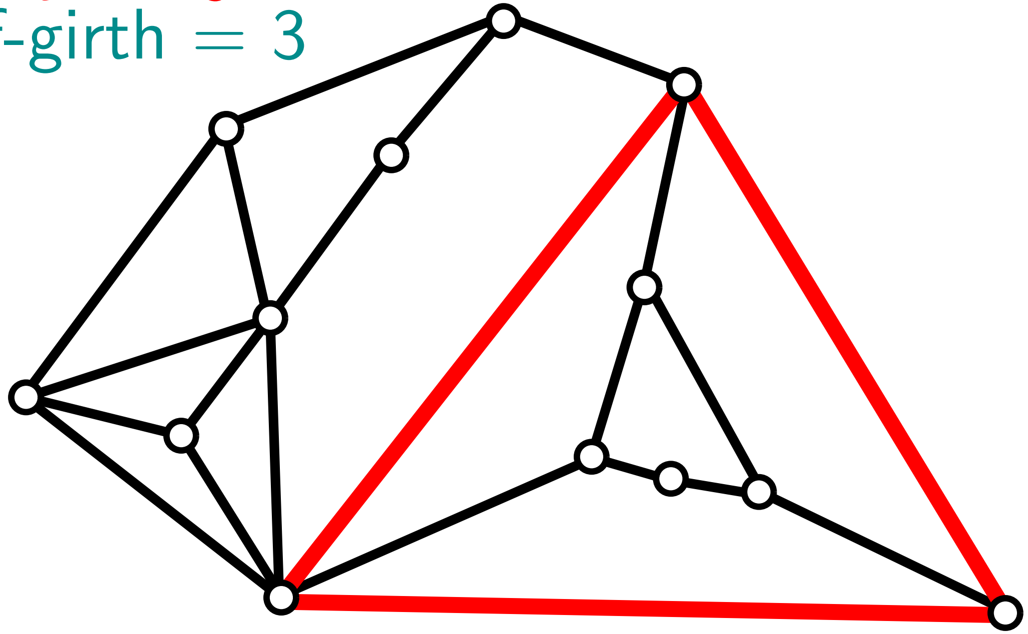
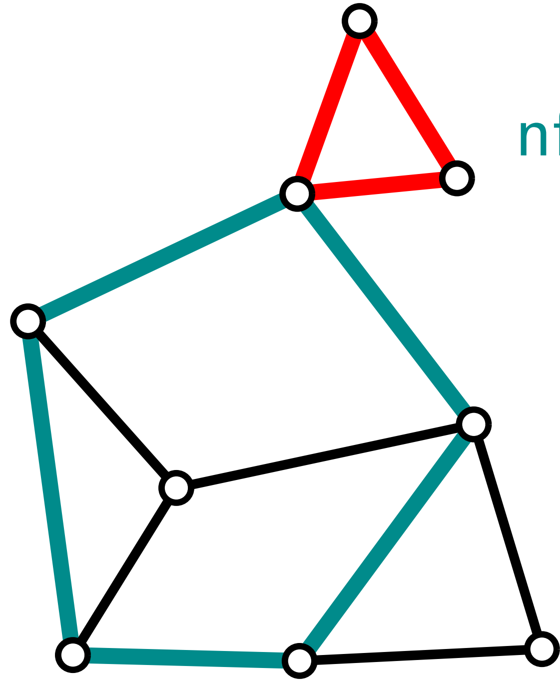
non-facial girth = length of a shortest **non-facial** cycle





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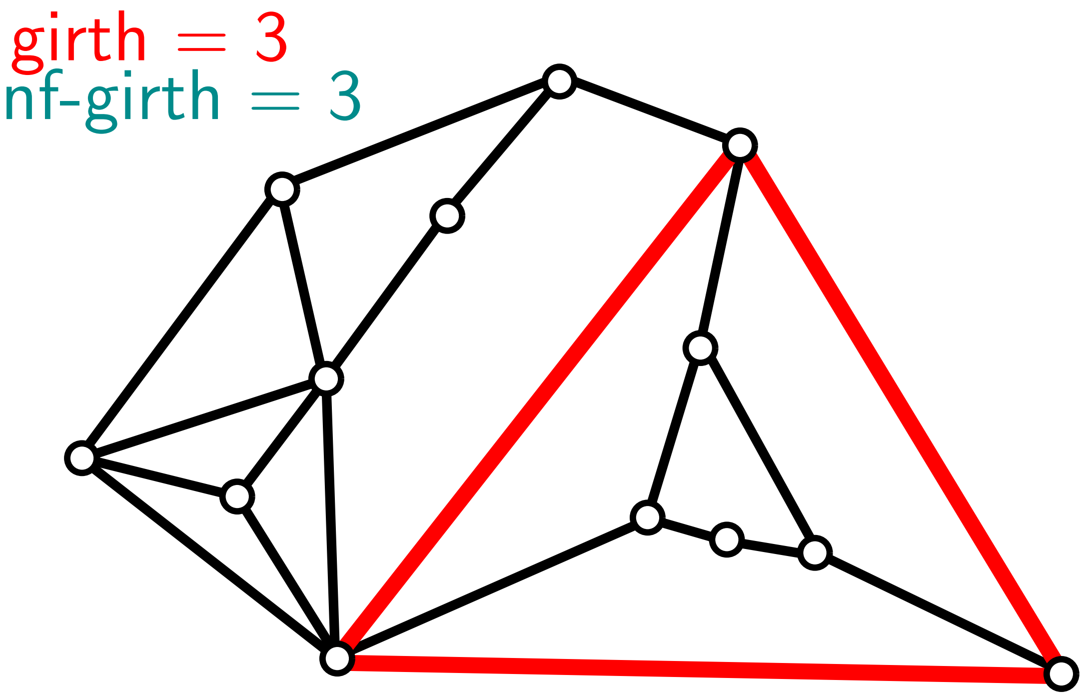

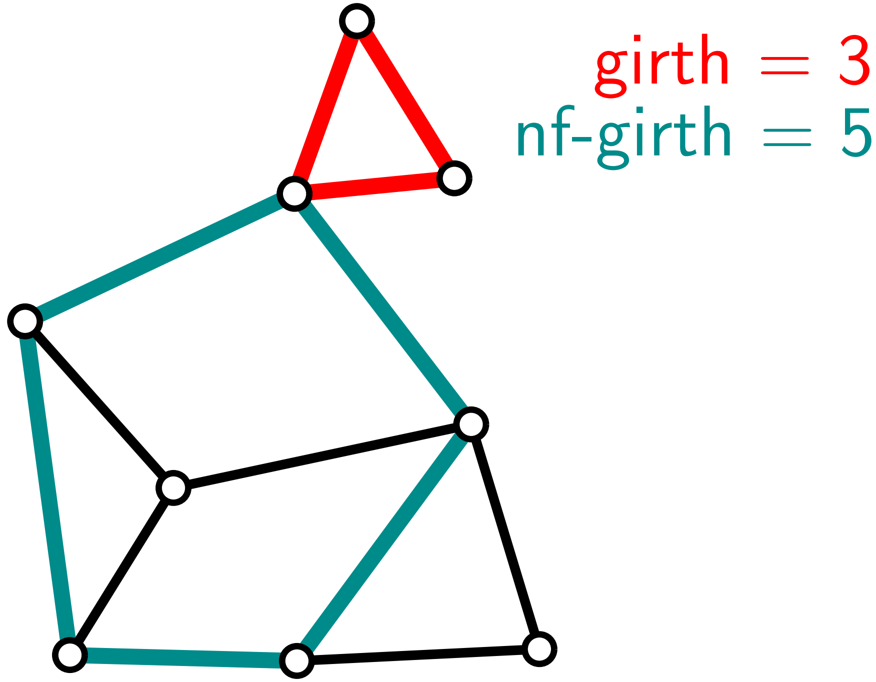

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Def: d-**irreducible map** = map such that **girth** $\geq d$ and **nf-girth** $> d$

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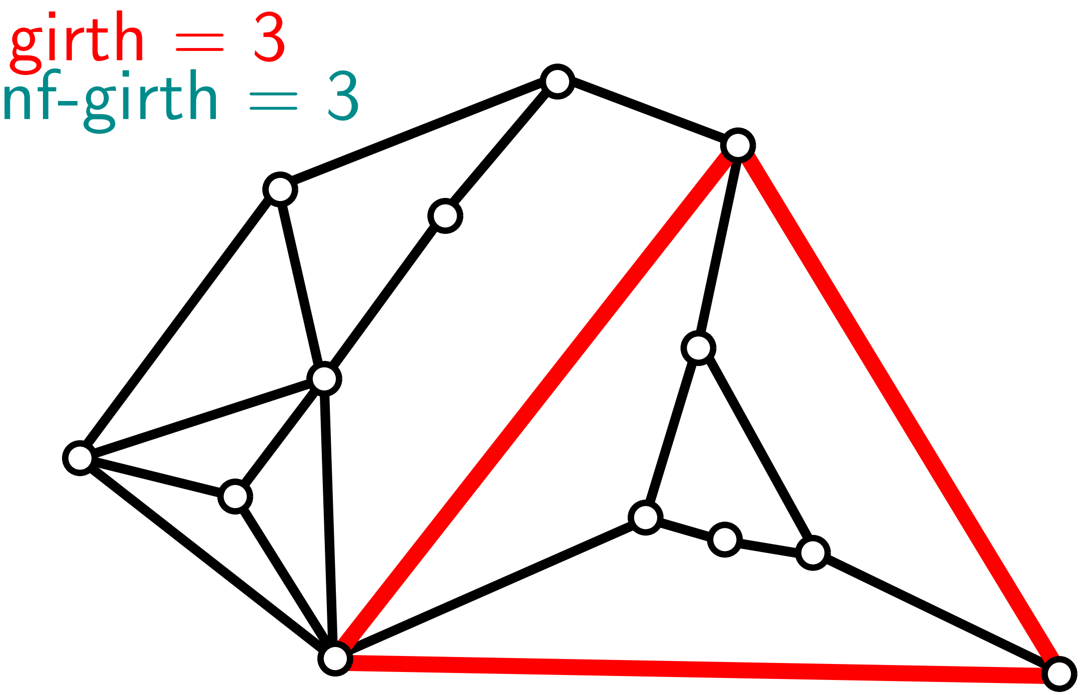

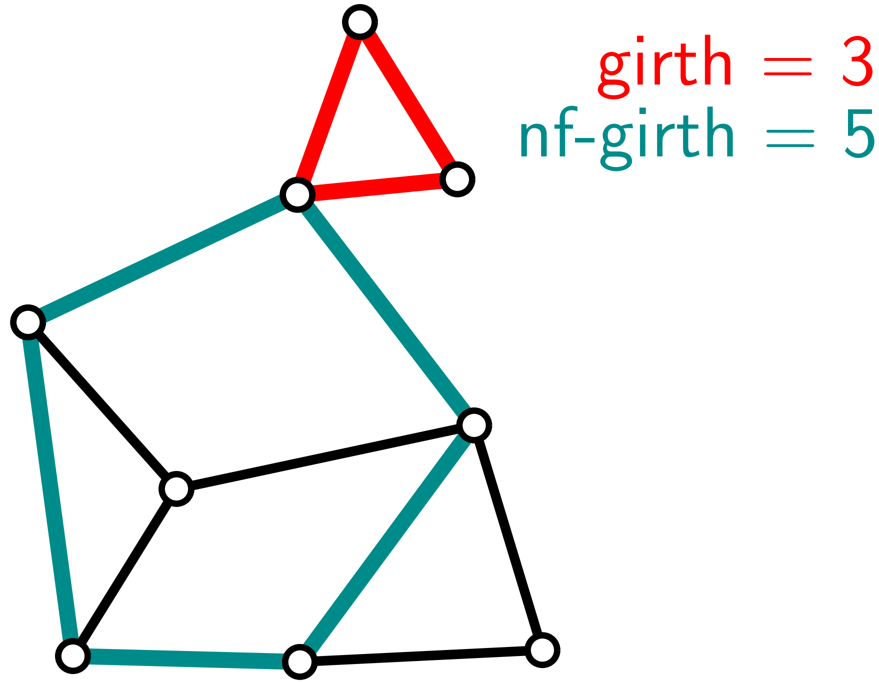

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Def: d-**irreducible map** = map such that **girth** $\geq d$ and **nf-girth** $> d$

Rk: letting $\mathcal{F}^{(d)}$:= maps of girth $\geq d$ $\qquad \mathcal{G}^{(d)}$:= d-irreducible maps

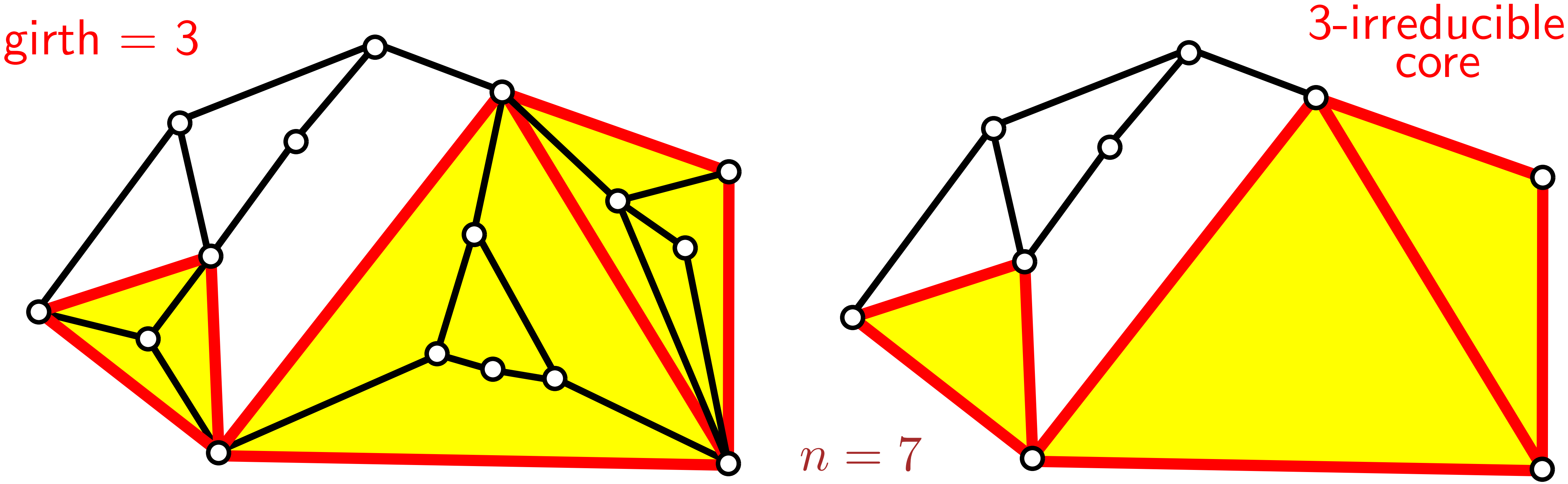
then $\quad \mathcal{F}^{(d)} \supset \mathcal{G}^{(d)} \supset \mathcal{F}^{(d+1)}$

Substitution approach

[Bouttier-Guitter'13]

$\mathcal{F}_n^{(d)}$:= maps of girth $\geq d$ outer degree n

$\mathcal{G}_n^{(d)}$:= d-irreducible maps outer degree n



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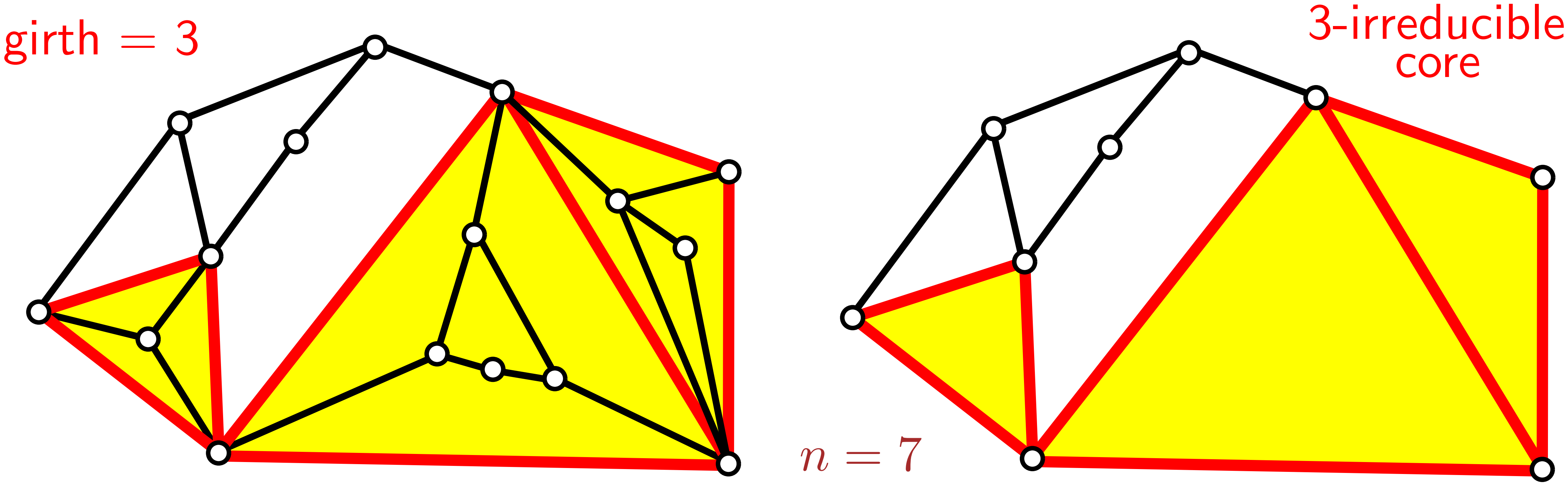

$$\begin{cases} F_n^{(d)}(x_d, x_{d+1}, \ldots) = G_n^{(d)}(X_d, x_{d+1}, \ldots) \\ F_n^{(d+1)}(x_{d+1}, \ldots) = G_n^{(d)}(0, x_{d+1}, \ldots) \end{cases}$$

with $X_d = F_d^{(d)}(x_d, \ldots)$

$\Rightarrow$ can carry algebraic expressions by induction on d, starting from $F_n^{(1)}$

Counting maps by girth and face-degrees

- Substitution method

  [Bouttier-Guitter'13]

  metric irreducible maps: [Budd'22] [Budd-Castro'25]

- Slice decomposition

  [Bouttier-Guitter'13] [Bouttier-Guitter-Manet'24]

  related problem: maps with tight boundaries

  [Bouttier-Guitter-Miermont'22]

- Bijections with families of trees via orientations

  [Bernardi-F'12] [Albenque-Poulalhon'15] [Bernardi-F-Liang'23]

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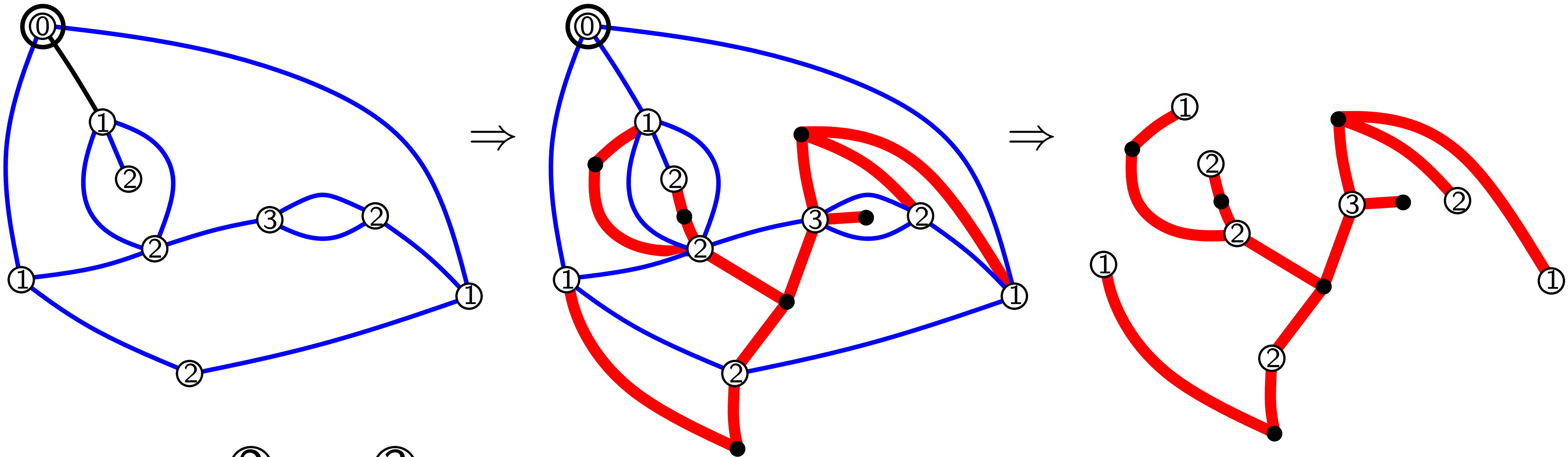
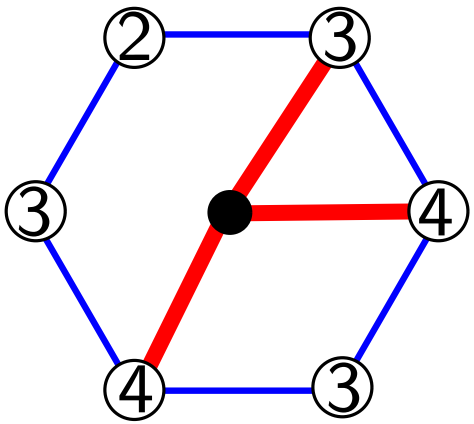
Bijections for d-angulations of girth d via orientations

BDG bijection

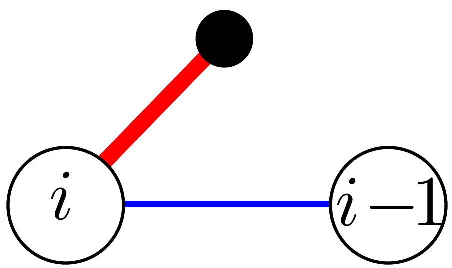
[Bouttier-Di Francesco-Guitter'04]

extends Cori-Vauquelin-Schaeffer bijection





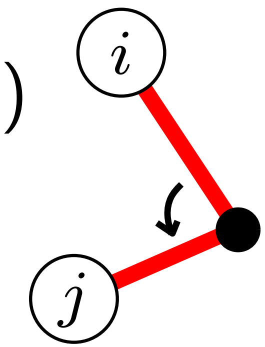
Local rule



Conditions:

(i) $\exists$ vertex label 1

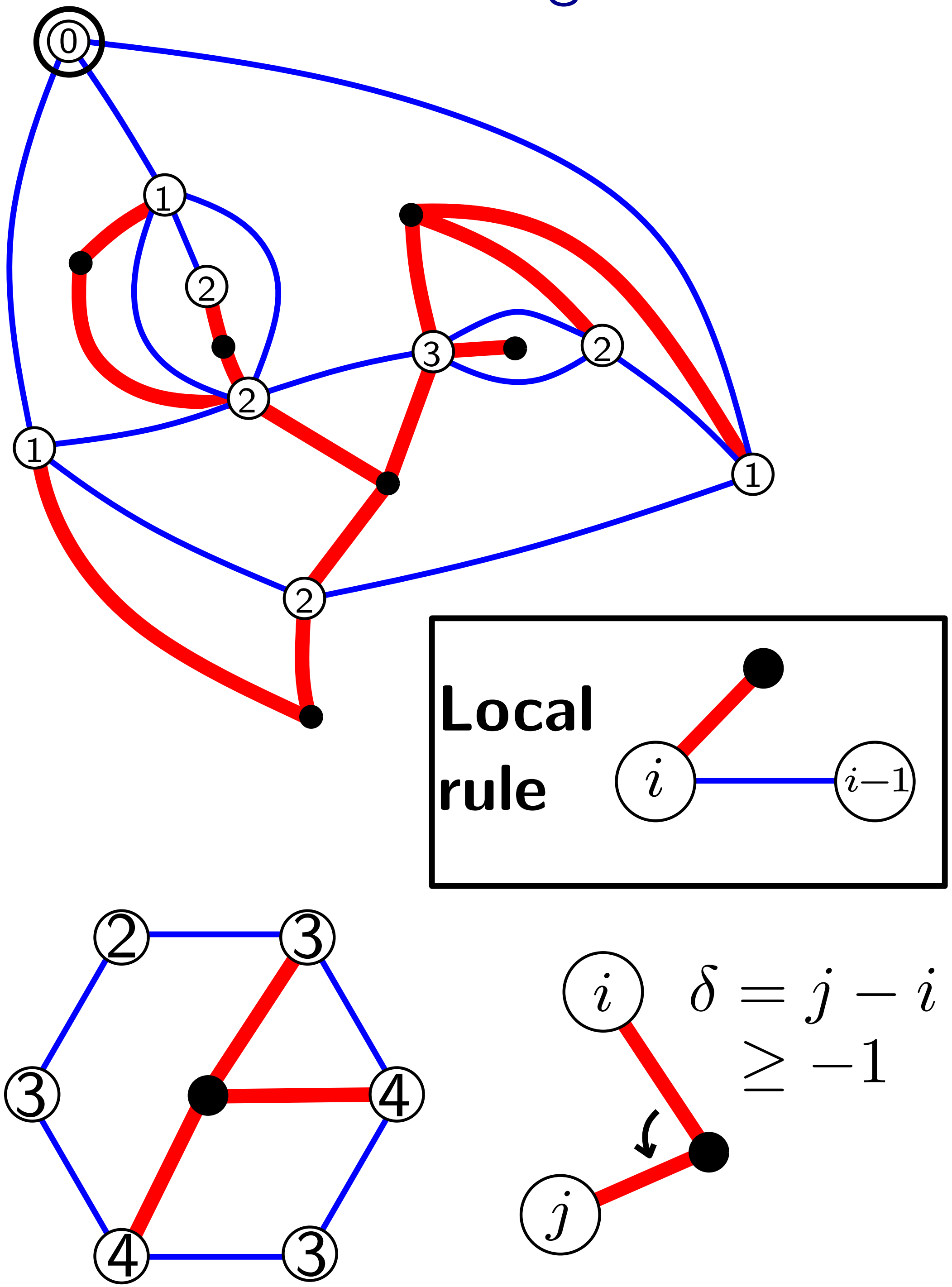
(ii)



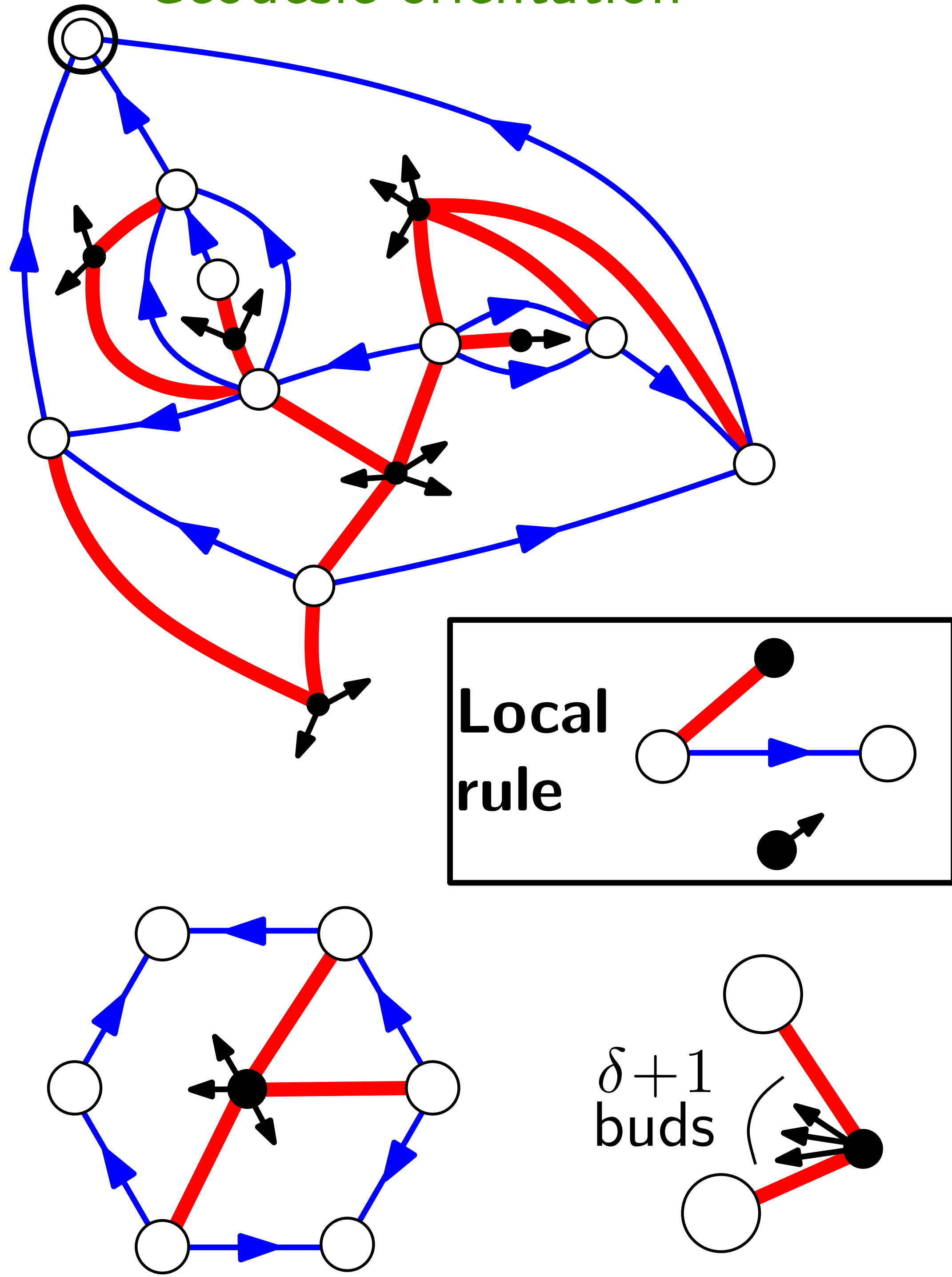
$j \geq i-1$

Reformulation on geodesic orientation

Distance labelling



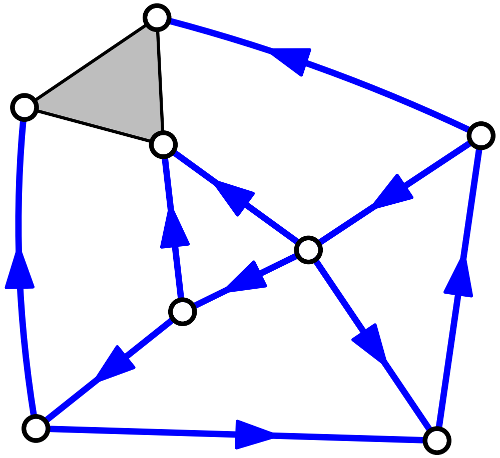
Geodesic orientation



Extension to minimal sink-orientations

sink-orientation : d-gonal sink $(d \geq 0)$, **accessible** (all vertices can reach sink)

minimal : **no clockwise cycle**



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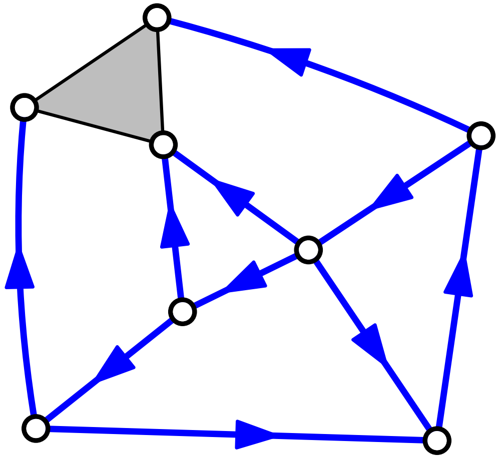

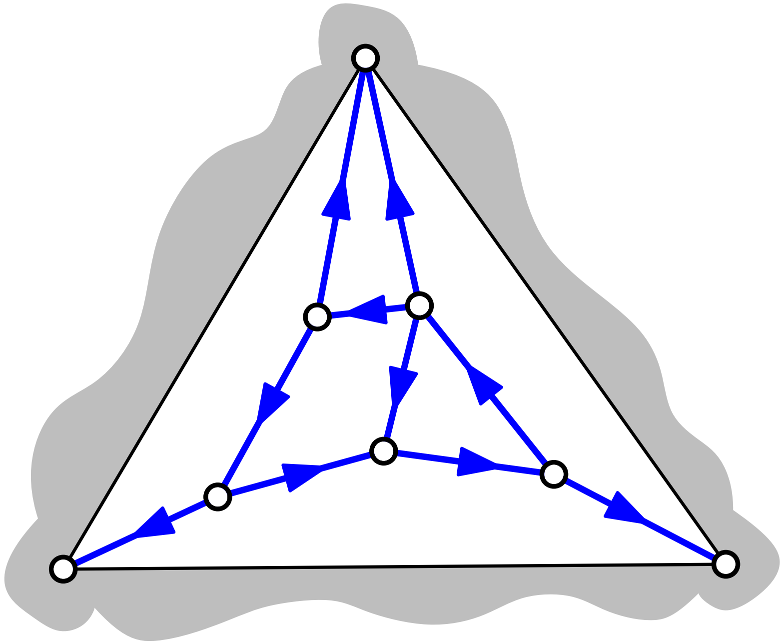

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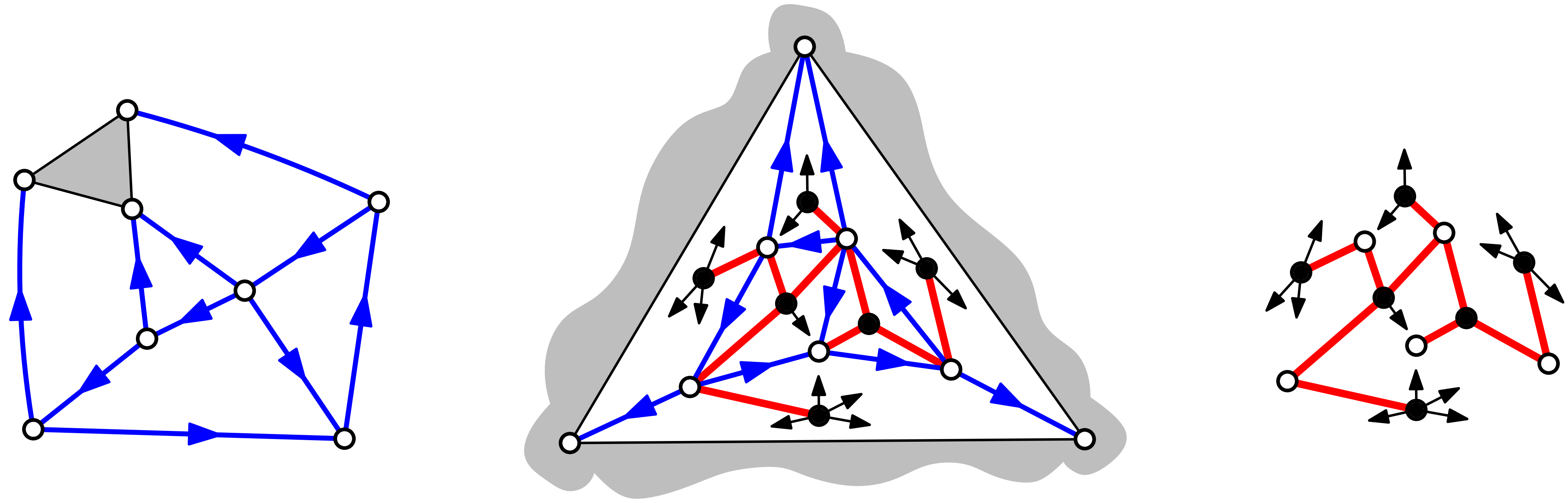

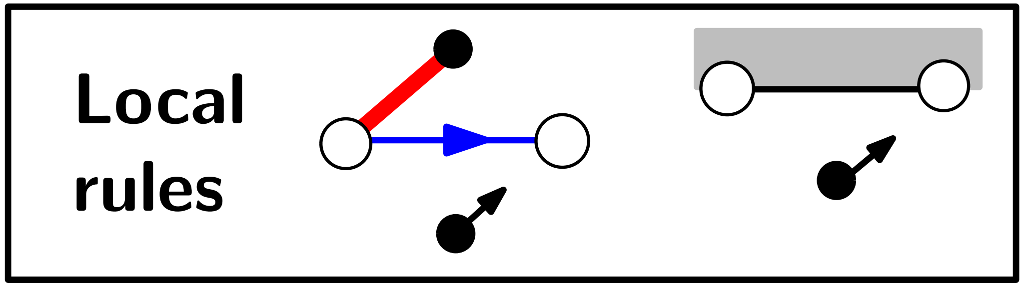

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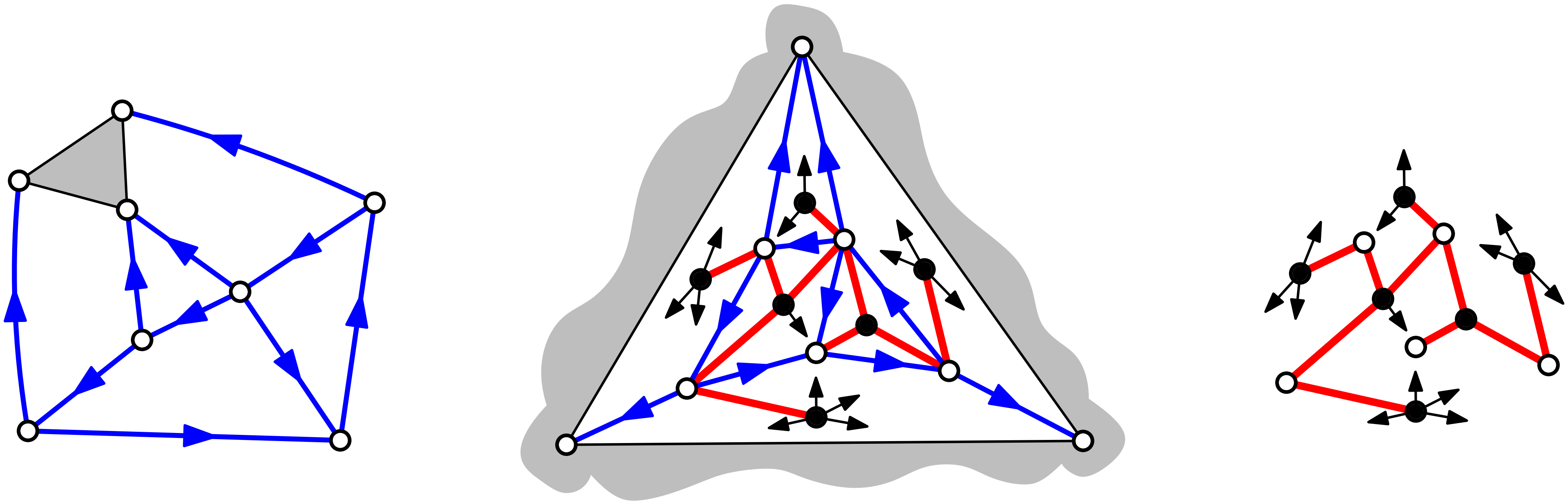
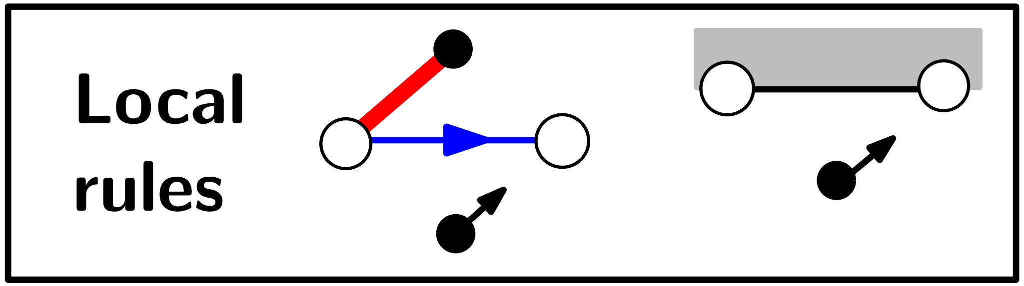


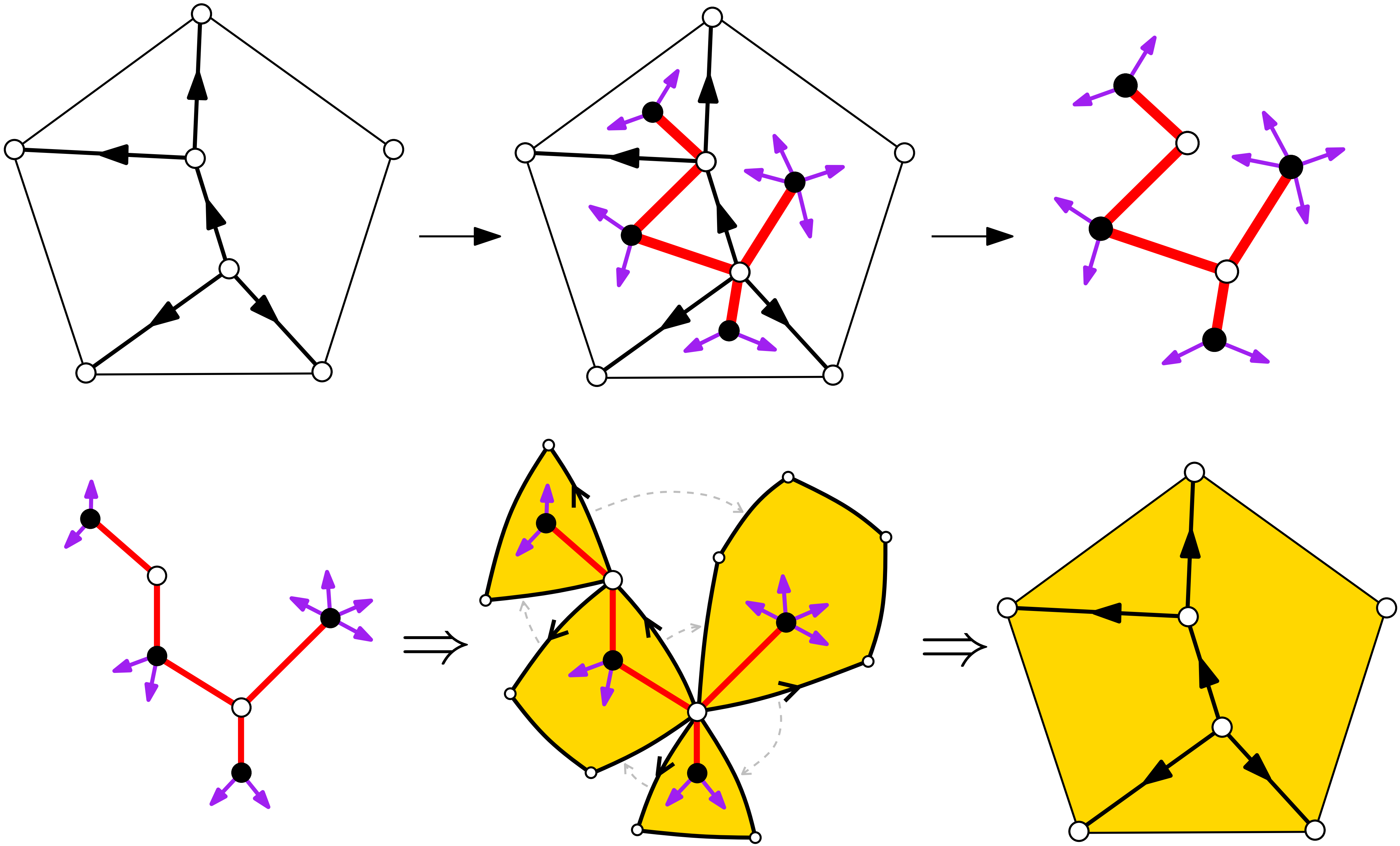


d ⟷ excess = # buds − # edges

degrees of inner faces ⟷ degrees of black nodes

degrees of inner vertices ⟷ degrees of white nodes

Inverse bijection

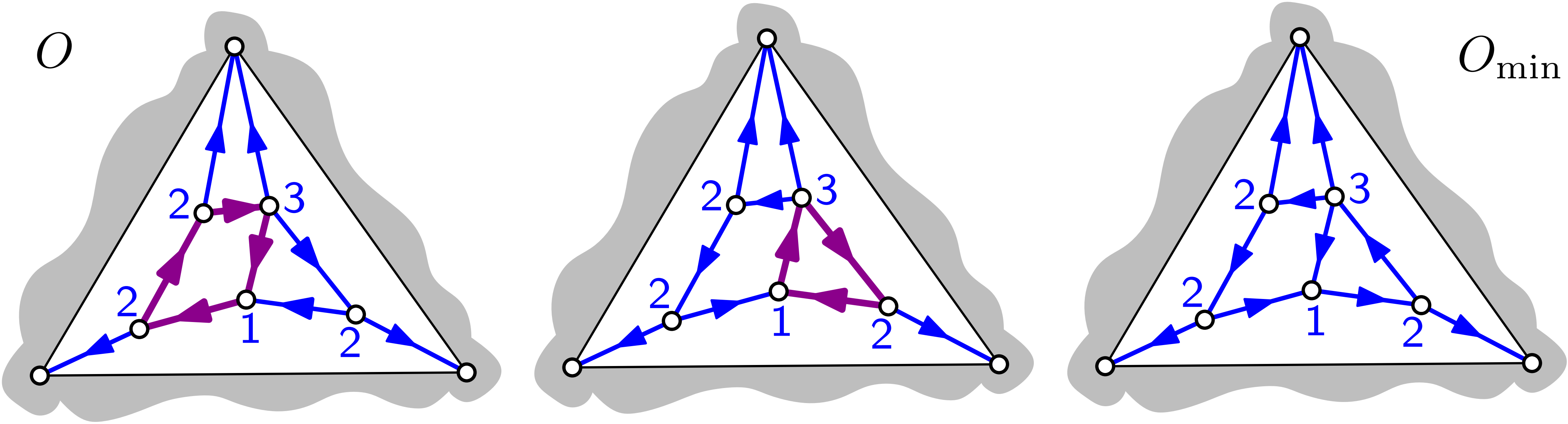


closure operations cf [Schaeffer'99]

Minimizing a sink-orientation

[Propp'93], [Felsner'04]

For O a sink-orientation, there is a unique minimal sink-orientation $O_{\min}$ with same vertex outdegrees



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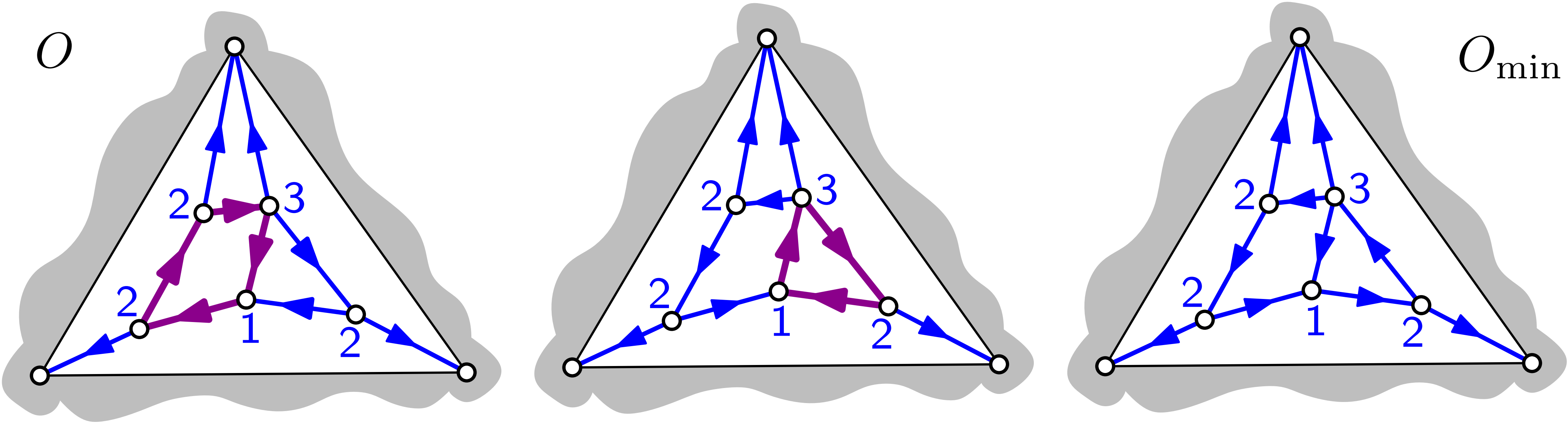
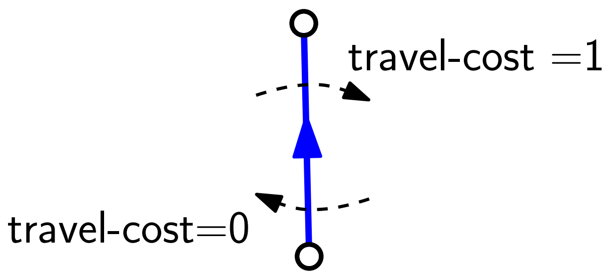
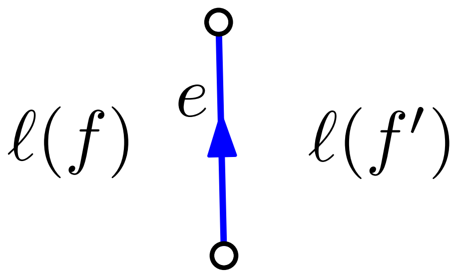


Edges to be returned on O can be obtained from a dual distance-labeling

[Khuller, Naor, Klein'93]





$$\ell(f') - \ell(f) \in \{0, 1\}$$

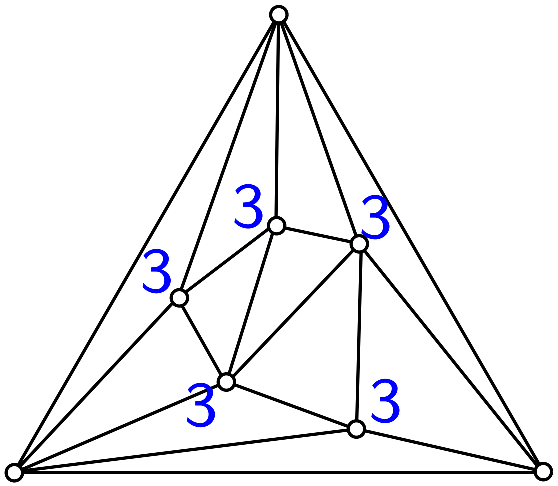
return e

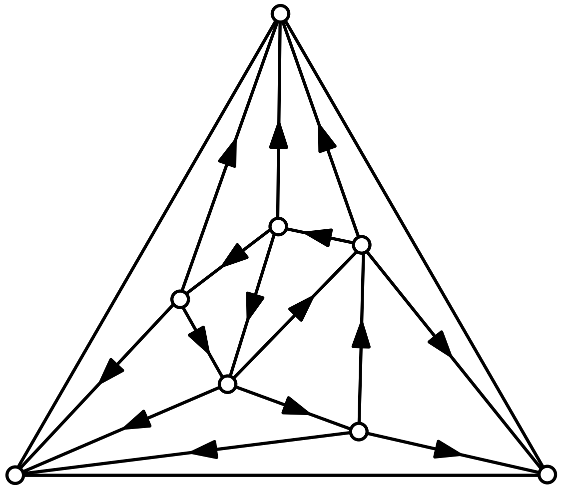
Sink-orientations for simple triangulations

3-orientation = sink-orientation where inner vertices have outdegree 3

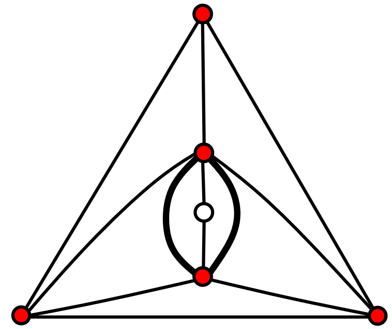
[Schnyder'89] every simple triangulation admits a 3-orientation

(can be computed in linear time from vertex-shelling [Brehm'00])





Rk: triangulation needs to be simple



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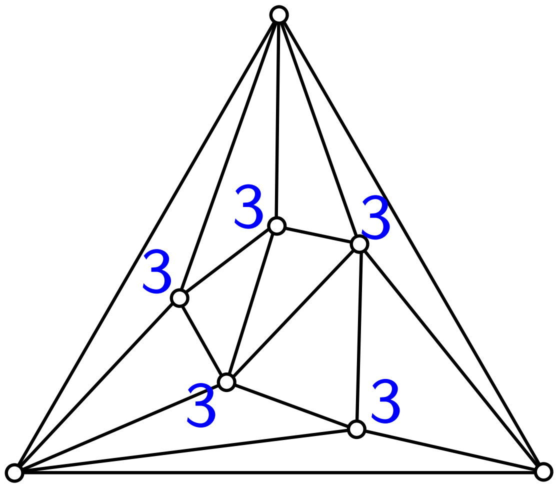

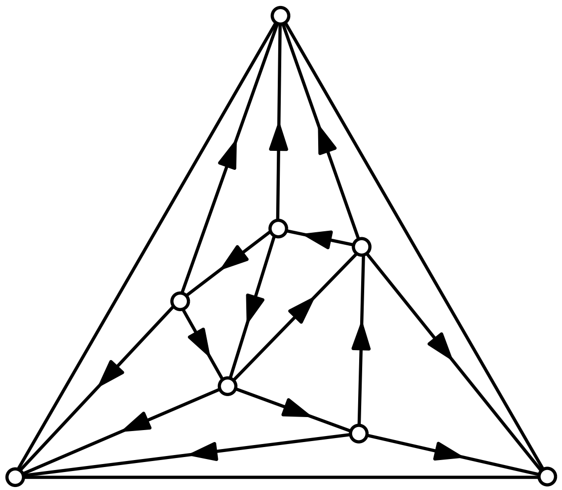

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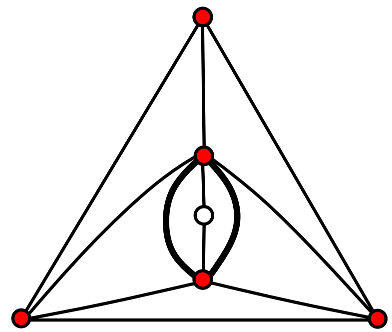
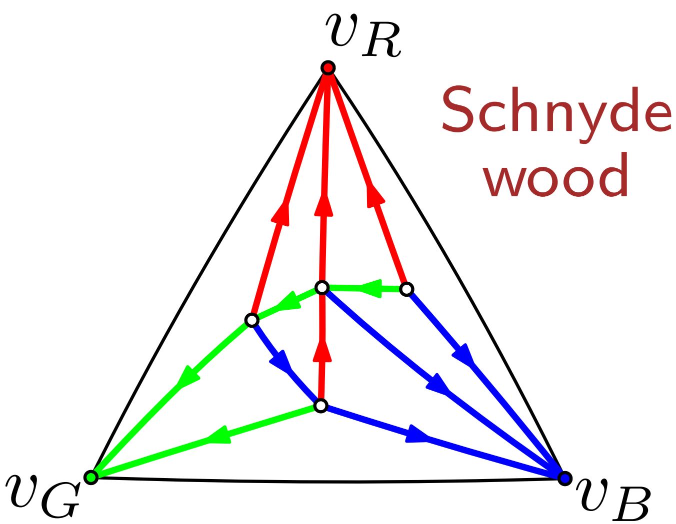
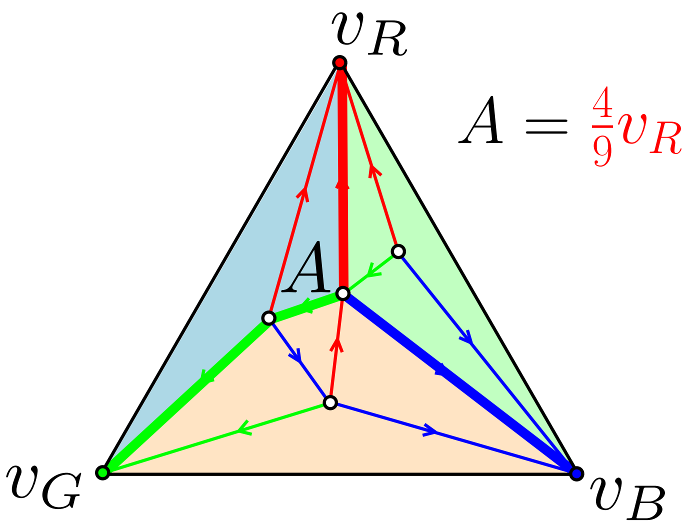


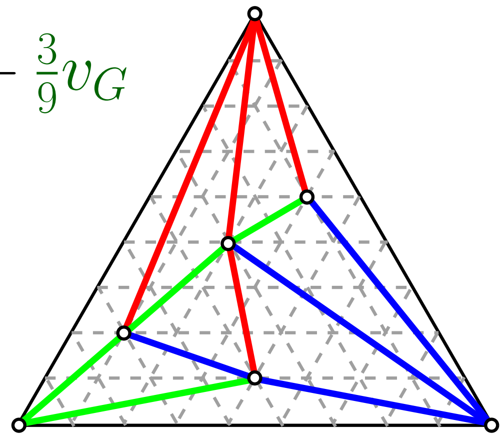
originally applied to straight-line drawing [Schnyder'90]



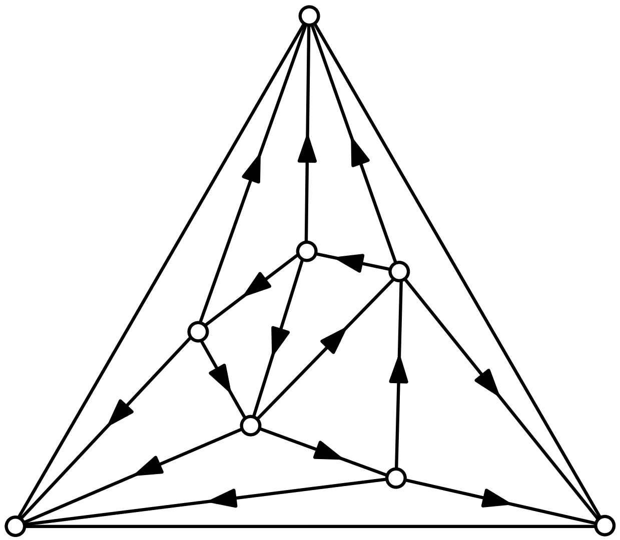
Schnyder wood



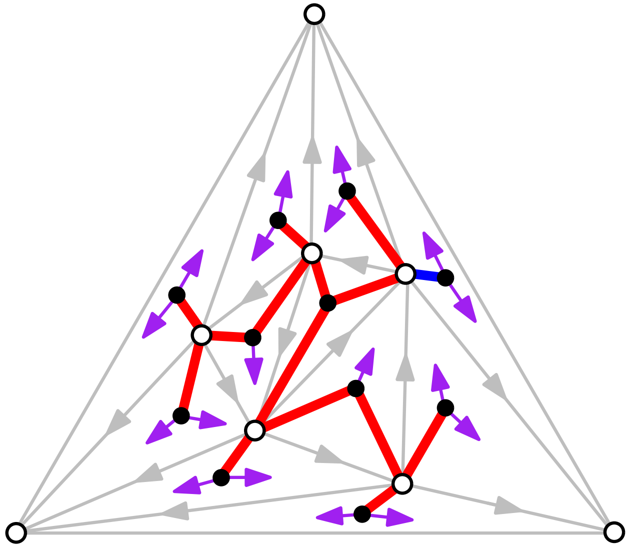
$$A = \frac{4}{9}v_R + \frac{2}{9}v_B + \frac{3}{9}v_G$$

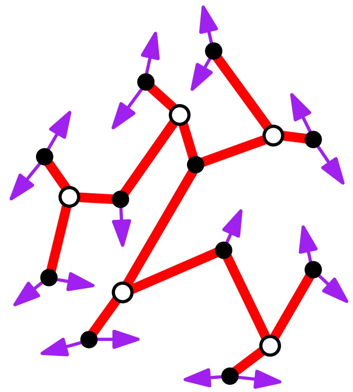


Bijection for simple triangulations



face-degrees 3
vertex-outdegrees 3





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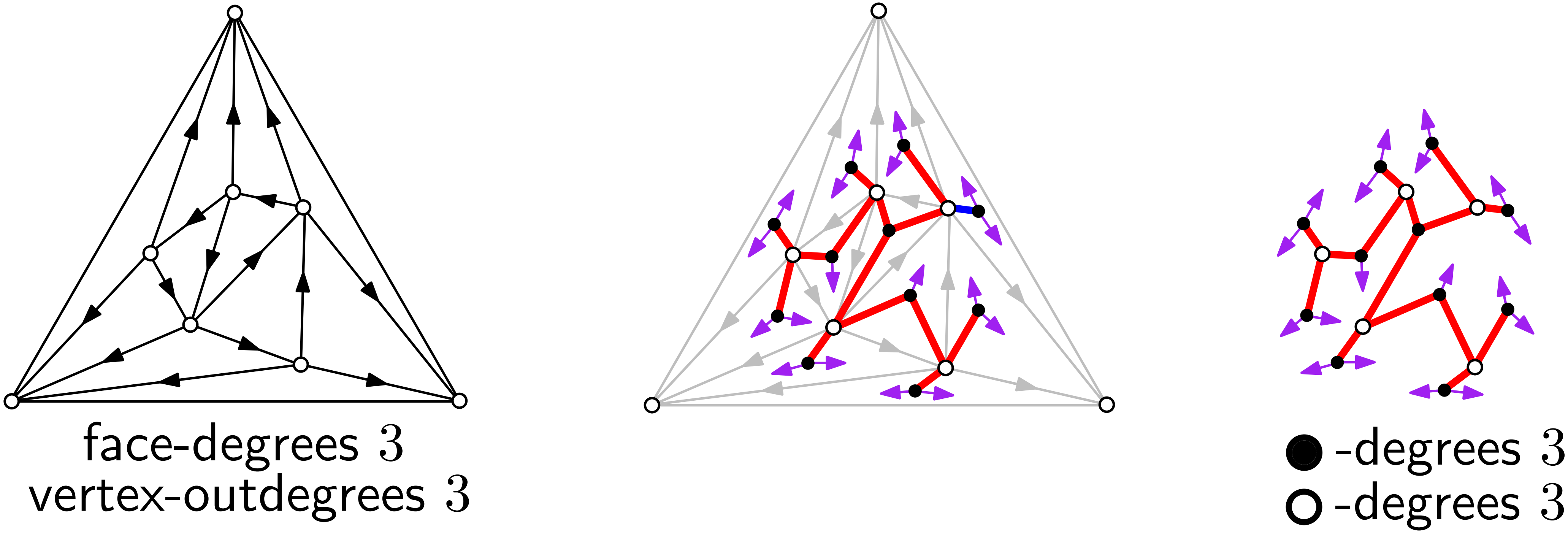


recover [F, Poulalhon, Schaeffer'08]

different bijection in [Poulalhon, Schaeffer'03]

give bijective proofs for

$$\#\text{ rooted simple triangulations with } n+3 \text{ vertices} = \frac{2}{n(n+1)}\binom{4n+1}{n-1}$$

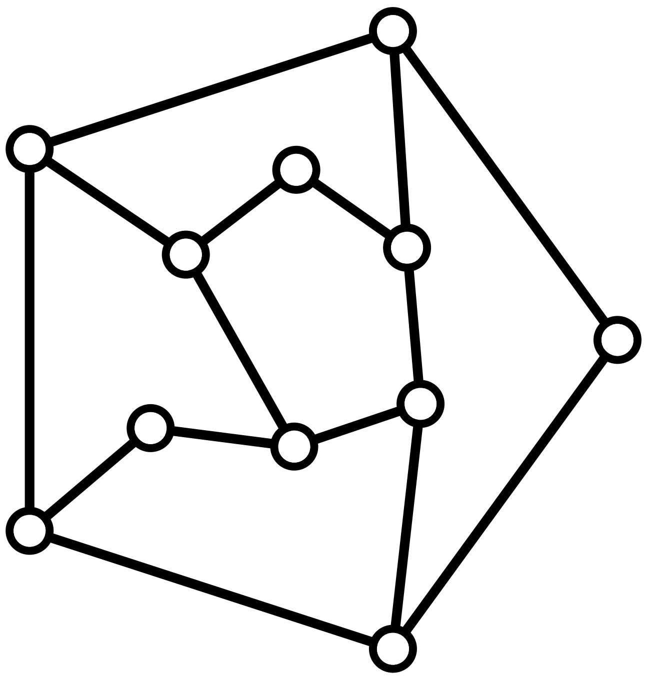
[Tutte'62]

Sink-orientations for d-angulations of girth d

Fact: A d-angulation satisfies $\frac{\#\text{inner edges}}{\#\text{inner vertices}} = \frac{d}{d-2}$

Natural candidate for outdegree function:

$$\alpha : \; v \mapsto \frac{d}{d-2} \text{ for each internal vertex } v\ldots$$



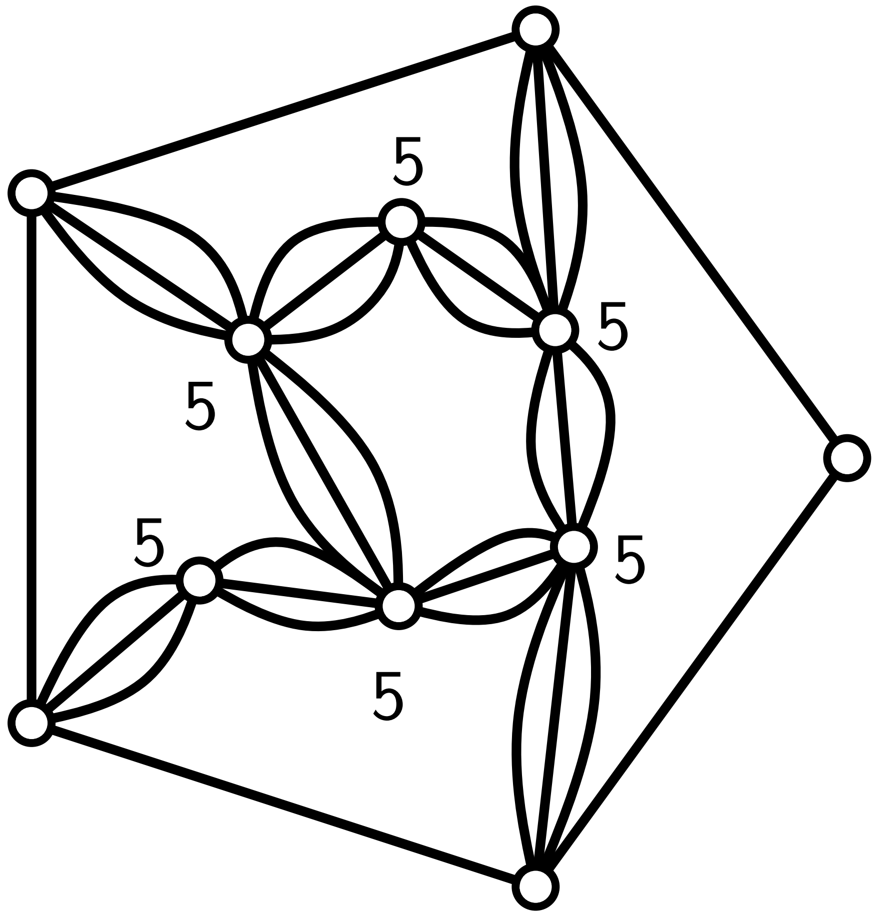
Sink-orientations for d-angulations of girth d

Fact: A d-angulation satisfies $\frac{\#\text{inner edges}}{\#\text{inner vertices}} = \frac{d}{d-2}$

Idea: look for an orientation of $(d-2)G$ with outdegree function

$$\alpha : \ v \mapsto d \text{ for each internal vertex } v.$$

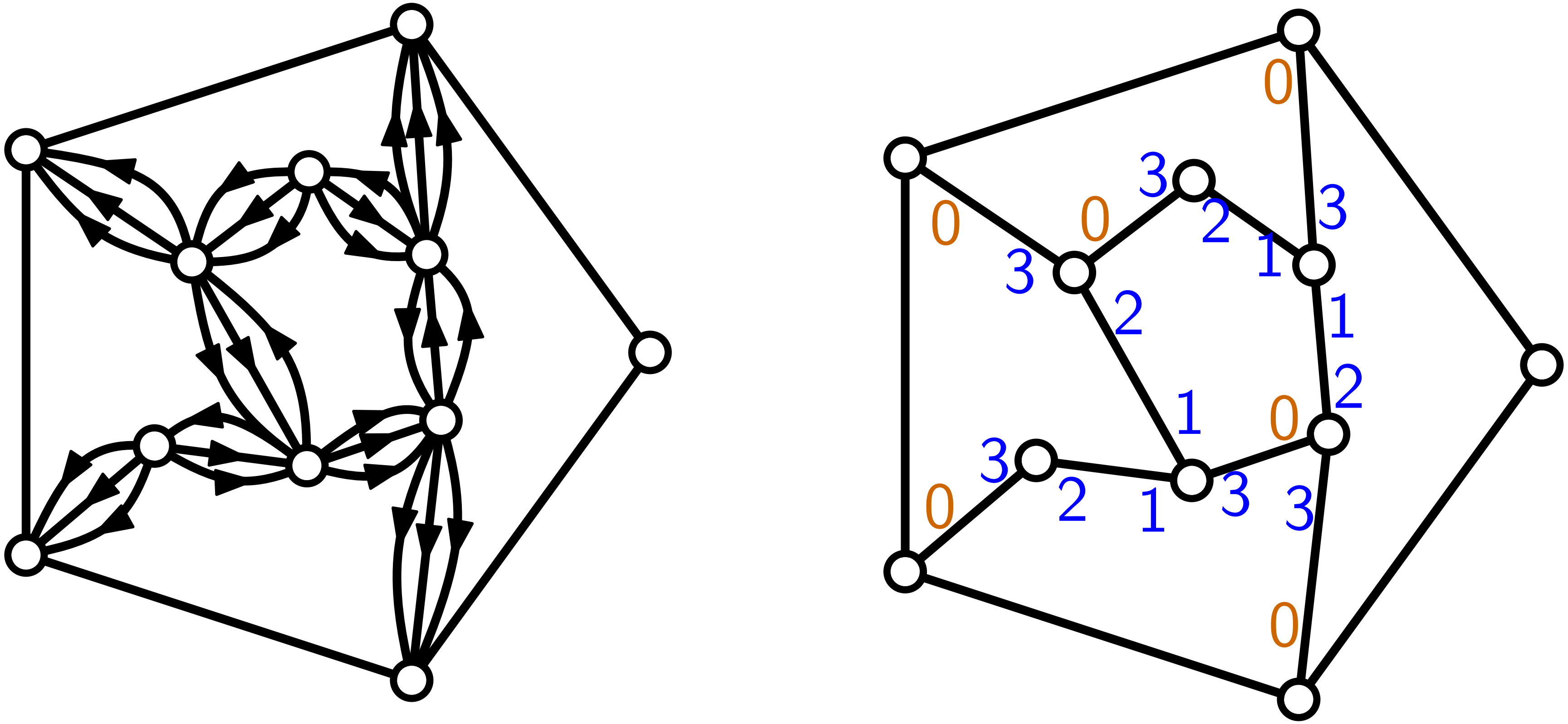
call $d/(d-2)$-orientation such an orientation



Sink-orientations for d-angulations of girth d

[Bernardi-F'10]: Let G be a d-angulation. Then G admits a $d/(d-2)$- orientation if and only if G has girth d.

$d = 5$



flow-formulation
inner edges: total flow $= d - 2$
inner vertices: total outflow $= d$

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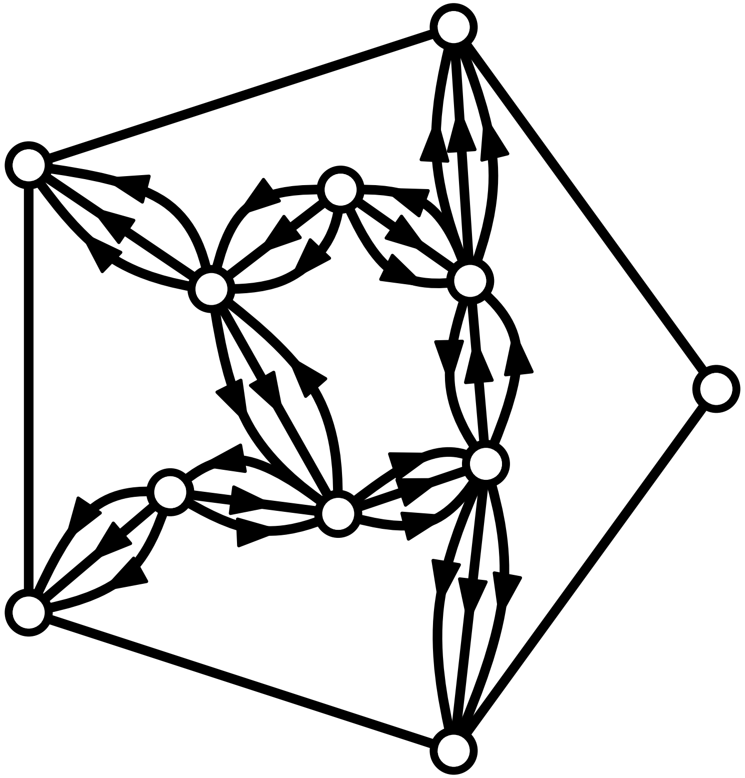
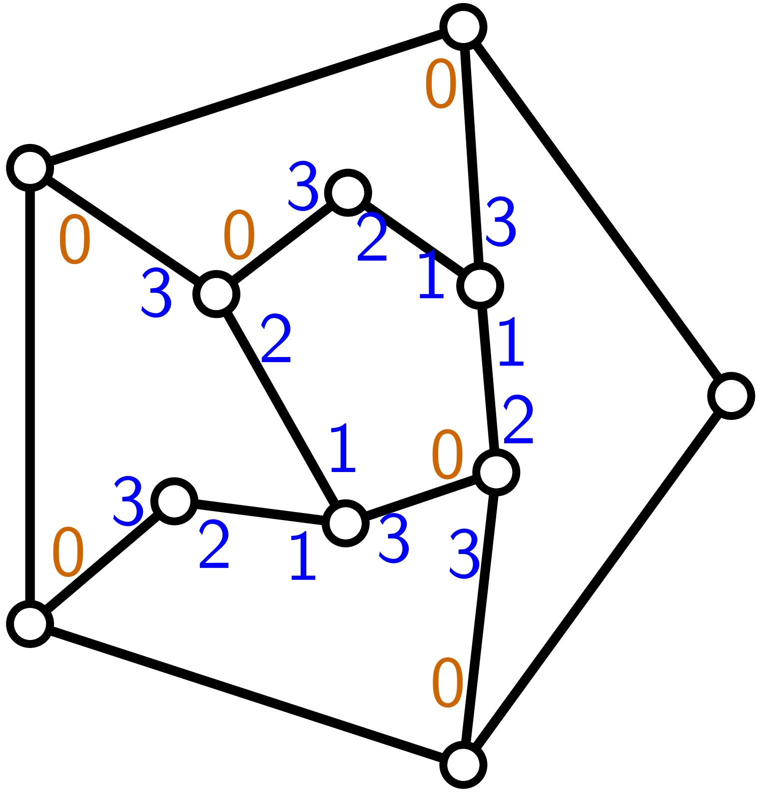
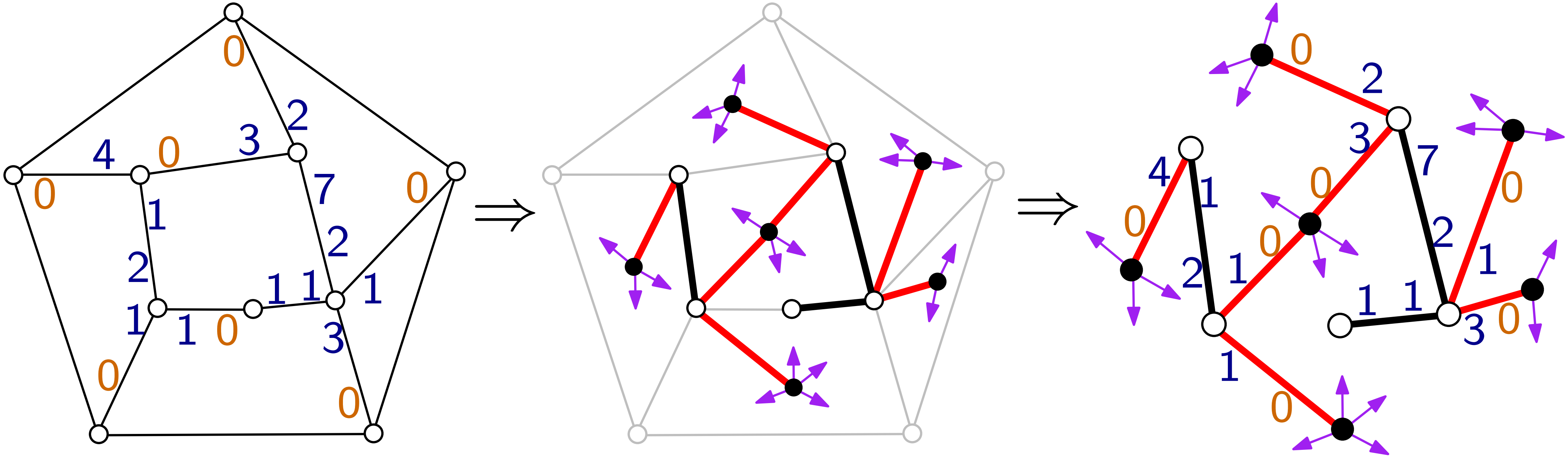


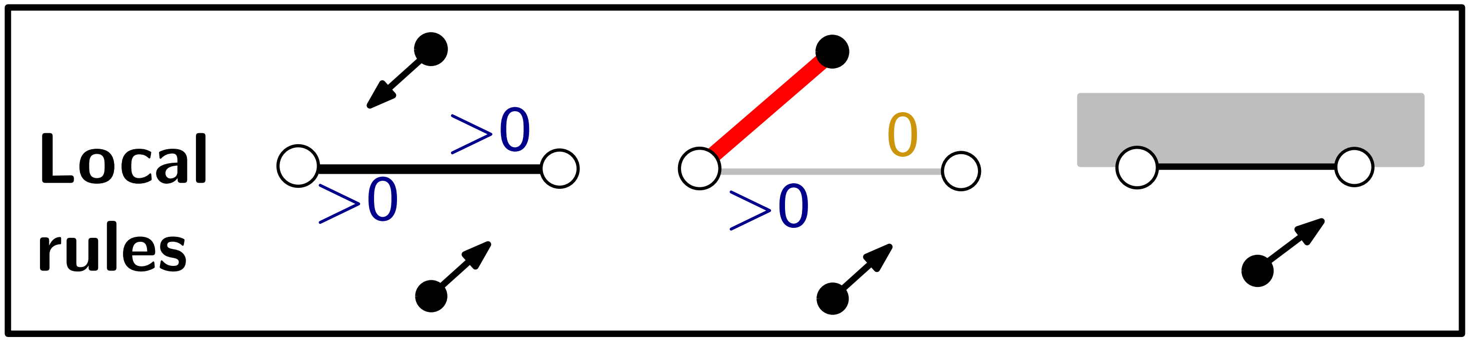


flow-formulation
inner edges: total flow $= d - 2$
inner vertices: total outflow $= d$

Proof: General existence criterion (cf Hall's marriage theorem)
& every map $G = (V, E)$ of girth $\geq d$ satisfies $(d-2)|E| \leq d|V| - 2d$

General bijection in the flow-formulation



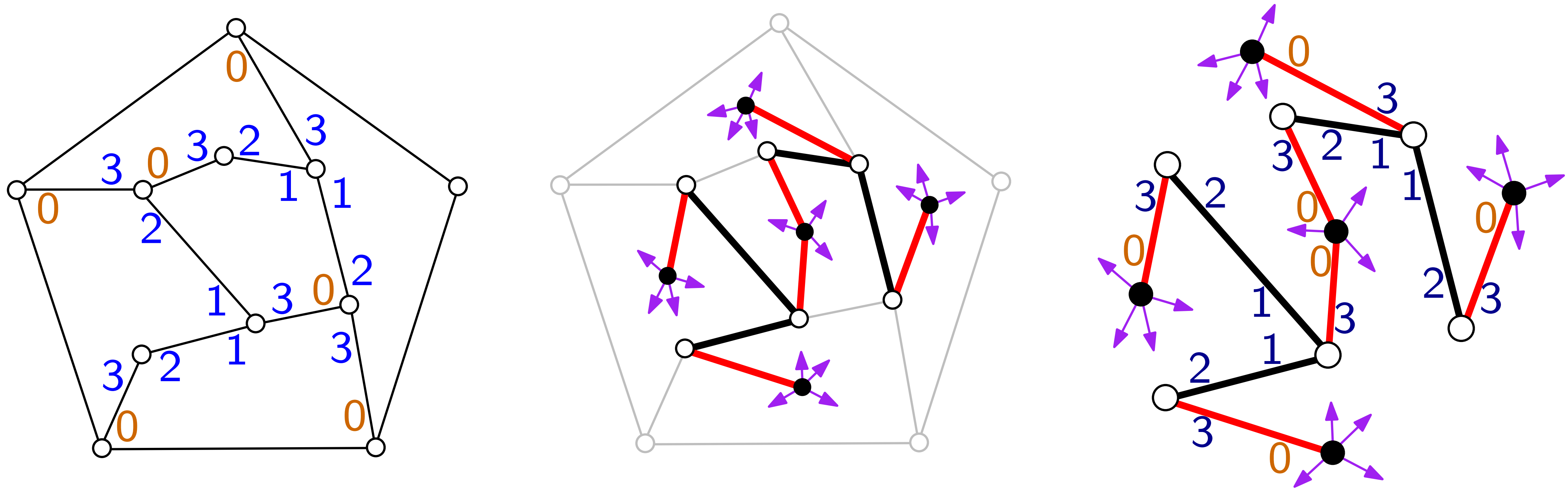


degrees of inner faces ⟷ degrees of black nodes

total flows at inner vertices ⟷ total weights at white nodes

total flows at inner edges ⟷ total weights at edges

Specialization to d-angulations of girth d



Bijection **d-angulations of girth d** $\leftrightarrow$ **weighted mobiles** such that

- each $\bullet$ has degree d
- each $\circ$ has total weight d
- each edge has total weight $d-2$ (weight>0 at $\circ$, weight$=0$ at $\bullet$)

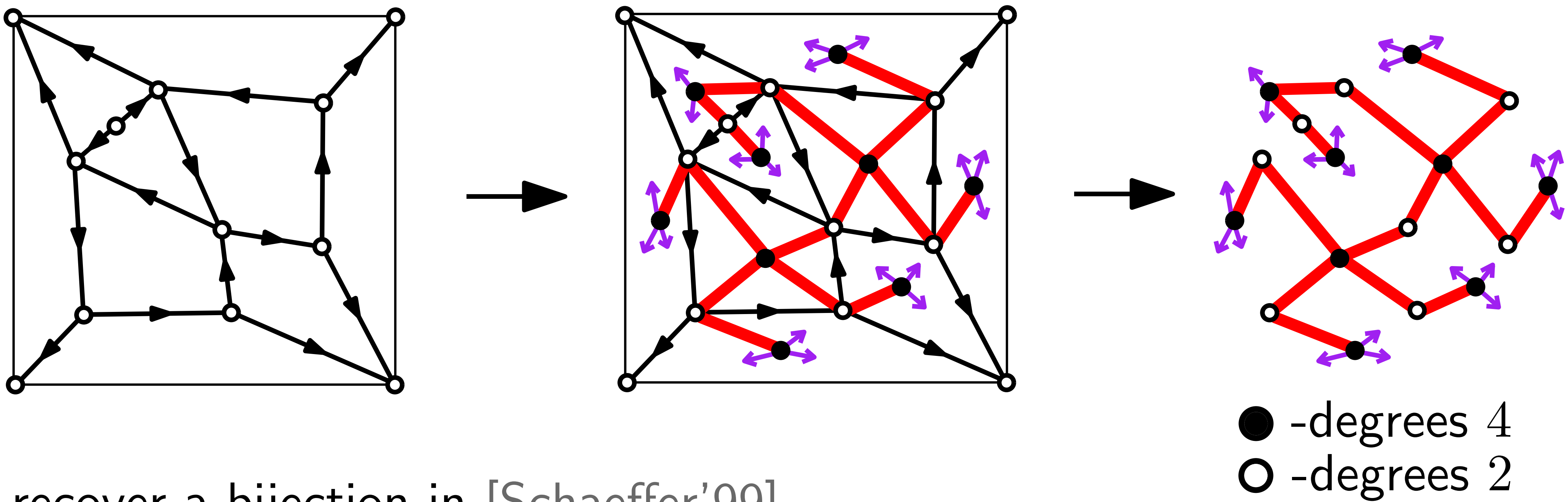
$\Rightarrow$ generating function expressed in terms of algebraic system

[Albenque, Poulalhon'13]: other bijection with weighted blossoming trees

Simplification in the bipartite case

- For d even, $d = 2b$, we have $\dfrac{d}{d-2} = \dfrac{b}{b-1}$
- Can work with $b/(b-1)$-orientations:
  - edges have weight $b-1$
  - vertices have indegree b

Example: $b = 2$, simple quadrangulations

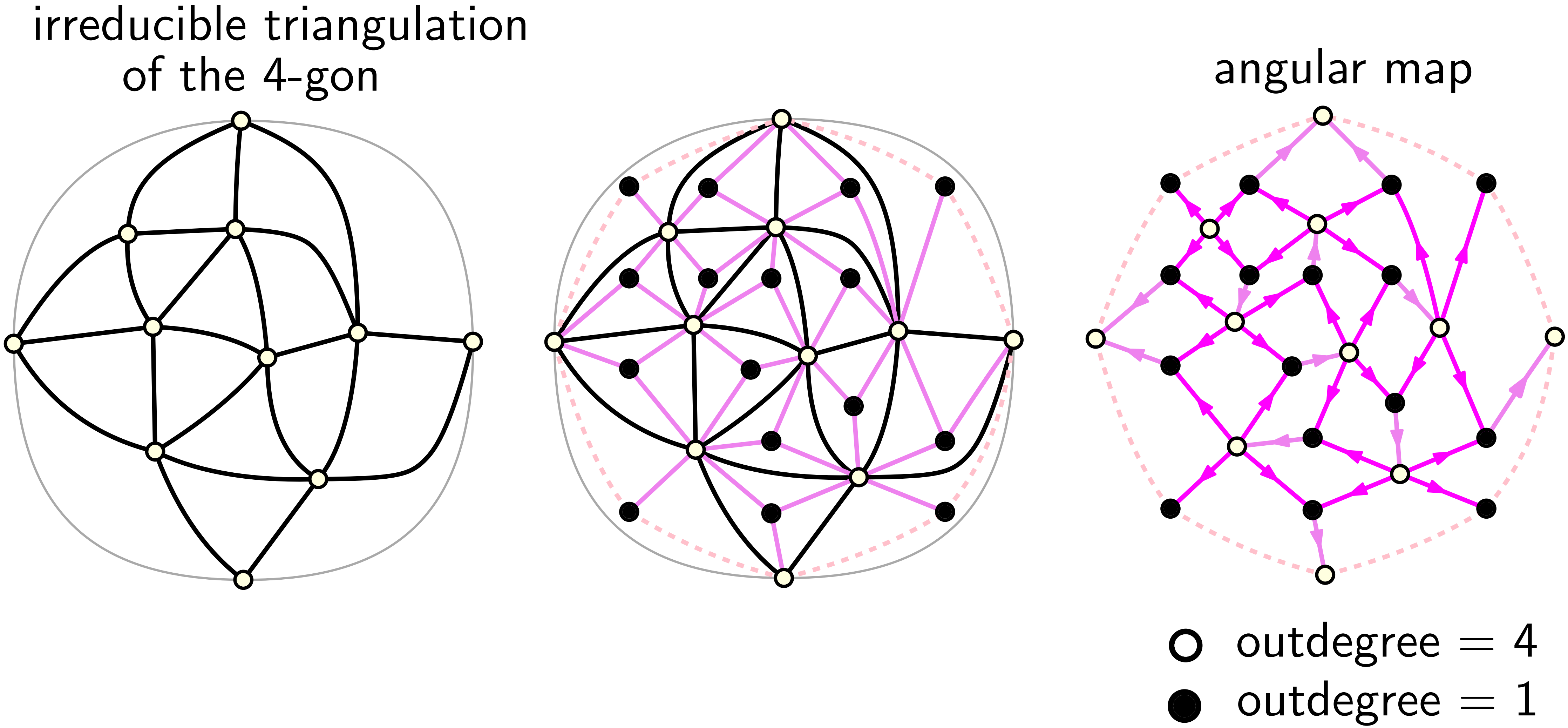


recover a bijection in [Schaeffer'99]

Orientations for irreducible d-angulations

Sink-orientations for irreducible triangulations

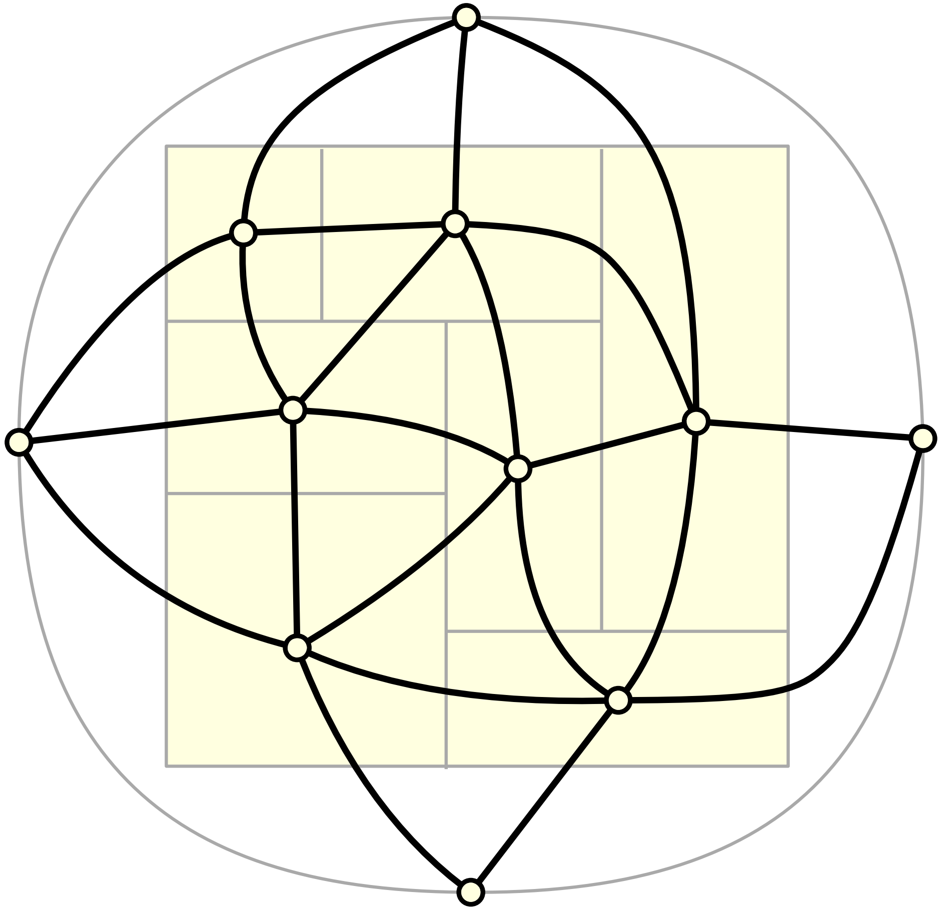
[F'07]

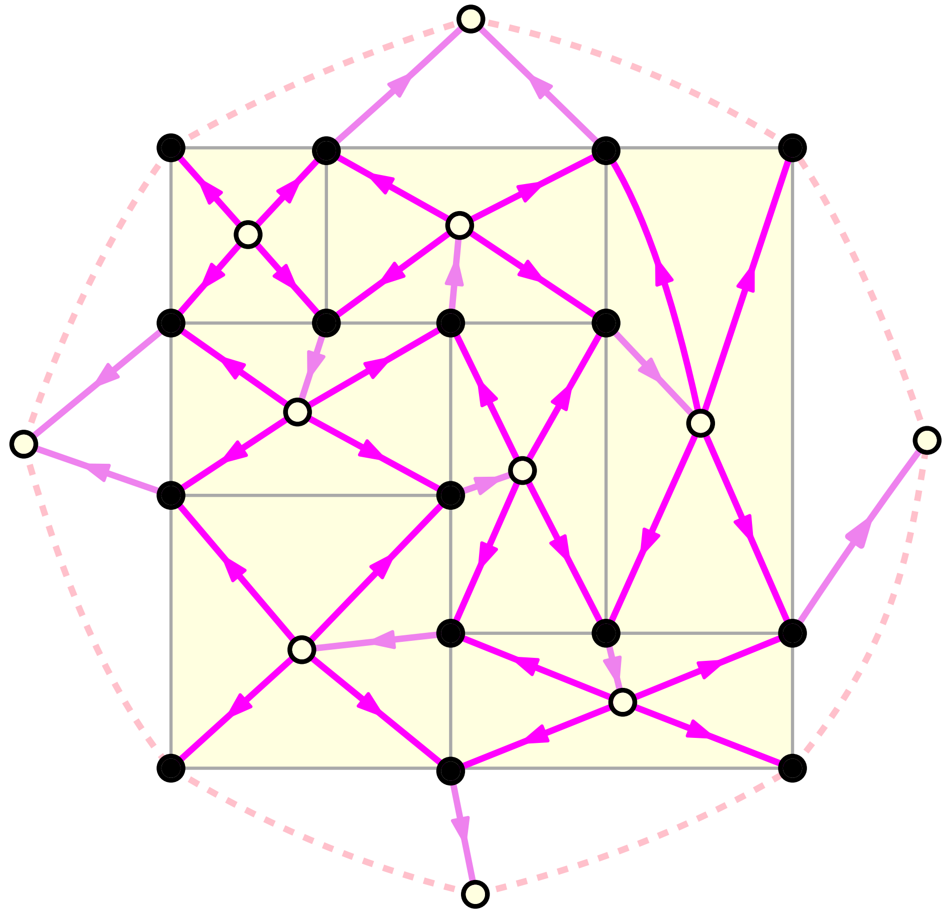


Sink-orientations for irreducible triangulations

[F'07]

cf duality with rectangulations [He'93]



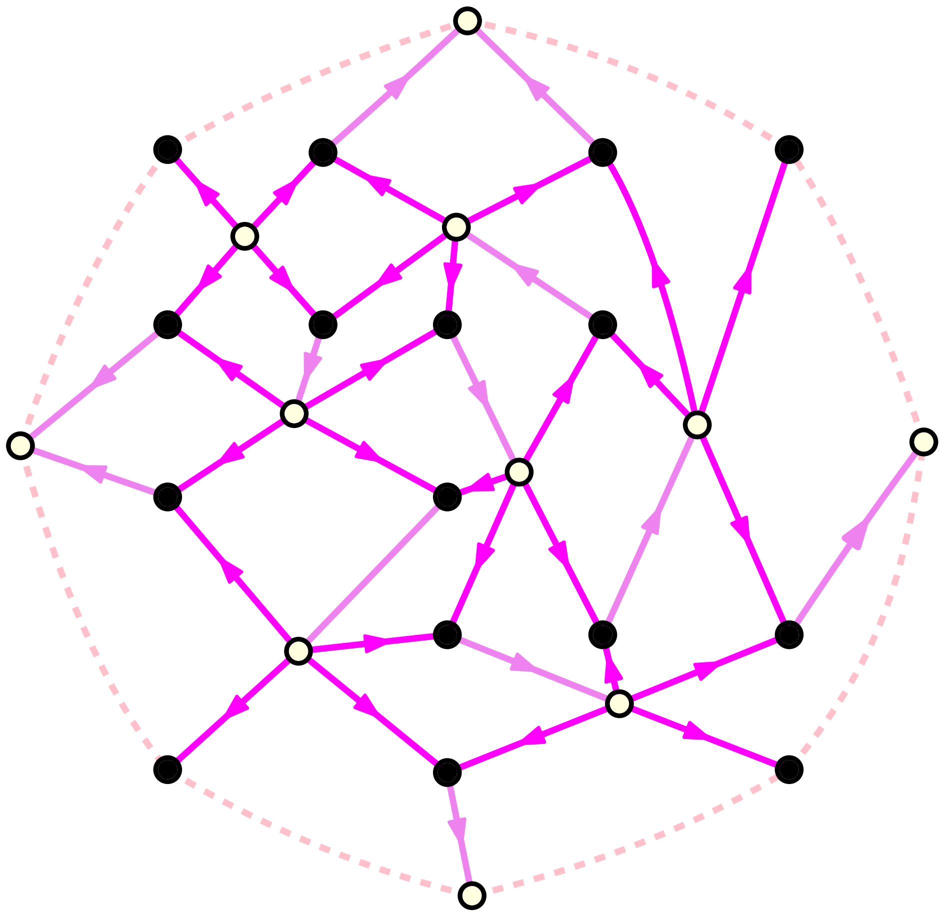


○ outdegree = 4

● outdegree = 1

Bijection with unrooted ternary trees [F'07]

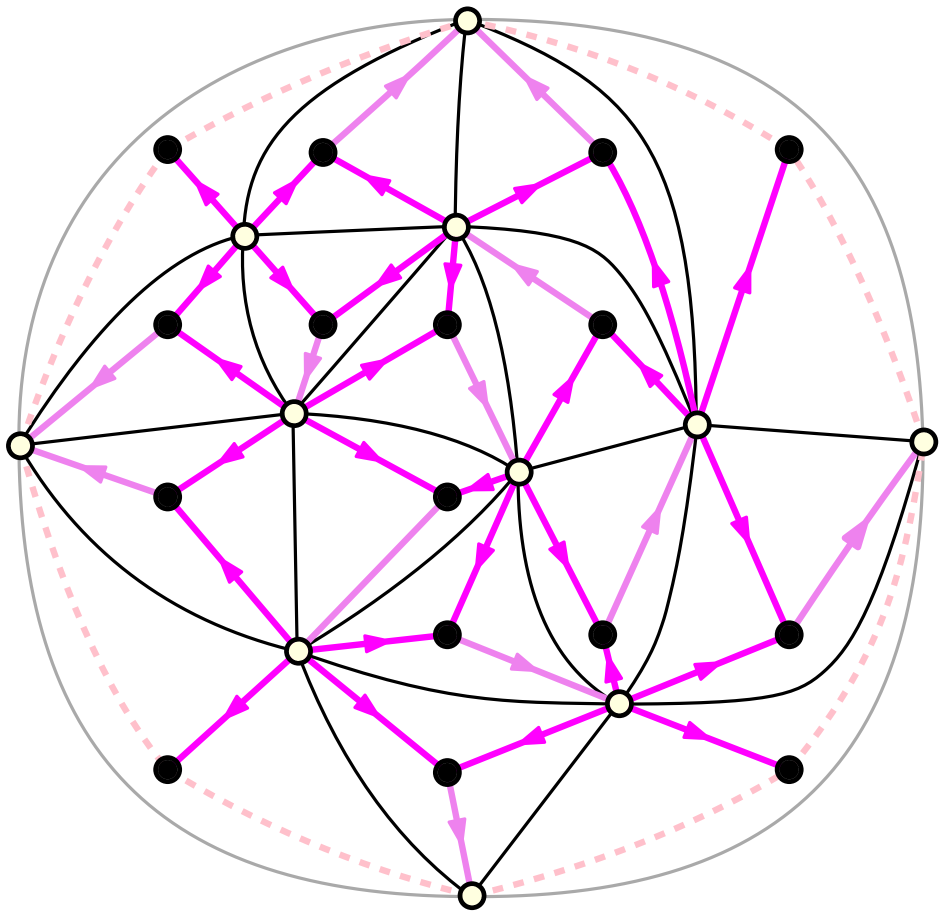
start from minimal angular orientation



Bijection with unrooted ternary trees

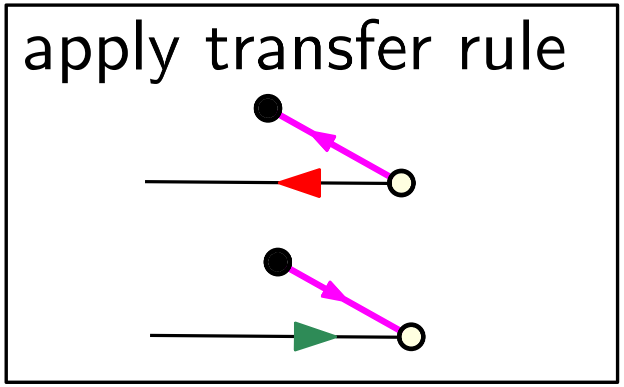
[F'07]

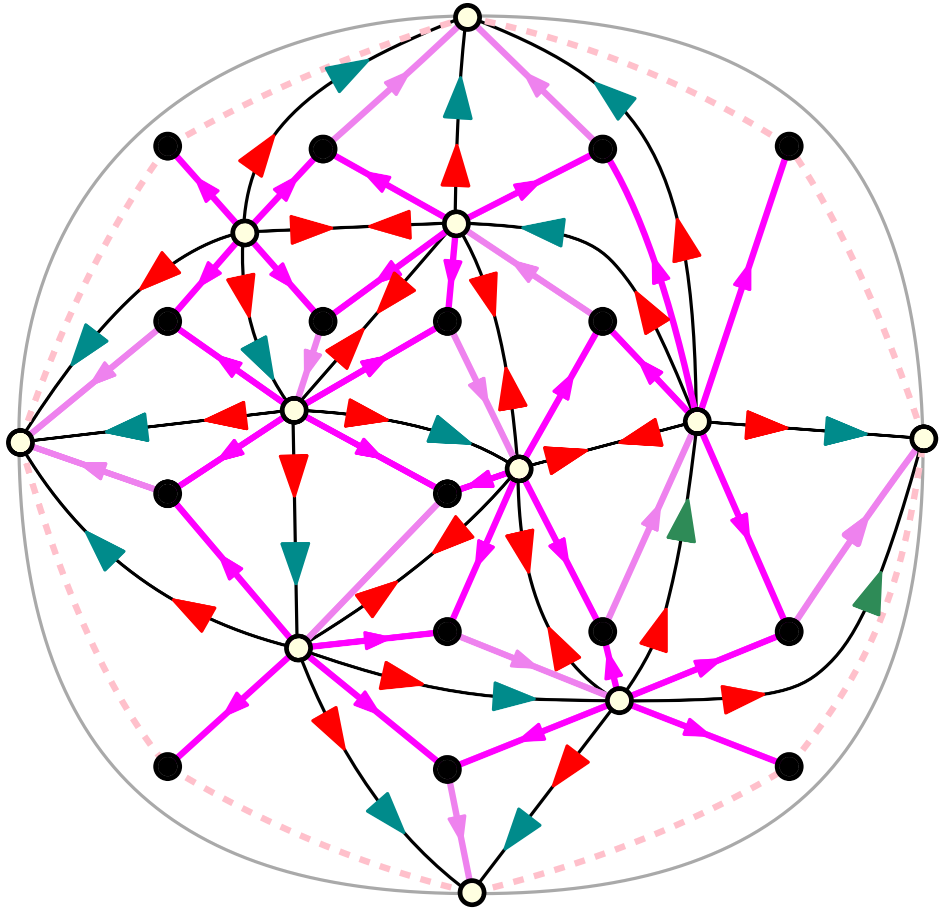
superimpose triangulation



Bijection with unrooted ternary trees

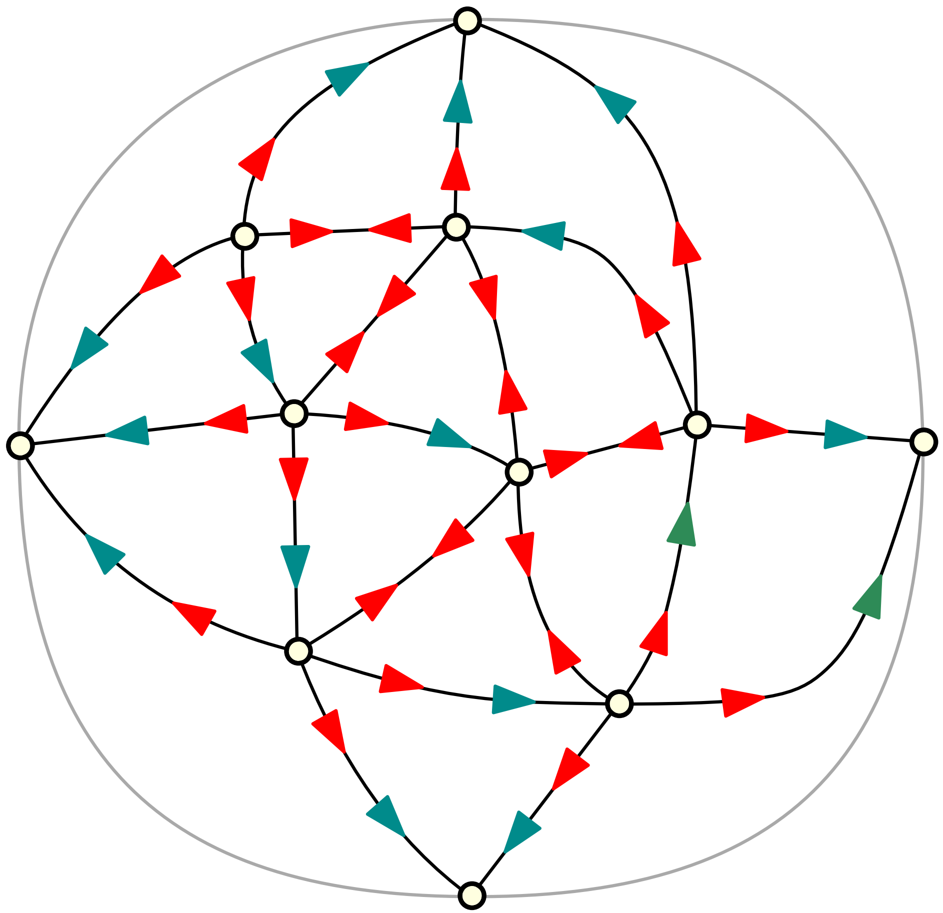
[F'07]





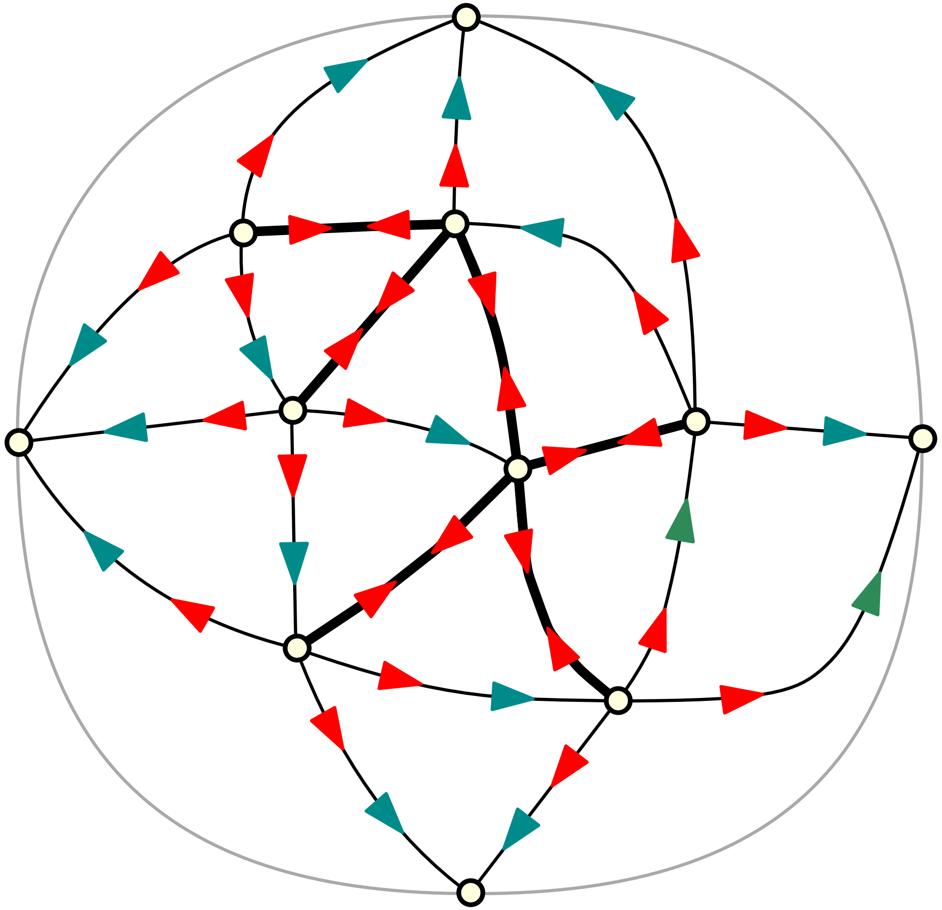
Bijection with unrooted ternary trees [F'07]

erase angular map



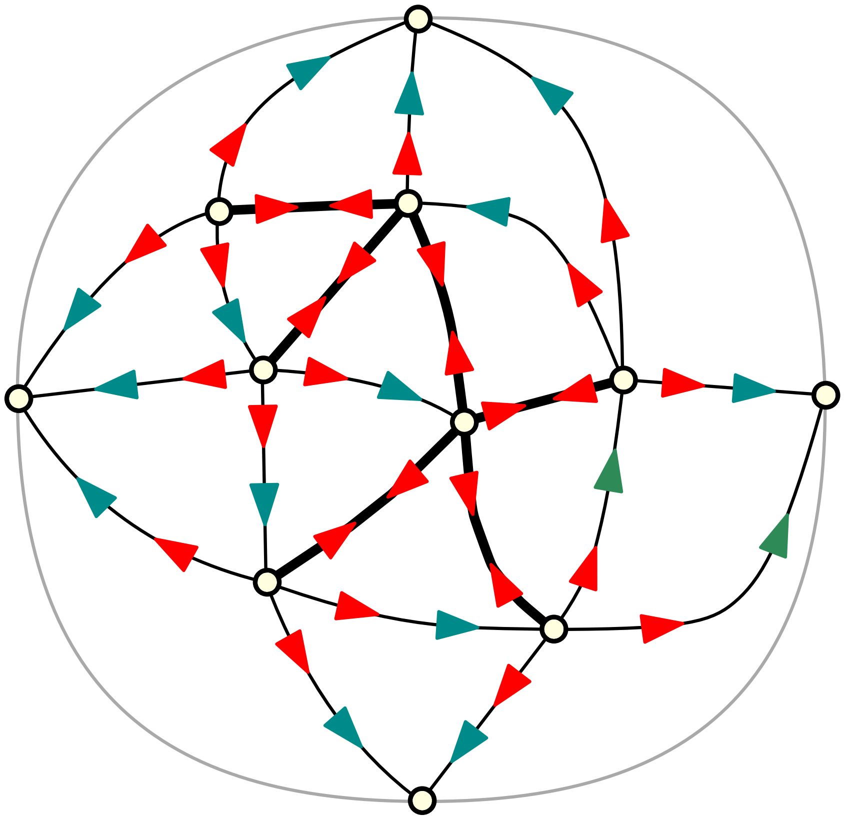
Bijection with unrooted ternary trees

[F'07]

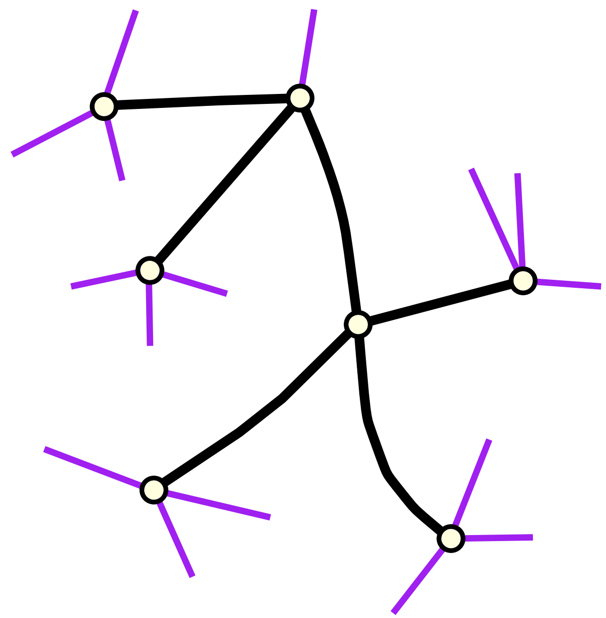


Bi-directed edges form spanning tree

Bijection with unrooted ternary trees

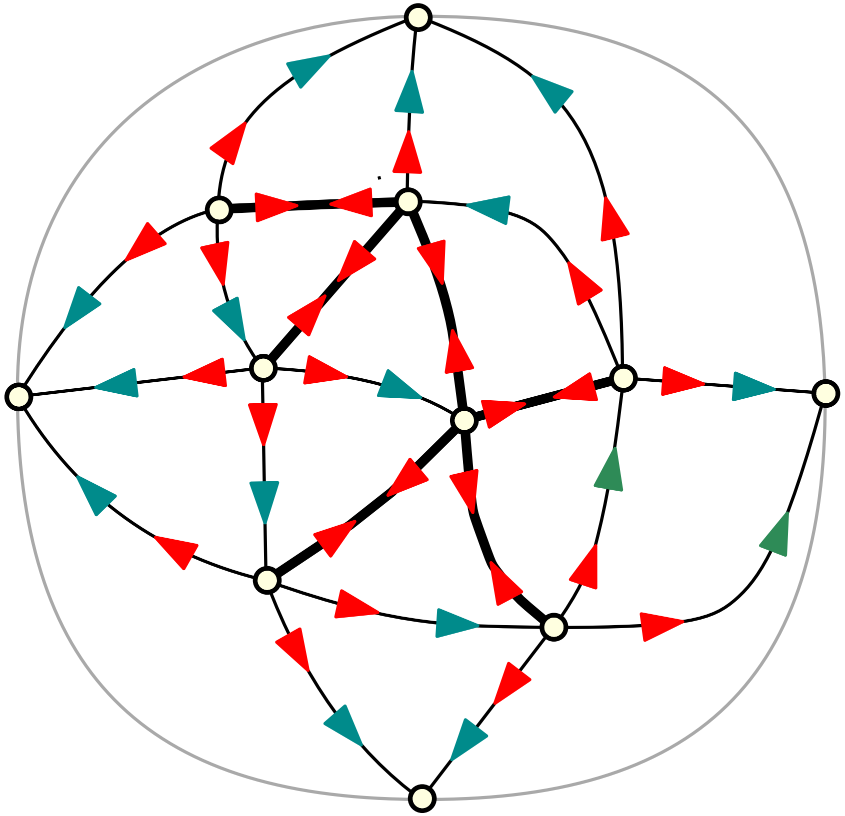


Tree-biorientation



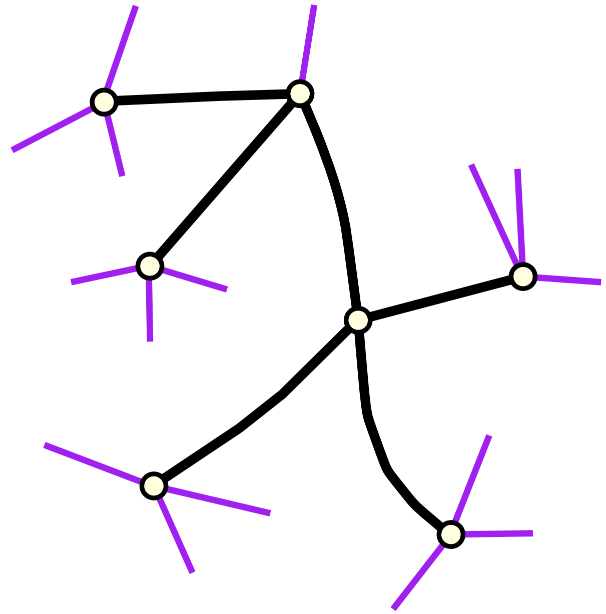
unrooted ternary tree

Bijection with unrooted ternary trees [F'07]



Tree-biorientation

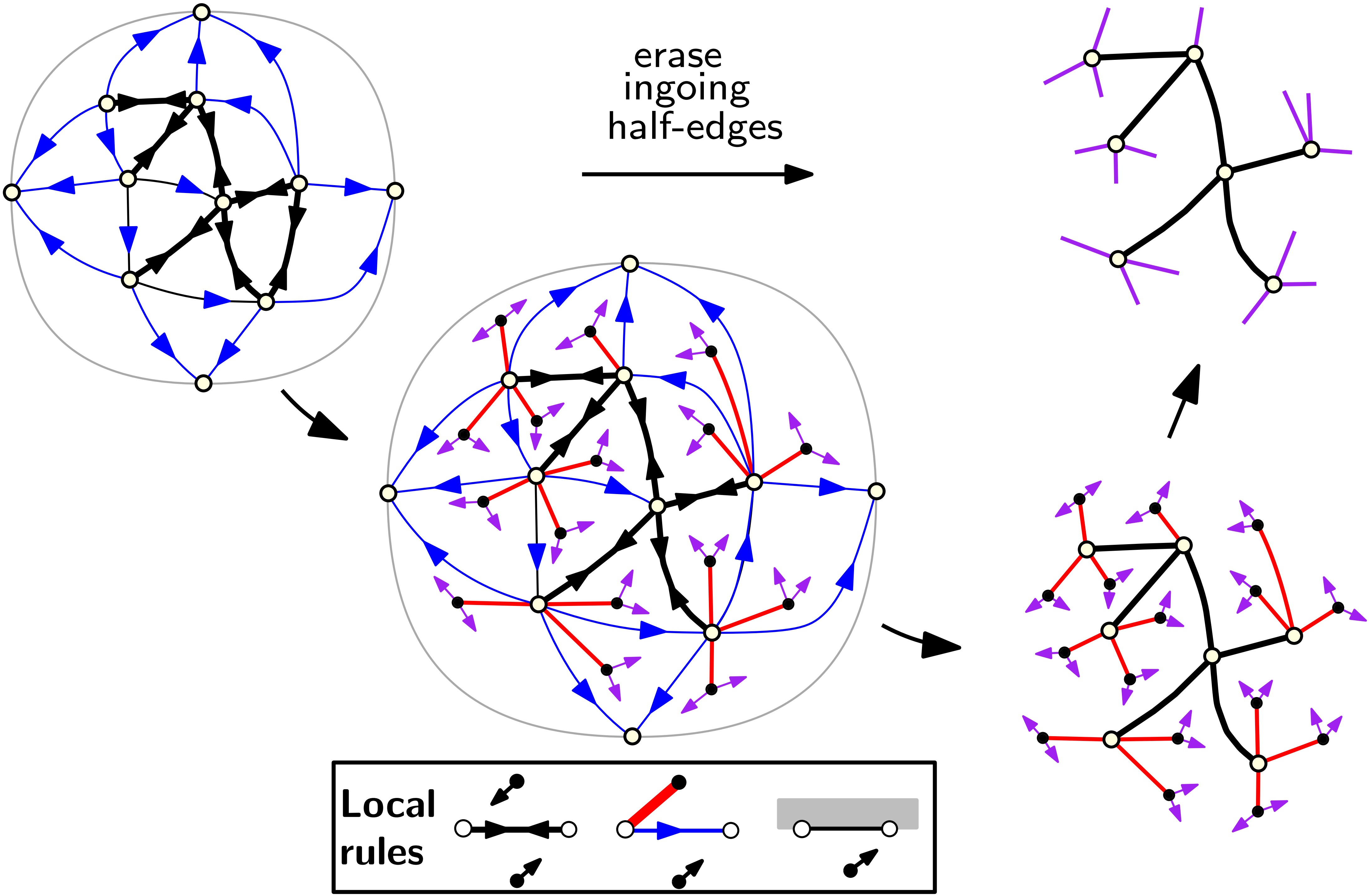
erase ingoing half-edges



unrooted ternary tree

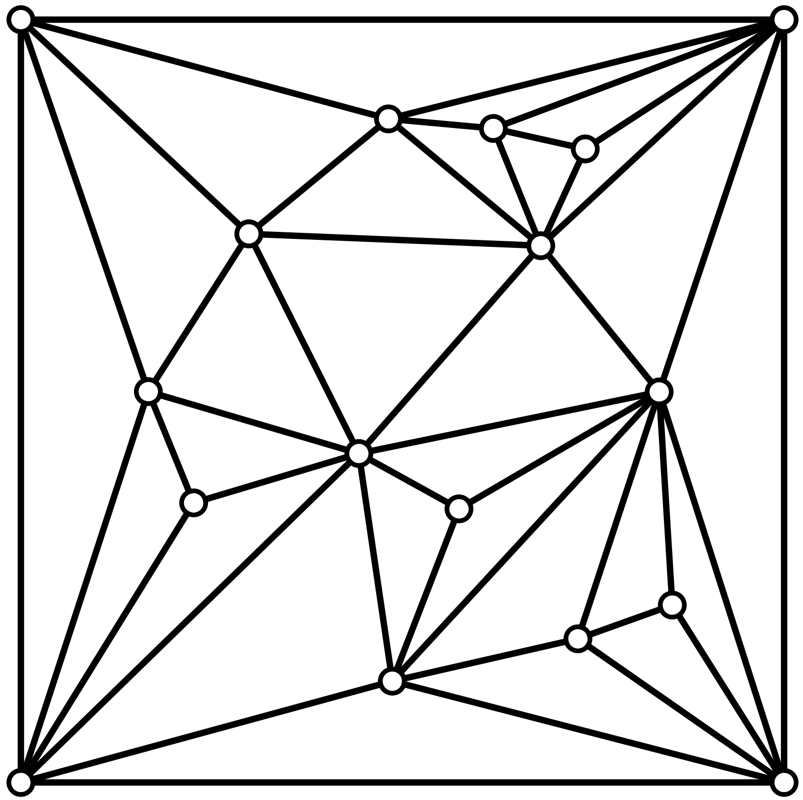
similar bijection for irreducible quadrangulations (of the 6-gon)
with unrooted binary trees [F-Poulalhon-Schaeffer'05]

Bijection with unrooted ternary trees [F'07]

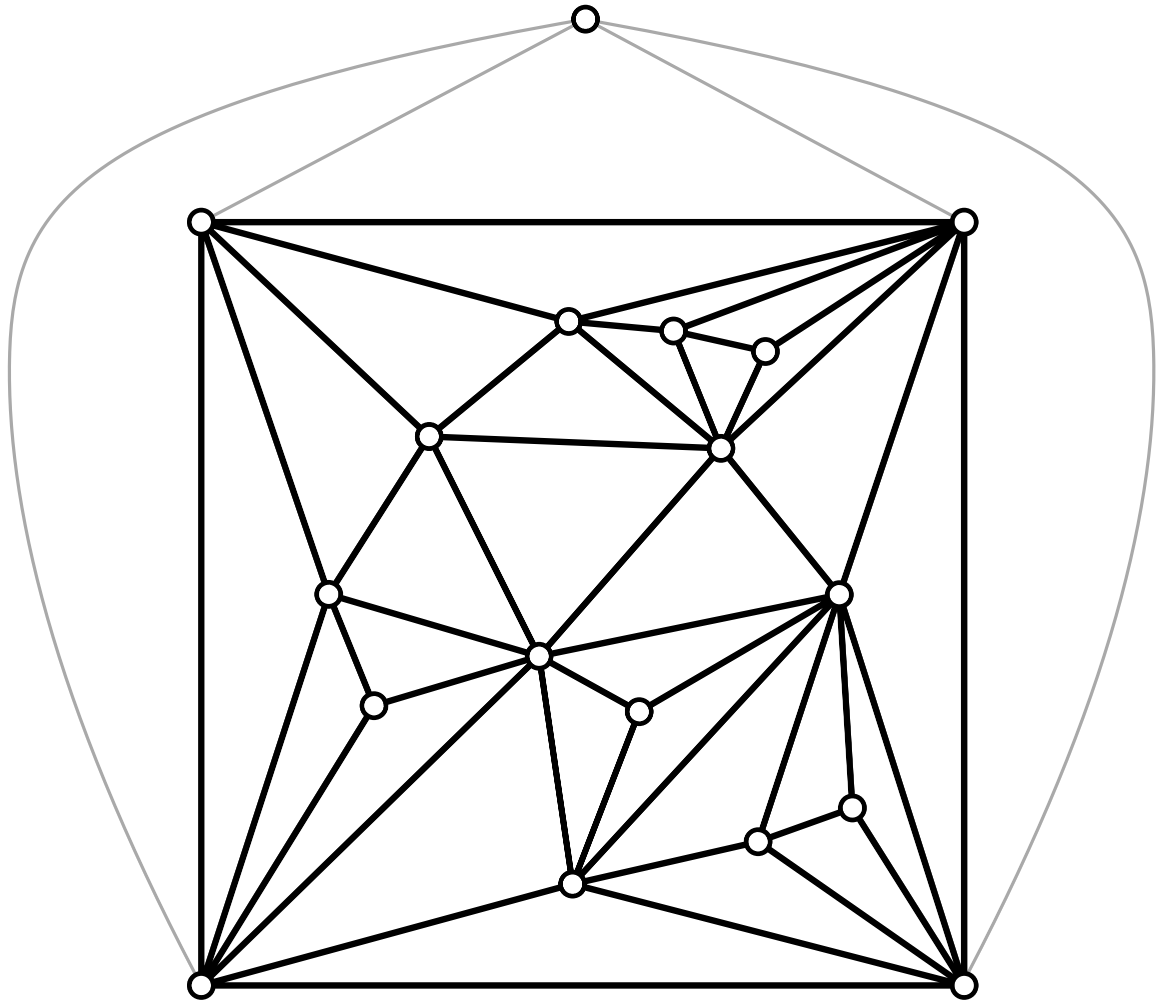


Relation of 3-orientations to irreducible core

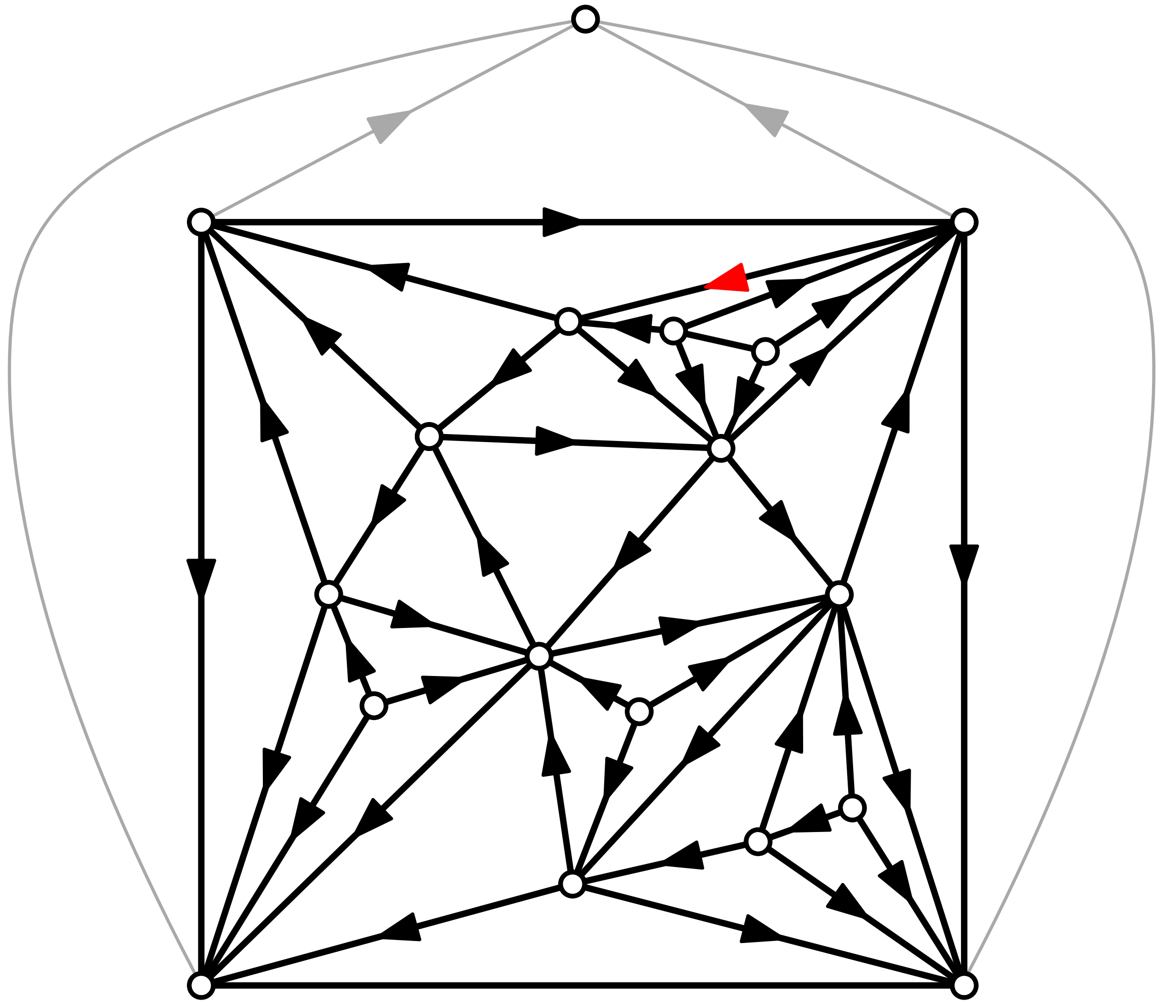
T a simple triangulation of the 4-gon



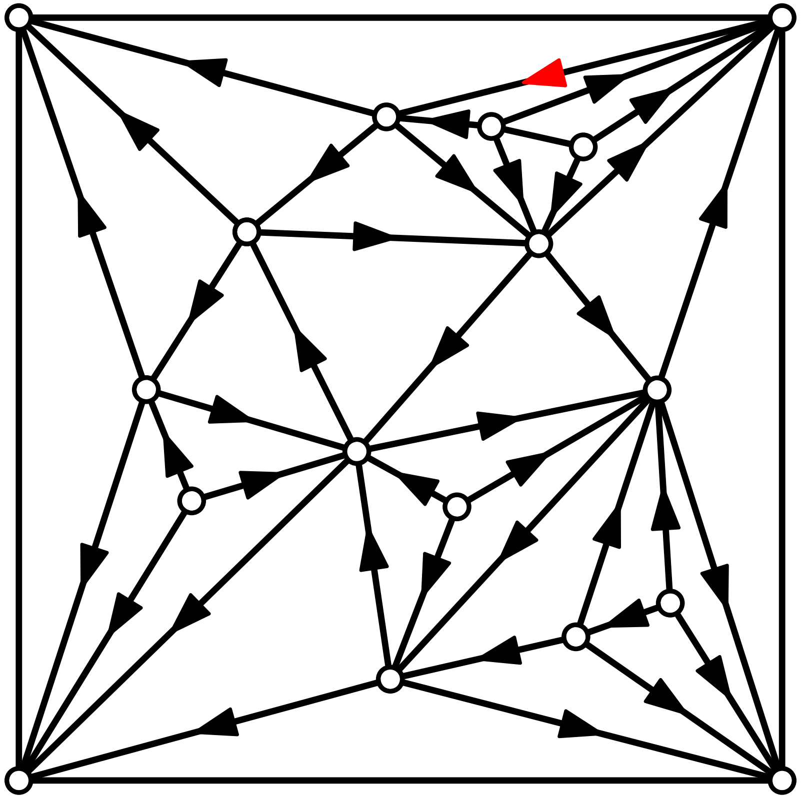
Relation of 3-orientations to irreducible core



Relation of 3-orientations to irreducible core

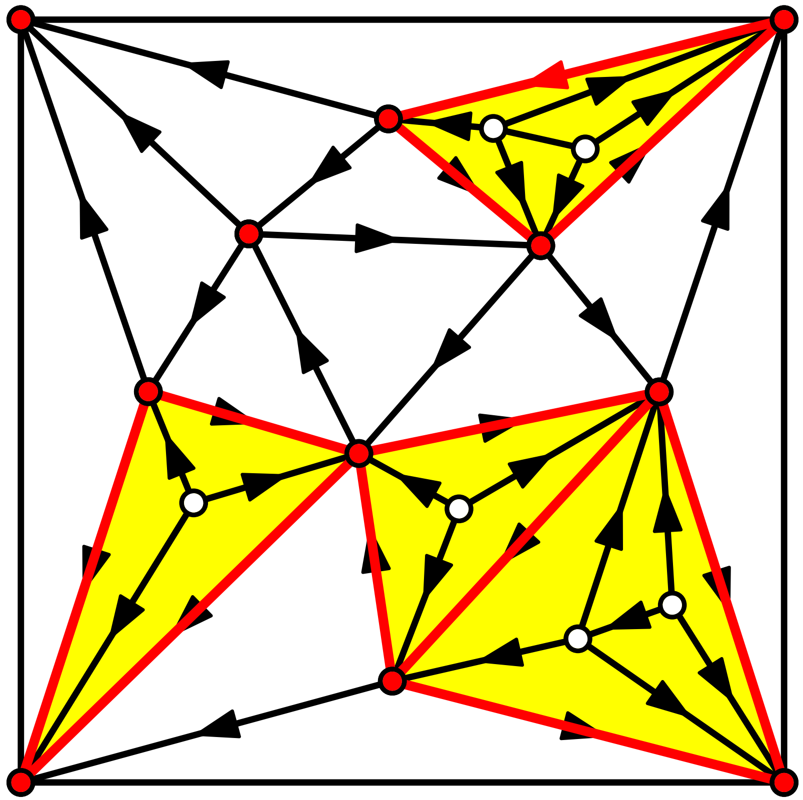


Relation of 3-orientations to irreducible core



3-orientation of T

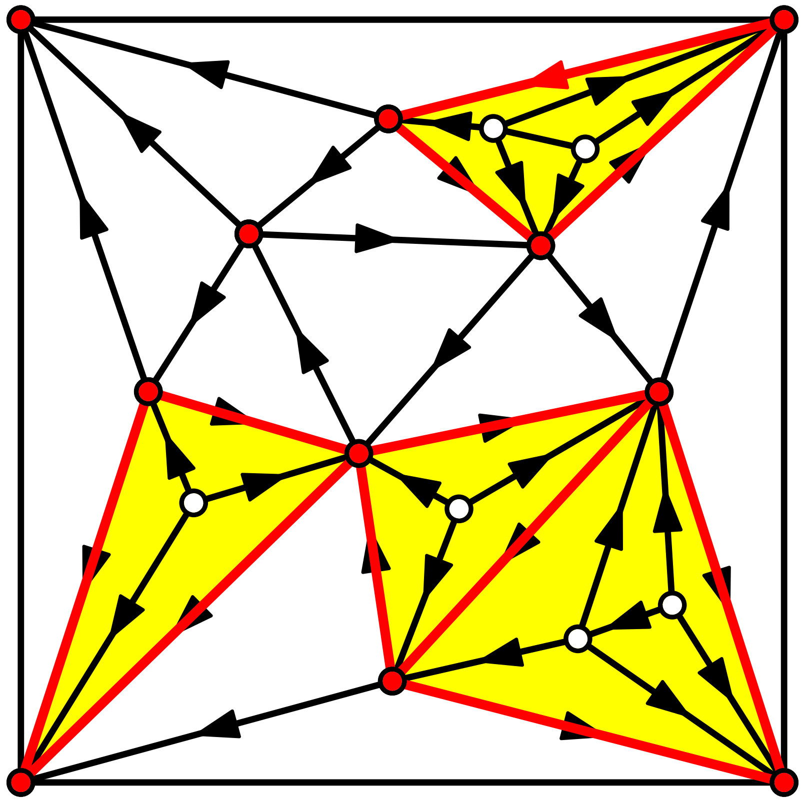
Relation of 3-orientations to irreducible core



3-orientation of T

Vertices that can be reached from the outer 4-gon
are those of the irreducible core

Relation of 3-orientations to irreducible core



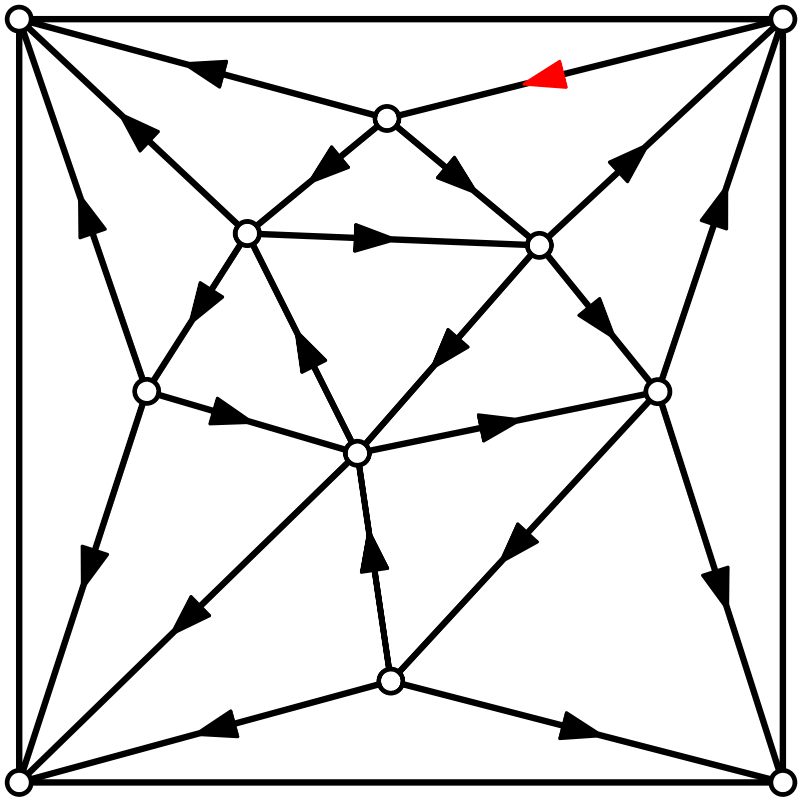
3-orientation of T

Vertices that can be reached from the outer 4-gon
are those of the irreducible core

$\Rightarrow$ T is irreducible iff 3-orientation is **co-accessible**

From 3-orientation to angular orientation

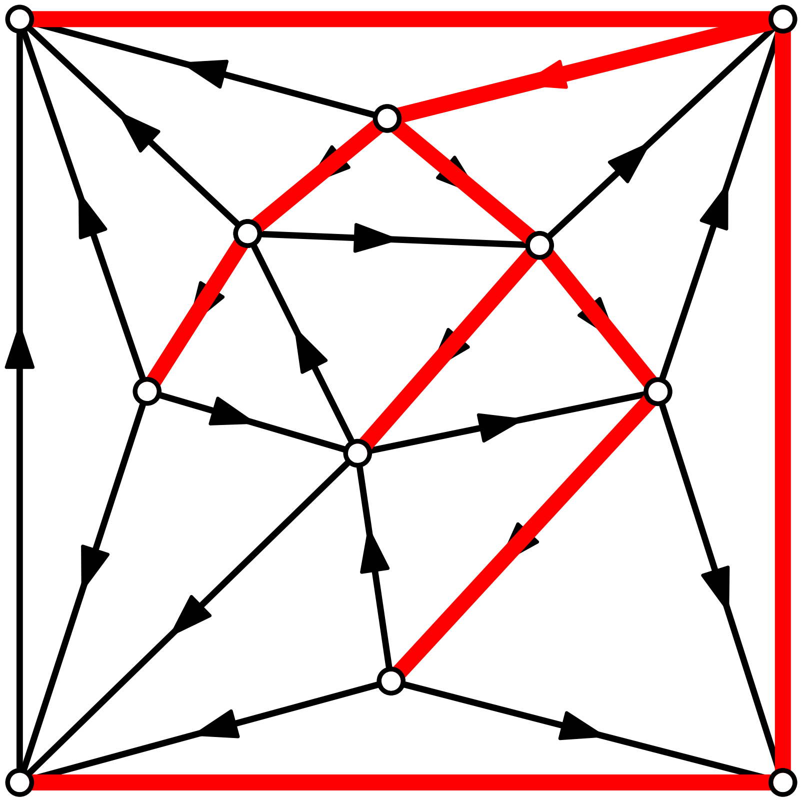
T an irreducible triangulation of the 4-gon endowed with a 3-orientation



From 3-orientation to angular orientation

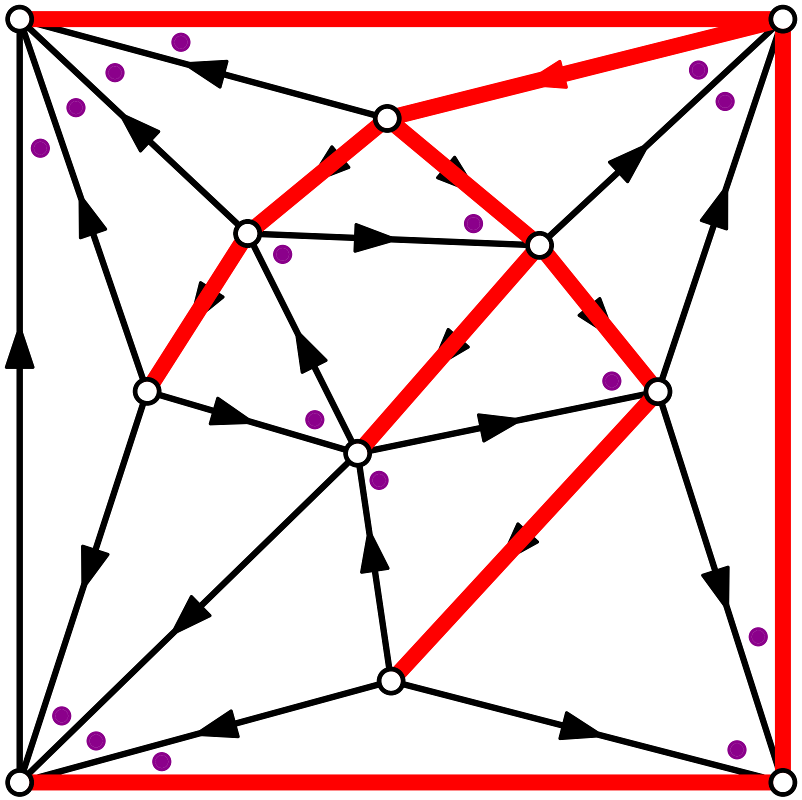
T an irreducible triangulation of the 4-gon endowed with a 3-orientation

Fix a **co-accessibility** spanning tree

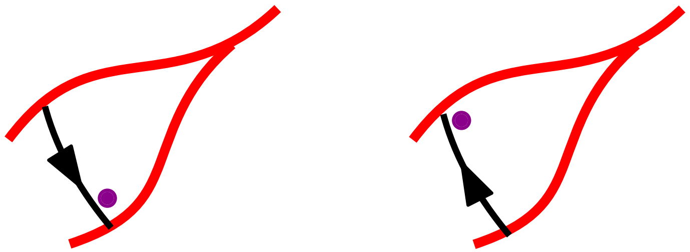


From 3-orientation to angular orientation

T an irreducible triangulation of the 4-gon endowed with a 3-orientation
Fix a **co-accessibility** spanning tree



mark a corner for each external edge



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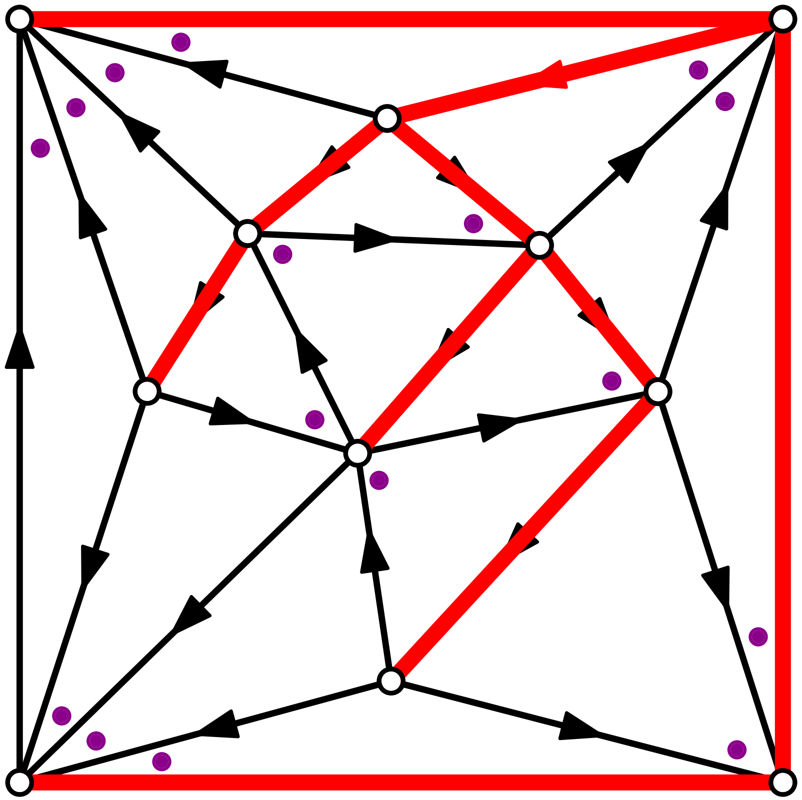

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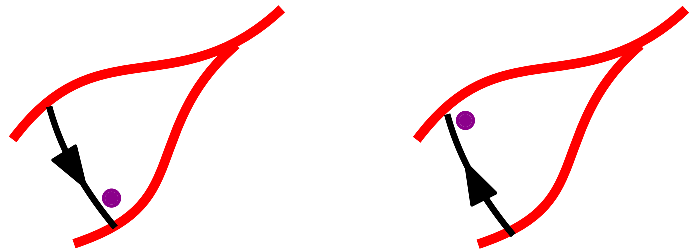
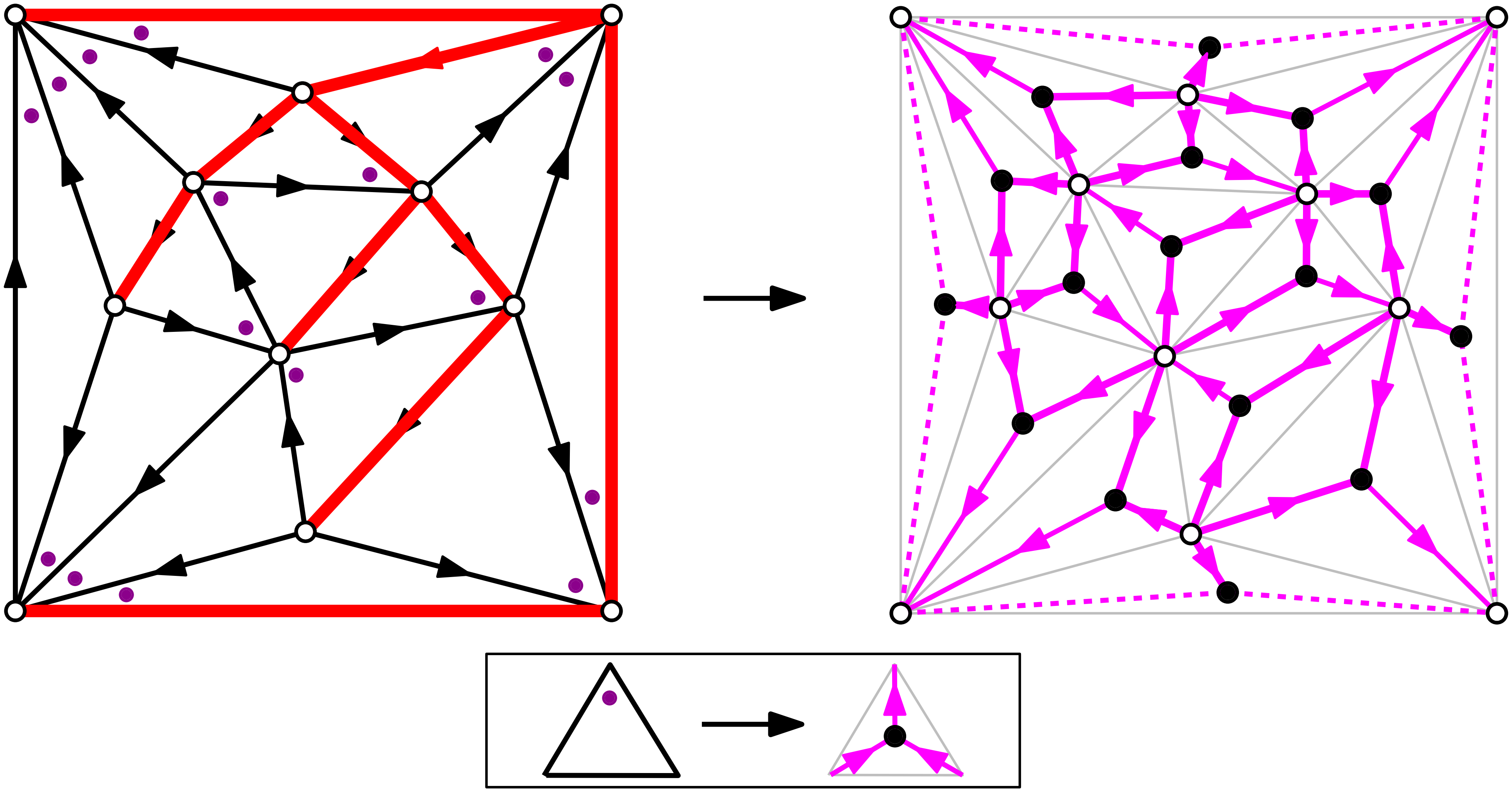


each internal face has 1 marked corner
each internal vertex has 4 unmarked corners

From 3-orientation to angular orientation

T an irreducible triangulation of the 4-gon endowed with a 3-orientation
Fix a **co-accessibility** spanning tree



each internal face has 1 marked corner
each internal vertex has 4 unmarked corners

Extension to all girths

odd d

d-angulation girth d

$d/(d-2)$-orientation

co-accessibility

irred. d-angulation of $(d+1)$-gon

angular orientation

Extension to all girths

odd d

d-angulation girth d

$d/(d-2)$-orientation

co-accessibility

irred. d-angulation of $(d+1)$-gon

angular orientation

even d, $d = 2b$

d-angulation girth d

$b/(b-1)$-orientation

irred. d-angulation of $(d+2)$-gon

angular orientation

Extension to all girths

odd d

d-angulation girth d

$d/(d-2)$-orientation

↓

irred. d-angulation of $(d+1)$-gon

angular orientation

↓

girth $d+1$?

even d, $d = 2b$

d-angulation girth d

$b/(b-1)$-orientation

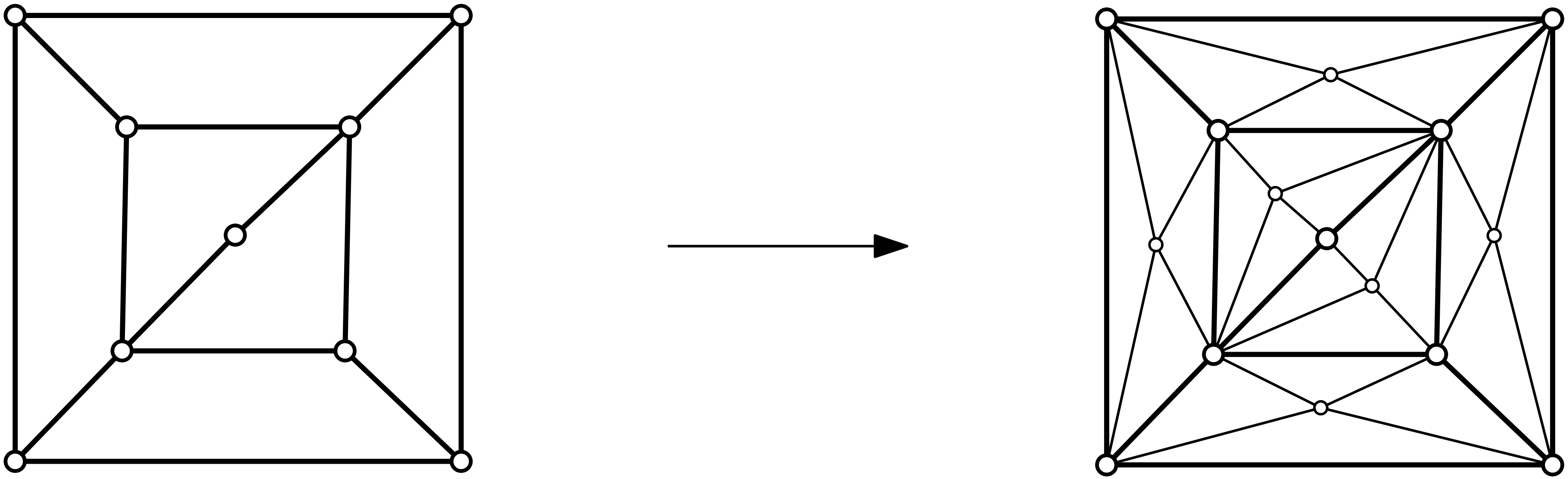
↓

irred. d-angulation of $(d+2)$-gon

angular orientation

co-accessibility

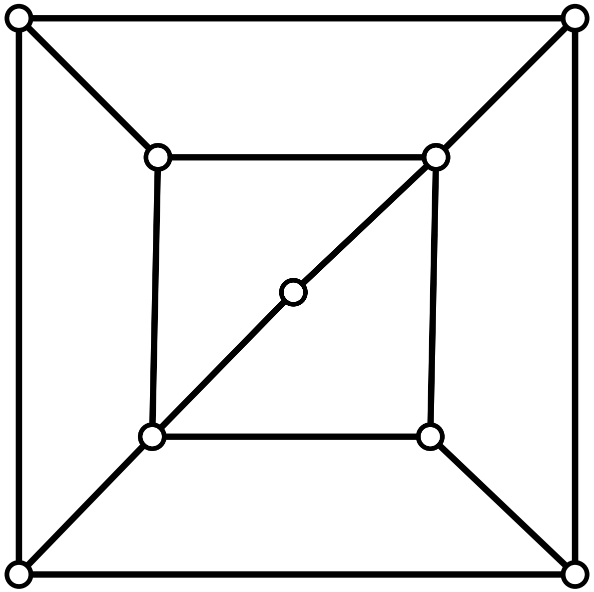
Girth 4 from 3-irreducible



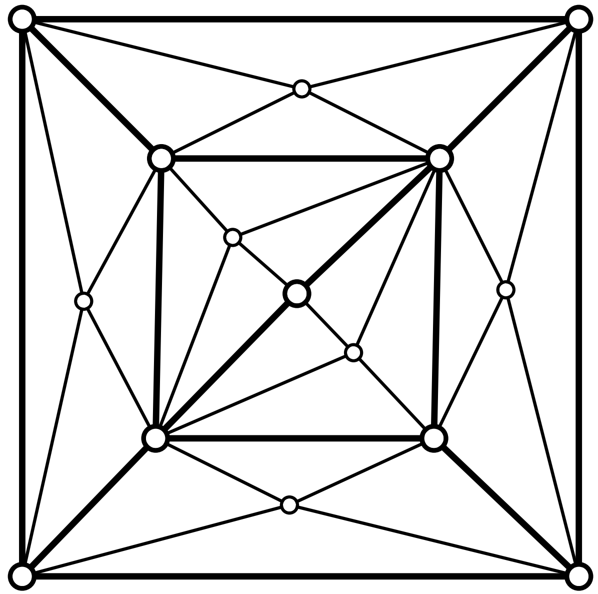
simple quadrangulation

irreducible triangulaton

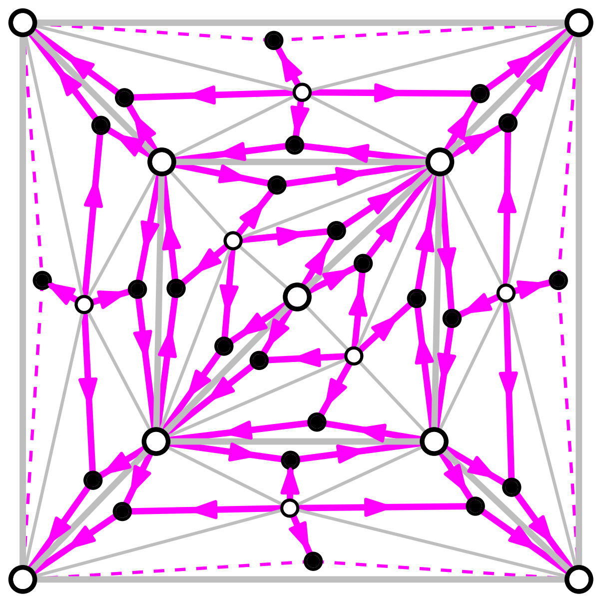
Girth 4 from 3-irreducible



simple quadrangulation

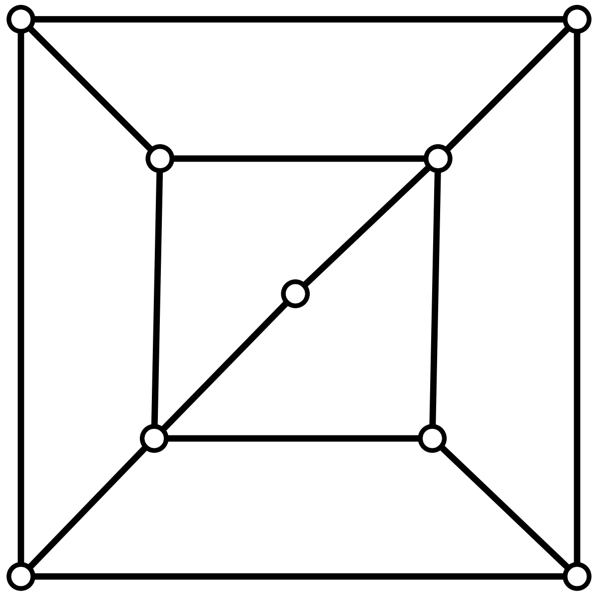


irreducible triangulaton

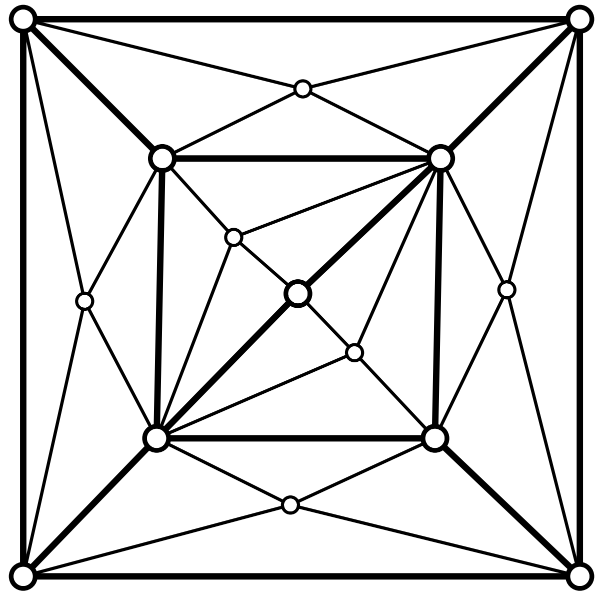


angular orientation

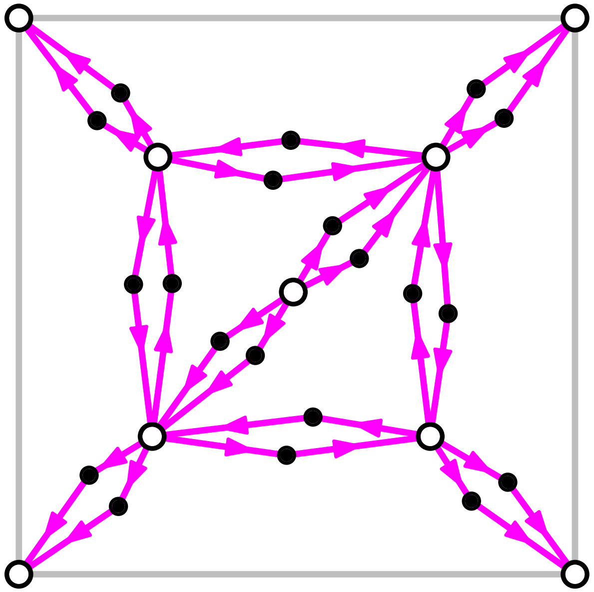
Girth 4 from 3-irreducible

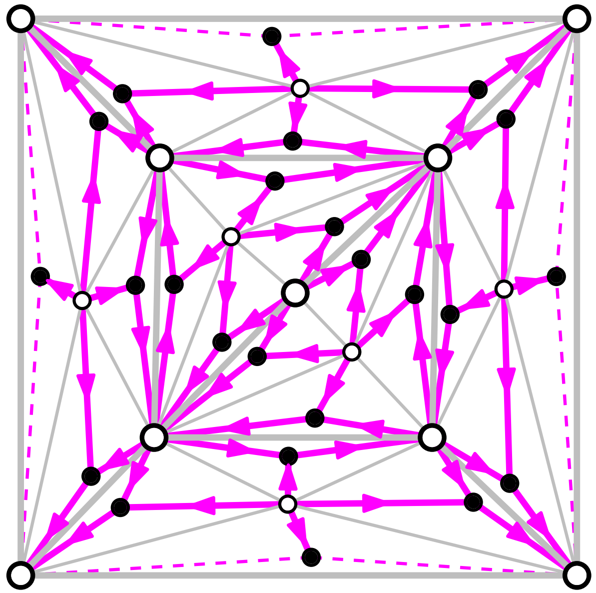


simple quadrangulation



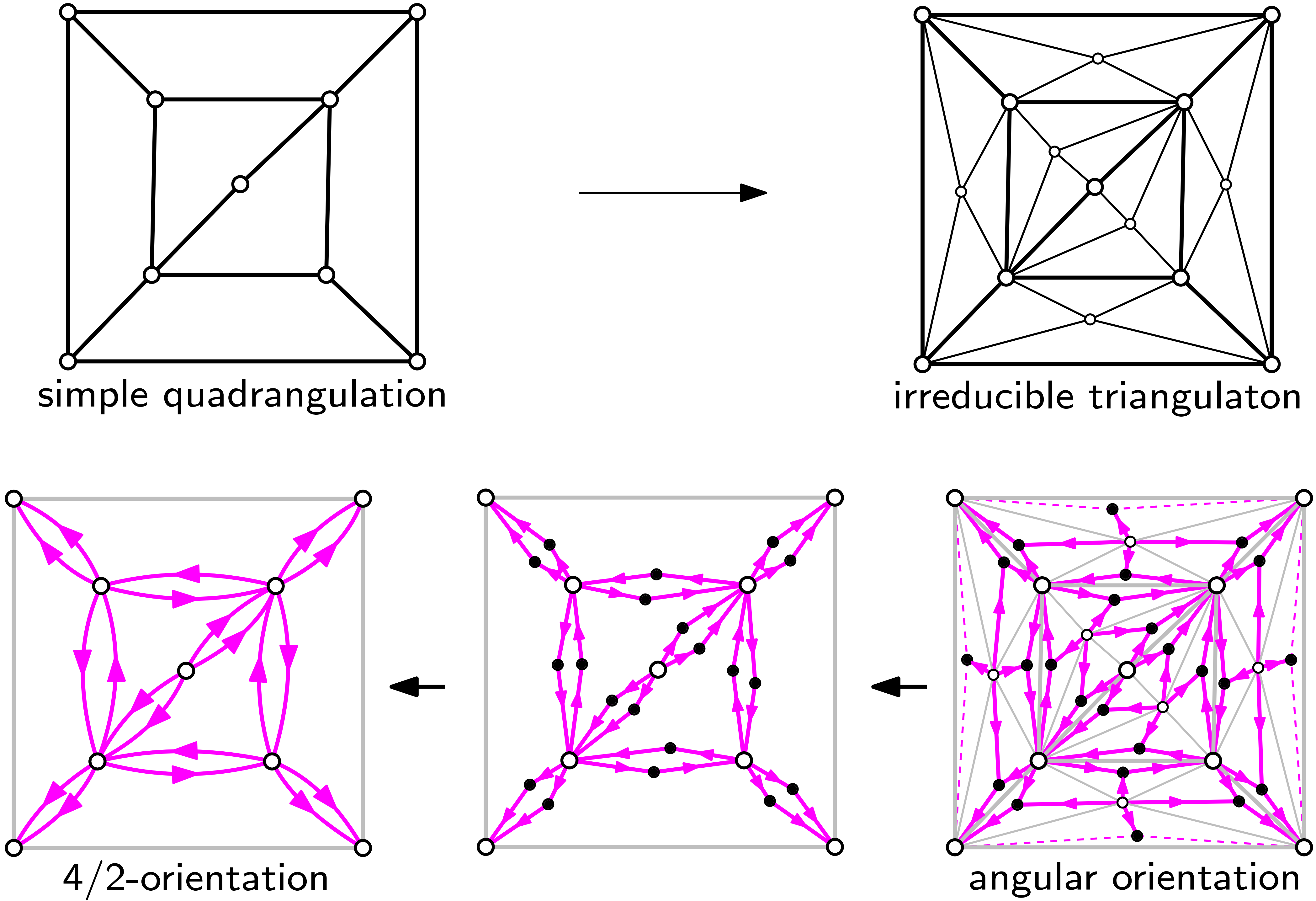
irreducible triangulaton





angular orientation

Girth 4 from 3-irreducible



Extension to all girths

odd d

d-angulation girth d
$d/(d-2)$-orientation

↓ co-accessibility

irred. d-angulation of $(d+1)$-gon
angular orientation

↓ d-angulate inner faces

$(d+1)$-angulation girth $d+1$
$(d+1)/(d-1)$-orientation

even d, $d=2b$

d-angulation girth d
$b/(b-1)$-orientation

↓ co-accessibility

irred. d-angulation of $(d+2)$-gon
angular orientation

↓ d-angulate inner faces

$(d+2)$-angulation girth $d+2$
$(b+1)/b$-orientation

Extension to all girths

odd d	even d, $d = 2b$
d-angulation girth d	d-angulation girth d
$d/(d-2)$-orientation	$b/(b-1)$-orientation
↓	↓

co-accessibility

irred. d-angulation of $(d+1)$-gon	irred. d-angulation of $(d+2)$-gon
angular orientation	angular orientation
↓	↓

d-angulate inner faces

$(d+1)$-angulation girth $d+1$	$(d+2)$-angulation girth $d+2$
$(d+1)/(d-1)$-orientation	$(b+1)/b$-orientation

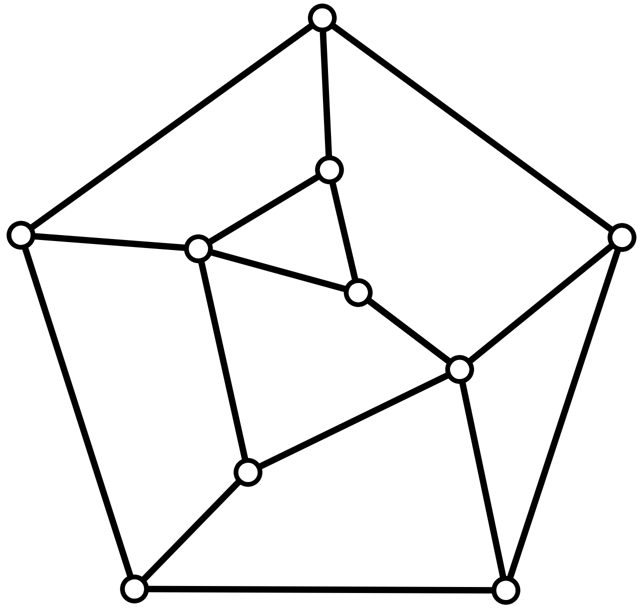
[Bouttier-Guitter-Manet'24]

computation from slice-decomposition

d-maps

[Bernardi-F-Liang'23]

d-map $=$ map with outer face degree d
non-facial girth d face-degrees $\leq d$

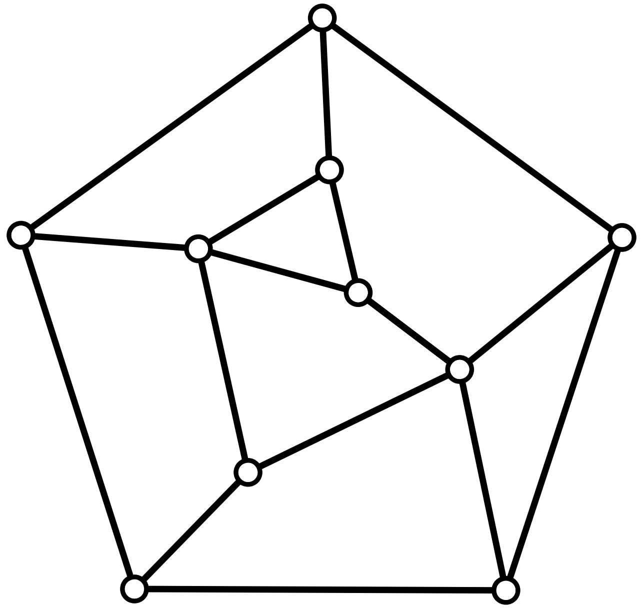


a 5-map

d-maps

[Bernardi-F-Liang'23]

d-map = map with outer face degree d
non-facial girth d face-degrees $\leq d$



a 5-map

d-maps
d-orientations

co-accessibility →

$(d+1)$-maps
face-degrees $\leq d$
$(d+1)$-orientations

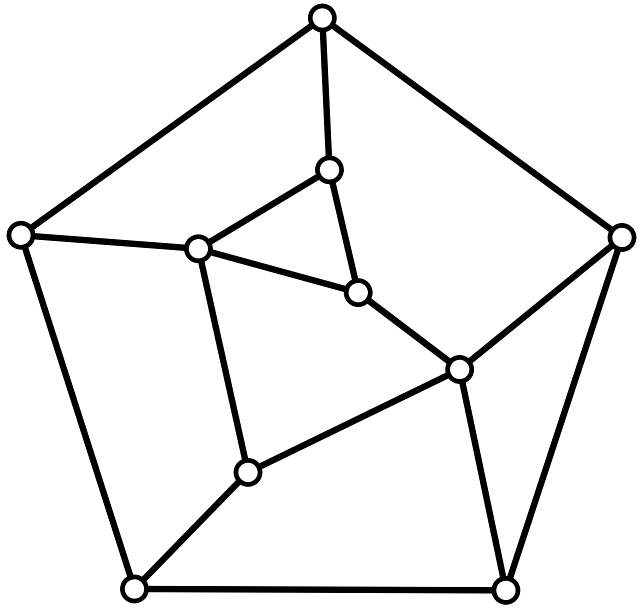
patch in $(d+1)$-faces →

$(d+1)$-maps
$(d+1)$-orientations

d-maps

[Bernardi-F-Liang'23]

d-map $=$ map with outer face degree d
non-facial girth d face-degrees $\leq d$



a 5-map

d-maps
d-orientations

co-accessibility →

$(d+1)$-maps
face-degrees $\leq d$
$(d+1)$-orientations

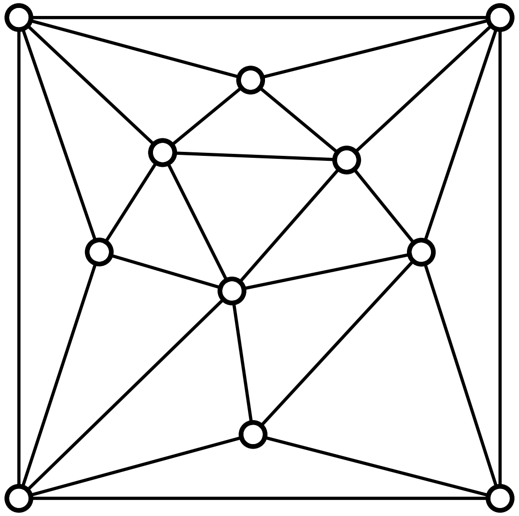
patch in $(d+1)$-faces →

$(d+1)$-maps
$(d+1)$-orientations

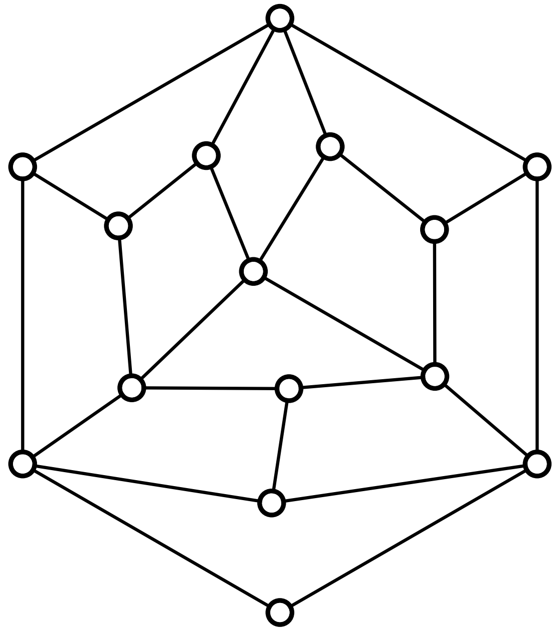
alternative incarnations of d-orientations as d-woods, d-corner labelings,...

Example with strong irreducibility

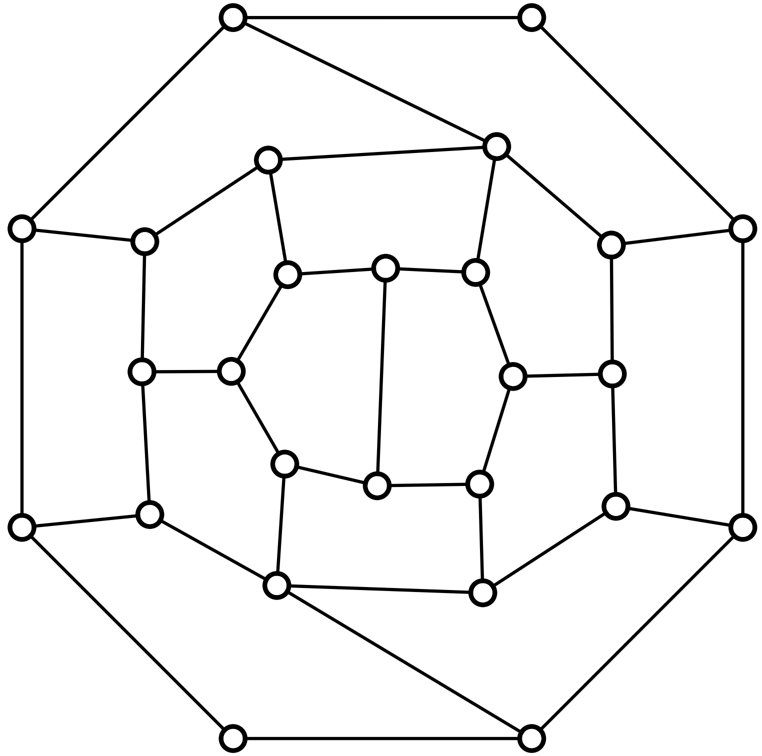
d-angulation of the $2d-2$-gon, nf-girth=$2d-2$ (none for $d \geq 6$)



$d = 3$



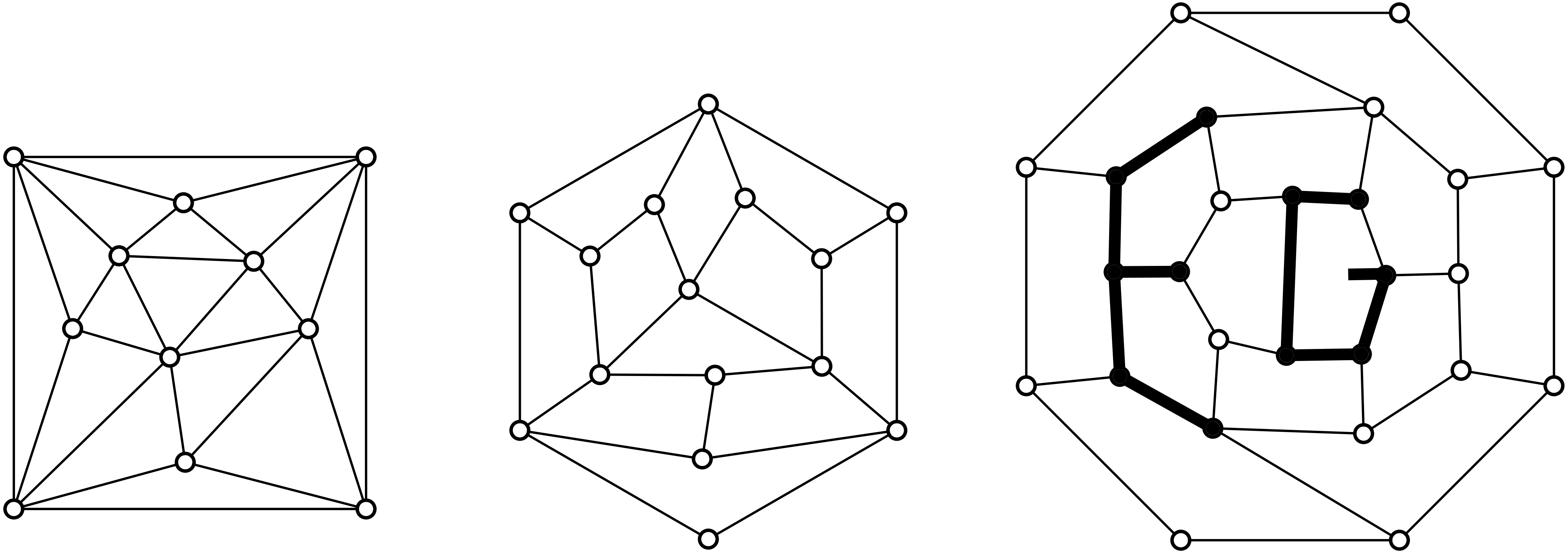
$d = 4$



$d = 5$

Example with strong irreducibility

d-angulation of the $2d-2$-gon, nf-girth$=2d-2$ (none for $d \geq 6$)



$d=3$ $\quad$ $d=4$ $\quad$ $d=5$

Thank you