The intrinsic metric and simple random walk on 2D critical percolation clusters

Jason Miller

Cambridge

based on joint works with Valeria Ambrosio, Irina Dankovic, Maarten Markering, and Yizheng Yuan

May 16, 2025









Fix  $p \in [0,1]$ ; keep each edge based on an independent coin toss with P[Heads] = p and  $P[\text{Tails}] = 1 - p \Rightarrow$  random subgraph of  $Z^2$  (Broadbent-Hammersley, 1957).



Goal: understand the resulting random graph when taking a scaling limit

Fix  $p \in [0,1]$ ; keep each edge based on an independent coin toss with P[Heads] = p and  $P[\text{Tails}] = 1 - p \Rightarrow$  random subgraph of  $Z^2$  (Broadbent-Hammersley, 1957).



Goal: understand the resulting random graph when taking a scaling limit

Sub-critical: clusters are small, no interesting behavior



- Goal: understand the resulting random graph when taking a scaling limit
- Sub-critical: clusters are small, no interesting behavior
- Super-critical: unique infinite cluster, macroscopic geometry is Euclidean



- Goal: understand the resulting random graph when taking a scaling limit
- Sub-critical: clusters are small, no interesting behavior
- Super-critical: unique infinite cluster, macroscopic geometry is Euclidean
- Critical: non-trivial limiting geometry, fractal behavior



 $CLE_6$  gasket. Scaling limit of critical percolation (Smirnov for  $\Delta$ -lattice).











- Supercritical percolation
  - behaves like the Euclidean metric (Gärtner-Molchanov, Antal-Pisztora, Garet-Marchand)

- Supercritical percolation
  - behaves like the Euclidean metric (Gärtner-Molchanov, Antal-Pisztora, Garet-Marchand)
- 2D Critical percolation

- Supercritical percolation
  - behaves like the Euclidean metric (Gärtner-Molchanov, Antal-Pisztora, Garet-Marchand)
- 2D Critical percolation
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest left-right open crossing of an n × n box is n<sup>α</sup> – what is α?



- Supercritical percolation
  - behaves like the Euclidean metric (Gärtner-Molchanov, Antal-Pisztora, Garet-Marchand)
- 2D Critical percolation
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest left-right open crossing of an n × n box is n<sup>α</sup> – what is α?
  - Damron-Hanson-Sosoe: exponent for shortest crossing is < 4/3</p>



- Supercritical percolation
  - behaves like the Euclidean metric (Gärtner-Molchanov, Antal-Pisztora, Garet-Marchand)
- 2D Critical percolation
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest left-right open crossing of an n × n box is n<sup>α</sup> – what is α?
  - Damron-Hanson-Sosoe: exponent for shortest crossing is < 4/3</p>
  - Pose-Schrenk-Araujo-Hermann: numerical work suggesting geodesics are SLE<sub>κ</sub> with κ ≅ 1.04



- Supercritical percolation
  - behaves like the Euclidean metric (Gärtner-Molchanov, Antal-Pisztora, Garet-Marchand)
- 2D Critical percolation
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest left-right open crossing of an n × n box is n<sup>α</sup> – what is α?
  - Damron-Hanson-Sosoe: exponent for shortest crossing is < 4/3</p>
  - Pose-Schrenk-Araujo-Hermann: numerical work suggesting geodesics are SLE<sub>κ</sub> with κ ≅ 1.04
- High dimensional critical percolation
  - Blanc-Renaudie-Broutin-Nachmias: scaling limit of critical percolation on the hypercube
  - Chatterjee-Chinmay-Hanson-Sosoe: forthcoming work on Z<sup>d</sup>











Many works study random walk scaling limits in related settings.

 Supercritical percolation: convergence to Brownian motion (Barlow, Berger-Biskup, Mathieu-Piatnitski)

- Supercritical percolation: convergence to Brownian motion (Barlow, Berger-Biskup, Mathieu-Piatnitski)
- 2D Critical percolation:
  - Kesten: subdiffusivity of random walk w.r.t. Euclidean metric
  - **Ghanguly-Lee**: subdiffusivity of random walk w.r.t. chemical distance metric

- Supercritical percolation: convergence to Brownian motion (Barlow, Berger-Biskup, Mathieu-Piatnitski)
- 2D Critical percolation:
  - Kesten: subdiffusivity of random walk w.r.t. Euclidean metric
  - **Ghanguly-Lee:** subdiffusivity of random walk w.r.t. chemical distance metric
- Uniform spanning tree (Barlow-Croydon-Kumagai)

- Supercritical percolation: convergence to Brownian motion (Barlow, Berger-Biskup, Mathieu-Piatnitski)
- 2D Critical percolation:
  - Kesten: subdiffusivity of random walk w.r.t. Euclidean metric
  - **Ghanguly-Lee:** subdiffusivity of random walk w.r.t. chemical distance metric
- Uniform spanning tree (Barlow-Croydon-Kumagai)
- High dimensional percolation (d > 6):

- Supercritical percolation: convergence to Brownian motion (Barlow, Berger-Biskup, Mathieu-Piatnitski)
- 2D Critical percolation:
  - Kesten: subdiffusivity of random walk w.r.t. Euclidean metric
  - ▶ Ghanguly-Lee: subdiffusivity of random walk w.r.t. chemical distance metric
- Uniform spanning tree (Barlow-Croydon-Kumagai)
- High dimensional percolation (d > 6):
  - Ben Arous-Cabezas-Fribergh: scaling limit results for simplified models

Many works study random walk scaling limits in related settings.

- Supercritical percolation: convergence to Brownian motion (Barlow, Berger-Biskup, Mathieu-Piatnitski)
- 2D Critical percolation:
  - Kesten: subdiffusivity of random walk w.r.t. Euclidean metric
  - ▶ Ghanguly-Lee: subdiffusivity of random walk w.r.t. chemical distance metric
- Uniform spanning tree (Barlow-Croydon-Kumagai)
- High dimensional percolation (d > 6):
  - Ben Arous-Cabezas-Fribergh: scaling limit results for simplified models
  - Kozma-Nachmias: Alexander-Orbach conjecture (spectral dimension is 4/3), i.e.

$$d_s := -2 \times \lim_{n \to \infty} \frac{\log p_{2n}(0,0)}{\log n} = \frac{4}{3}, \quad \text{i.e.,} \quad p_{2n}(0,0) = n^{-2/3 + o(1)}$$

where  $p_n(x, y)$  is the *n*-step transition kernel for random walk on the IIC (0-containing critical percolation cluster conditioned to be infinite) for  $d \ge 11$ .



Sierpinski gasket



Sierpinski gasket



Sierpinski gasket






Sierpinski gasket

Barlow-Perkins (1988): Brownian motion on the Sierpinski gasket



Sierpinski gasket



Sierpinski carpet

- Barlow-Perkins (1988): Brownian motion on the Sierpinski gasket
- Barlow-Bass (1989), Kusuoka-Zhou (1992): Brownian motion on the Sierpinski carpet



Sierpinski gasket



Sierpinski carpet

- Barlow-Perkins (1988): Brownian motion on the Sierpinski gasket
- Barlow-Bass (1989), Kusuoka-Zhou (1992): Brownian motion on the Sierpinski carpet
- Barlow-Bass-Kumagai-Teplyaev (2010): equivalence of Sierpinski carpet Brownian motions



Sierpinski gasket



- Barlow-Perkins (1988): Brownian motion on the Sierpinski gasket
- Barlow-Bass (1989), Kusuoka-Zhou (1992): Brownian motion on the Sierpinski carpet
- Barlow-Bass-Kumagai-Teplyaev (2010): equivalence of Sierpinski carpet Brownian motions
- Important framework: Kigami's theory of resistance metrics



- Barlow-Perkins (1988): Brownian motion on the Sierpinski gasket
- Barlow-Bass (1989), Kusuoka-Zhou (1992): Brownian motion on the Sierpinski carpet
- Barlow-Bass-Kumagai-Teplyaev (2010): equivalence of Sierpinski carpet Brownian motions
- Important framework: Kigami's theory of resistance metrics

#### Intrinsic metric and random walk in critical percolation

- Goal: understand the limit of the intrinsic metric and random walk in 2D critical percolation
- Step 1: work directly in the continuum on CLE<sub>6</sub> and construct its intrinsic metric and its canonical Brownian motion.
- Step 2: show that these objects are the scaling limit of the corresponding discrete objects.





# $\operatorname{CLE}$ background

The conformal loop ensembles (CLE<sub>κ</sub>) are a countable collection of non-crossing loops in a simply connected domain D ⊆ C

- The conformal loop ensembles (CLE<sub>κ</sub>) are a countable collection of non-crossing loops in a simply connected domain D ⊆ C
- Indexed by κ ∈ [8/3,8]; κ = 8/3 get the empty collection of loops; κ = 8 get a single space-filling loop.

# $\operatorname{CLE}$ background

- The conformal loop ensembles (CLE<sub>κ</sub>) are a countable collection of non-crossing loops in a simply connected domain D ⊆ C
- Indexed by κ ∈ [8/3,8]; κ = 8/3 get the empty collection of loops; κ = 8 get a single space-filling loop.
- Locally each loop looks like one of Schramm's  $SLE_{\kappa}$  curves.
  - ▶  $\kappa \in (8/3, 4]$  loops are simple, do not intersect each other or  $\partial D$
  - $\kappa \in (4, 8)$  loops are self-intersecting, hit each other and  $\partial D$

- The conformal loop ensembles (CLE<sub>κ</sub>) are a countable collection of non-crossing loops in a simply connected domain D ⊆ C
- Indexed by κ ∈ [8/3,8]; κ = 8/3 get the empty collection of loops; κ = 8 get a single space-filling loop.
- Locally each loop looks like one of Schramm's  $SLE_{\kappa}$  curves.
  - ▶  $\kappa \in (8/3, 4]$  loops are simple, do not intersect each other or  $\partial D$
  - ▶  $\kappa \in (4,8)$  loops are self-intersecting, hit each other and  $\partial D$
- Describe the scaling limit of the interfaces in critical lattice models in two-dimensions:
  - Ising model (κ = 3), GFF level lines (κ = 4), FK-Ising model (κ = 16/3), percolation (κ = 6), uniform spanning tree (κ = 8)

- The conformal loop ensembles (CLE<sub>κ</sub>) are a countable collection of non-crossing loops in a simply connected domain D ⊆ C
- Indexed by κ ∈ [8/3,8]; κ = 8/3 get the empty collection of loops; κ = 8 get a single space-filling loop.
- Locally each loop looks like one of Schramm's  $SLE_{\kappa}$  curves.
  - ▶  $\kappa \in (8/3, 4]$  loops are simple, do not intersect each other or  $\partial D$
  - ▶  $\kappa \in (4,8)$  loops are self-intersecting, hit each other and  $\partial D$
- Describe the scaling limit of the interfaces in critical lattice models in two-dimensions:
  - Ising model (κ = 3), GFF level lines (κ = 4), FK-Ising model (κ = 16/3), percolation (κ = 6), uniform spanning tree (κ = 8)

Characterized by restriction, conformal invariance (Sheffield-Werner)

- The conformal loop ensembles (CLE<sub>κ</sub>) are a countable collection of non-crossing loops in a simply connected domain D ⊆ C
- Indexed by κ ∈ [8/3,8]; κ = 8/3 get the empty collection of loops; κ = 8 get a single space-filling loop.
- Locally each loop looks like one of Schramm's  $SLE_{\kappa}$  curves.
  - ▶  $\kappa \in (8/3, 4]$  loops are simple, do not intersect each other or  $\partial D$
  - ▶  $\kappa \in (4,8)$  loops are self-intersecting, hit each other and  $\partial D$
- Describe the scaling limit of the interfaces in critical lattice models in two-dimensions:
  - Ising model (κ = 3), GFF level lines (κ = 4), FK-Ising model (κ = 16/3), percolation (κ = 6), uniform spanning tree (κ = 8)
- Characterized by restriction, conformal invariance (Sheffield-Werner)
- Two constructions:
  - Exploration tree (Sheffield)
  - Cluster boundaries of a Poisson point process of Brownian loops (Sheffield-Werner)



 ${\rm CLE}_3$  carpet. Scaling limit of the critical Ising model (Smirnov).



 ${\rm CLE}_6$  gasket. Scaling limit of critical percolation (Smirnov for  $\Delta\text{-lattice}).$ 



This talk is about constructing the chemical distance metric and canonical Brownian motion defined in the CLE<sub>κ</sub> gasket.

### Recap

- This talk is about constructing the chemical distance metric and canonical Brownian motion defined in the CLE<sub>κ</sub> gasket.
- Continuous objects which should describe
  - the scaling limit of the graph distance metric and
  - simple random walk

on clusters of discrete models which converge to  ${\rm CLE}_{\kappa}$  for  $\kappa \in (4, 8)$ , e.g., critical percolation.

# $\begin{array}{l} \mbox{Approximations} \\ \blacktriangleright \ \Upsilon \sim {\rm CLE}_{\kappa} \ \mbox{gasket} \end{array}$



- $\Upsilon \sim \text{CLE}_{\kappa}$  gasket
- For a path ω: [0, 1] → Υ which does not cross a loop, let 𝔑<sub>ϵ</sub>(ω) be the Lebesgue measure of the ϵ-neighborhood of ω



- $\Upsilon \sim \text{CLE}_{\kappa}$  gasket
- For a path ω: [0, 1] → Υ which does not cross a loop, let 𝔑<sub>ϵ</sub>(ω) be the Lebesgue measure of the ϵ-neighborhood of ω



- Υ ~ CLE<sub>κ</sub> gasket
- For a path ω: [0, 1] → Υ which does not cross a loop, let 𝔑<sub>ϵ</sub>(ω) be the Lebesgue measure of the ϵ-neighborhood of ω
- For  $z, w \in \Upsilon$ , let

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega)$$

where the infimum is over all paths  $\omega$ in  $\Upsilon$  from z to w which do not cross a loop



- Υ ~ CLE<sub>κ</sub> gasket
- For a path ω: [0, 1] → Υ which does not cross a loop, let 𝔑<sub>ϵ</sub>(ω) be the Lebesgue measure of the ϵ-neighborhood of ω
- For  $z, w \in \Upsilon$ , let

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega)$$

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w which do not cross a loop

► Goal 1: show that ∂<sub>e</sub>(·, ·) properly renormalized is tight; subsequential limit defines a geodesic metric



- Υ ~ CLE<sub>κ</sub> gasket
- For a path ω: [0,1] → Υ which does not cross a loop, let 𝔑<sub>ϵ</sub>(ω) be the Lebesgue measure of the ϵ-neighborhood of ω
- For  $z, w \in \Upsilon$ , let

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega)$$

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w which do not cross a loop

- ► Goal 1: show that ∂<sub>e</sub>(·, ·) properly renormalized is tight; subsequential limit defines a geodesic metric
- Goal 2: show that the limit is unique, characterized by a list of axioms



- Υ̂ ~ CLE<sub>κ</sub> gasket
- For a path ω: [0,1] → Υ which does not cross a loop, let 𝔑<sub>ϵ</sub>(ω) be the Lebesgue measure of the ϵ-neighborhood of ω
- For  $z, w \in \Upsilon$ , let

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega)$$

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w which do not cross a loop

- ► Goal 1: show that ∂<sub>e</sub>(·, ·) properly renormalized is tight; subsequential limit defines a geodesic metric
- Goal 2: show that the limit is unique, characterized by a list of axioms
- Important: choice of domain

Optimizing  $\omega$ ,  $\mathfrak{N}_{\epsilon}(\omega) \sim \epsilon^{2-\alpha}$  for  $\alpha \in (1,2)$  but for  $\omega = \partial \mathbf{D}$ ,  $\mathfrak{N}_{\epsilon}(\omega) \sim \epsilon$ .









- Consider the space **K** of 5-tuples  $(X, d, D, \mu, \pi)$  where
  - $(X, d, \mu)$  is a compact metric measure space,
  - D is another metric on X continuous with respect to d,
  - ▶  $\pi: X \rightarrow \mathbf{C}$  is a projection map which is 1-Lipschitz with respect to d

equipped with a natural variant of the Gromov-Hausdorff-Prokhorov topology.

- Consider the space **K** of 5-tuples  $(X, d, D, \mu, \pi)$  where
  - $(X, d, \mu)$  is a compact metric measure space,
  - $\blacktriangleright$  D is another metric on X continuous with respect to d,
  - ▶  $\pi: X \to \mathbf{C}$  is a projection map which is 1-Lipschitz with respect to d

equipped with a natural variant of the Gromov-Hausdorff-Prokhorov topology.

d is the "nice", "easy to construct", "ambient" metric and D is the non-trivial metric we want to construct

- Consider the space **K** of 5-tuples  $(X, d, D, \mu, \pi)$  where
  - $(X, d, \mu)$  is a compact metric measure space,
  - $\blacktriangleright$  D is another metric on X continuous with respect to d,
  - ▶  $\pi: X \rightarrow \mathbf{C}$  is a projection map which is 1-Lipschitz with respect to d

equipped with a natural variant of the Gromov-Hausdorff-Prokhorov topology.

- d is the "nice", "easy to construct", "ambient" metric and D is the non-trivial metric we want to construct
- In the CLE<sub>κ</sub> gasket Υ, there are points that correspond to multiple points in its natural metric space ("prime ends"):



- Consider the space **K** of 5-tuples  $(X, d, D, \mu, \pi)$  where
  - $(X, d, \mu)$  is a compact metric measure space,
  - $\blacktriangleright$  D is another metric on X continuous with respect to d,
  - ▶  $\pi: X \rightarrow \mathbf{C}$  is a projection map which is 1-Lipschitz with respect to d

equipped with a natural variant of the Gromov-Hausdorff-Prokhorov topology.

- d is the "nice", "easy to construct", "ambient" metric and D is the non-trivial metric we want to construct
- In the CLE<sub>κ</sub> gasket Υ, there are points that correspond to multiple points in its natural metric space ("prime ends"):
  - Set  $X^*$  the set of points in  $\Upsilon$  not on a loop of  $\Gamma$ ,

 $d(x, y) = \inf\{\operatorname{diam}(\gamma) : \gamma \text{ connects } x \text{ to } y \text{ in } \Upsilon\},\$ 

X the completion of  $X^*$  with respect to d



- Consider the space **K** of 5-tuples  $(X, d, D, \mu, \pi)$  where
  - $(X, d, \mu)$  is a compact metric measure space,
  - $\blacktriangleright$  D is another metric on X continuous with respect to d,
  - ▶  $\pi: X \rightarrow \mathbf{C}$  is a projection map which is 1-Lipschitz with respect to d

equipped with a natural variant of the Gromov-Hausdorff-Prokhorov topology.

- d is the "nice", "easy to construct", "ambient" metric and D is the non-trivial metric we want to construct
- In the CLE<sub>κ</sub> gasket Υ, there are points that correspond to multiple points in its natural metric space ("prime ends"):
  - Set  $X^*$  the set of points in  $\Upsilon$  not on a loop of  $\Gamma$ ,

 $d(x, y) = \inf\{\operatorname{diam}(\gamma) : \gamma \text{ connects } x \text{ to } y \text{ in } \Upsilon\},\$ 

X the completion of  $X^*$  with respect to d

•  $\pi$  natural projection map  $X \rightarrow \mathbf{C}$ , and



- Consider the space **K** of 5-tuples  $(X, d, D, \mu, \pi)$  where
  - $(X, d, \mu)$  is a compact metric measure space,
  - D is another metric on X continuous with respect to d,
  - ▶  $\pi: X \rightarrow \mathbf{C}$  is a projection map which is 1-Lipschitz with respect to d

equipped with a natural variant of the Gromov-Hausdorff-Prokhorov topology.

- d is the "nice", "easy to construct", "ambient" metric and D is the non-trivial metric we want to construct
- In the CLE<sub>κ</sub> gasket Υ, there are points that correspond to multiple points in its natural metric space ("prime ends"):
  - Set  $X^*$  the set of points in  $\Upsilon$  not on a loop of  $\Gamma$ ,

 $d(x, y) = \inf\{\operatorname{diam}(\gamma) : \gamma \text{ connects } x \text{ to } y \text{ in } \Upsilon\},\$ 

X the completion of  $X^*$  with respect to d

- $\pi$  natural projection map  $X \rightarrow \mathbf{C}$ , and
- μ is the natural measure on the CLE<sub>κ</sub> gasket (M.-Schoug)





Cluster boundaries in a CLE<sub>κ</sub> look like SLE<sub>16/κ</sub> curves, take our domain D to be an "SLE<sub>16/κ</sub> loop"



- Cluster boundaries in a CLE<sub>κ</sub> look like SLE<sub>16/κ</sub> curves, take our domain D to be an "SLE<sub>16/κ</sub> loop"
- ▶ *D* a domain bounded by an  $SLE_{16/\kappa}$  loop,  $z \in \partial D$  a "typical point".



- Cluster boundaries in a CLE<sub>κ</sub> look like SLE<sub>16/κ</sub> curves, take our domain D to be an "SLE<sub>16/κ</sub> loop"
- ▶ *D* a domain bounded by an  $SLE_{16/\kappa}$  loop,  $z \in \partial D$  a "typical point".

▶ In the infinite volume limit near *z*,  $\partial D$  is described by a two-sided whole-plane  $SLE_{16/\kappa}$ 



- Cluster boundaries in a CLE<sub>κ</sub> look like SLE<sub>16/κ</sub> curves, take our domain D to be an "SLE<sub>16/κ</sub> loop"
- ▶ *D* a domain bounded by an  $SLE_{16/\kappa}$  loop,  $z \in \partial D$  a "typical point".
- ▶ In the infinite volume limit near *z*,  $\partial D$  is described by a two-sided whole-plane  $SLE_{16/\kappa}$
# The choice of normalization



- Cluster boundaries in a CLE<sub>κ</sub> look like SLE<sub>16/κ</sub> curves, take our domain D to be an "SLE<sub>16/κ</sub> loop"
- ▶ *D* a domain bounded by an  $SLE_{16/\kappa}$  loop,  $z \in \partial D$  a "typical point".
- ▶ In the infinite volume limit near *z*,  $\partial D$  is described by a two-sided whole-plane  $SLE_{16/\kappa}$
- Work in the infinite volume setting and set

$$\mathfrak{m}_{\epsilon} = \mathsf{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$$

# Intrinsic metric tightness theorem



#### Theorem (Ambrosio-M.-Yuan)

Fix  $\kappa \in (4,8)$ . Suppose that  $\Gamma_D$  is a  $CLE_{\kappa}$  in **D** and let *D* be the set of points surrounded by the loop  $\mathcal{L} \in \Gamma_D$  which surrounds 0. Given  $\mathcal{L}$ , let  $\Gamma$  be a  $CLE_{\kappa}$  in *D* with gasket  $\Upsilon$ . Then:

- The law of the map  $(z, w) \mapsto \mathfrak{m}_{\epsilon}^{-1} \mathfrak{d}_{\epsilon}(z, w)$  is tight in the space **K**.
- Any subsequential limit is a.s. a geodesic metric space on Υ.

## Intrinsic metric uniqueness theorem



#### Theorem (M.-Yuan)

Fix  $\kappa \in (4,8)$ . Suppose that  $\Gamma_D$  is a  $CLE_{\kappa}$  in **D** and let *D* be the set of points surrounded by the loop  $\mathcal{L} \in \Gamma_D$  which surrounds 0. Given  $\mathcal{L}$ , let  $\Gamma$  be a  $CLE_{\kappa}$  in *D* with gasket  $\Upsilon$ .

- There exists at most one metric ∂ on Ŷ which is local, geodesic, and conformally covariant.
- Every subsequential limit as  $\epsilon \to 0$  of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  satisfies these properties, so the limit exists.

 $\kappa=$  6 should describe the scaling limit of the intrinsic metric for 2D critical percolation.

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in E} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in E} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

Using the edge weights w as conductances defines a random walk X on G

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in \mathcal{E}} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

metric

walk

form

• Using the edge weights w as conductances defines a random walk X on G

► Facts: i) 
$$\mathcal{R}$$
 is a metric on  $V$ , ii)  $\underbrace{\mathcal{R}}_{\text{Resistance}} \longleftrightarrow \underbrace{w}_{\text{Virblet}} \longleftrightarrow \underbrace{X}_{\text{Random}}$ 

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in \mathcal{E}} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

• Using the edge weights w as conductances defines a random walk X on G

► Facts: i) 
$$\mathcal{R}$$
 is a metric on  $V$ , ii)  $\underbrace{\mathcal{R}}_{\substack{\text{Resistance}\\ \text{metric}}} \underbrace{w}_{\substack{W}} \longleftrightarrow \underbrace{X}_{\substack{\text{Random}\\ \text{form}}} \underbrace{W}_{\substack{W \\ \text{walk}}}$ 

Kigami's theory of resistance metrics generalizes this to the continuum

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in E} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

Using the edge weights w as conductances defines a random walk X on G

► Facts: i) 
$$\mathcal{R}$$
 is a metric on  $V$ , ii)  $\underset{\substack{\mathcal{R} \\ \text{Resistance} \\ \text{metric}}}{\mathcal{R}} \longleftrightarrow \underset{\substack{W \\ \text{orr}}}{\overset{W}{\longleftrightarrow}} \longleftrightarrow \underset{\substack{X \\ \text{walk} \\ \text{orr}}}{\overset{W}{\longleftrightarrow}}$ 

Kigami's theory of resistance metrics generalizes this to the continuum

▶ A metric  $\mathcal{R}$  on a set F is called a *resistance metric* if for every  $V \subseteq F$  finite, there exists a graph G = (V, E) with edge weights w so that  $\mathcal{R}|_{V \times V}$  is equal to the resistance metric associated with those weights

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in E} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

Using the edge weights w as conductances defines a random walk X on G

► Facts: i) 
$$\mathcal{R}$$
 is a metric on  $V$ , ii)  $\underbrace{\mathcal{R}}_{\text{Resistance}} \longleftrightarrow \underbrace{w}_{\text{form}} \longleftrightarrow \underbrace{X}_{\text{walk}}_{\text{form}}$ 

Kigami's theory of resistance metrics generalizes this to the continuum

- A metric  $\mathcal{R}$  on a set F is called a *resistance metric* if for every  $V \subseteq F$  finite, there exists a graph G = (V, E) with edge weights w so that  $\mathcal{R}|_{V \times V}$  is equal to the resistance metric associated with those weights
- **Kigami**: if  $\mathcal{R}$  is a resistance metric on F so that
  - ▶ (*F*, *R*) is compact and
  - $\mu$  is a finite Borel measure on F with full support,

then  $(\mathcal{R}, \mu)$  determine a Dirichlet form hence a  $\mu$ -symmetric Markov process on F.

• Given a graph G = (V, E) with edge weights w, the effective resistance is

$$\mathcal{R}(x,y) = \left(\inf\left\{\sum_{e \in E} w(e)(\nabla h(e))^2 : h \colon V \to \mathbf{R}, \ h(x) = 0, \ h(y) = 1\right\}\right)^{-1} \quad \forall x, y \in V.$$

Using the edge weights w as conductances defines a random walk X on G

► Facts: i) 
$$\mathcal{R}$$
 is a metric on  $V$ , ii)  $\underbrace{\mathcal{R}}_{\text{Resistance}} \longleftrightarrow \underbrace{w}_{\text{form}} \longleftrightarrow \underbrace{X}_{\text{walk}}_{\text{form}}$ 

Kigami's theory of resistance metrics generalizes this to the continuum

- A metric  $\mathcal{R}$  on a set F is called a *resistance metric* if for every  $V \subseteq F$  finite, there exists a graph G = (V, E) with edge weights w so that  $\mathcal{R}|_{V \times V}$  is equal to the resistance metric associated with those weights
- ▶ Kigami: if *R* is a resistance metric on *F* so that
  - ▶ (*F*, *R*) is compact and
  - $\mu$  is a finite Borel measure on F with full support,

then  $(\mathcal{R}, \mu)$  determine a Dirichlet form hence a  $\mu$ -symmetric Markov process on F. Works for low-dimensional fractals (e.g., Sierpinski gasket / carpet) but not, say, for  $\mathbf{R}^d$  with  $d \geq 2$ .

#### Resistance metric existence and uniqueness



**Ambrosio-M.-Yuan**: for a natural family of graph approximations  $\Upsilon_{\epsilon}$  to  $\Upsilon$  with associated effective resistance  $\mathcal{R}_{\epsilon}$ ,

- ▶ The law of the map  $(z, w) \mapsto \mathfrak{m}_{\epsilon}^{-1} \mathcal{R}_{\epsilon}(z, w)$  is tight in the space **K**
- Any subsequential limit is a resistance metric on Υ hence defines a Markov process on Υ symmetric w.r.t. the CLE<sub>κ</sub> gasket measure.

#### Resistance metric existence and uniqueness



**Ambrosio-M.-Yuan**: for a natural family of graph approximations  $\Upsilon_{\epsilon}$  to  $\Upsilon$  with associated effective resistance  $\mathcal{R}_{\epsilon}$ ,

- ▶ The law of the map  $(z, w) \mapsto \mathfrak{m}_{\epsilon}^{-1} \mathcal{R}_{\epsilon}(z, w)$  is tight in the space K
- Any subsequential limit is a resistance metric on Υ hence defines a Markov process on Υ symmetric w.r.t. the CLE<sub>κ</sub> gasket measure.

#### M.-Yuan

- There exists at most one resistance metric R on Υ which induces a Dirichlet form on Υ which is local and scale covariant.
- Every subsequential limit as e → 0 of m<sub>e</sub><sup>-1</sup>R<sub>e</sub> satisfies these properties, so the limit exists.

#### Resistance metric existence and uniqueness



**Ambrosio-M.-Yuan**: for a natural family of graph approximations  $\Upsilon_{\epsilon}$  to  $\Upsilon$  with associated effective resistance  $\mathcal{R}_{\epsilon}$ ,

- ▶ The law of the map  $(z, w) \mapsto \mathfrak{m}_{\epsilon}^{-1} \mathcal{R}_{\epsilon}(z, w)$  is tight in the space K
- Any subsequential limit is a resistance metric on Υ hence defines a Markov process on Υ symmetric w.r.t. the CLE<sub>κ</sub> gasket measure.

#### M.-Yuan

- There exists at most one resistance metric R on Υ which induces a Dirichlet form on Υ which is local and scale covariant.
- Every subsequential limit as e → 0 of m<sub>e</sub><sup>-1</sup>R<sub>e</sub> satisfies these properties, so the limit exists.

**Consequence**: existence and uniqueness of the canonical  $CLE_{\kappa}$  Brownian motion;  $\kappa = 6$  should describe the scaling limit of simple random walk on 2D critical percolation

For critical percolation on the  $\Delta$ -lattice, we have:

**Smirnov**: the interfaces converge to CLE<sub>6</sub>



For critical percolation on the  $\Delta$ -lattice, we have:

- **Smirnov**: the interfaces converge to CLE<sub>6</sub>
- Garban-Pete-Schramm: the cluster measure converges to a continuum measure on CLE<sub>6</sub>



For critical percolation on the  $\Delta$ -lattice, we have:

- ▶ Smirnov: the interfaces converge to CLE<sub>6</sub>
- Garban-Pete-Schramm: the cluster measure converges to a continuum measure on CLE<sub>6</sub>

**Croydon**: given a sequence of metric measure spaces  $(X_n, \mathcal{R}_n, \mu_n)$  where  $\mathcal{R}_n$  is a resistance metric on  $X_n$ , the Gromov-Hausdorff-Prokhorov convergence  $(X_n, \mathcal{R}_n, \mu_n) \rightarrow (X, \mathcal{R}, \mu)$  implies weak convergence of the associated processes (provided all spaces are compact).



For critical percolation on the  $\Delta$ -lattice, we have:

- ▶ Smirnov: the interfaces converge to CLE<sub>6</sub>
- Garban-Pete-Schramm: the cluster measure converges to a continuum measure on CLE<sub>6</sub>

**Croydon**: given a sequence of metric measure spaces  $(X_n, \mathcal{R}_n, \mu_n)$  where  $\mathcal{R}_n$  is a resistance metric on  $X_n$ , the Gromov-Hausdorff-Prokhorov convergence  $(X_n, \mathcal{R}_n, \mu_n) \rightarrow (X, \mathcal{R}, \mu)$  implies weak convergence of the associated processes (provided all spaces are compact).

#### Theorem (Dankovic-Markering-M.-Yuan)

For critical percolation on the  $\Delta$ -lattice,

- $\blacktriangleright$  the intrinsic metric converges to the  ${\rm CLE}_6$  intrinsic metric and
- ▶ simple random walk converges to the CLE<sub>6</sub> Brownian motion

jointly with the convergence of the interfaces to  ${\rm CLE}_6$  and the cluster measure to the  ${\rm CLE}_6$  gasket measure.



- Can we say anything about the shortest path exponent in 2D critical percolation?
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest path exponent (it is < 4/3 by Damron-Hanson-Sosoe)</p>

- Can we say anything about the shortest path exponent in 2D critical percolation?
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest path exponent (it is < 4/3 by Damron-Hanson-Sosoe)</p>

▶ Alexander-Orbach conjecture: spectral dimension of the IIC is 4/3, i.e.

$$d_s := -2 \times \lim_{n \to \infty} \frac{\log p_{2n}(0,0)}{\log n} = \frac{4}{3}, \quad \text{i.e.,} \quad p_{2n}(0,0) = n^{-2/3 + o(1)}$$

where  $p_n(x, y)$  is the *n*-step transition kernel for random walk on the IIC (0-containing critical percolation cluster conditioned to be infinite).

- Can we say anything about the shortest path exponent in 2D critical percolation?
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest path exponent (it is < 4/3 by Damron-Hanson-Sosoe)</p>

▶ Alexander-Orbach conjecture: spectral dimension of the IIC is 4/3, i.e.

$$d_s := -2 \times \lim_{n \to \infty} \frac{\log p_{2n}(0,0)}{\log n} = \frac{4}{3}, \quad \text{i.e.,} \quad p_{2n}(0,0) = n^{-2/3 + o(1)}$$

where  $p_n(x, y)$  is the *n*-step transition kernel for random walk on the IIC (0-containing critical percolation cluster conditioned to be infinite).

Proved by Kozma-Nachmias for d ≥ 11; expected to be true for d > 6 (missing ingredient: construction of the IIC for 6 < d < 11)</p>

- Can we say anything about the shortest path exponent in 2D critical percolation?
  - Problem 3.3 of Schramm's 2006 ICM contribution: shortest path exponent (it is < 4/3 by Damron-Hanson-Sosoe)</p>

▶ Alexander-Orbach conjecture: spectral dimension of the IIC is 4/3, i.e.

$$d_s := -2 \times \lim_{n \to \infty} \frac{\log p_{2n}(0,0)}{\log n} = \frac{4}{3}, \quad \text{i.e.,} \quad p_{2n}(0,0) = n^{-2/3 + o(1)}$$

where  $p_n(x, y)$  is the *n*-step transition kernel for random walk on the IIC (0-containing critical percolation cluster conditioned to be infinite).

- ▶ Proved by Kozma-Nachmias for  $d \ge 11$ ; expected to be true for d > 6 (missing ingredient: construction of the IIC for 6 < d < 11)
- Expected to be false for  $2 \le d \le 5$ , but numerical simulations show it is remarkably close to being true:

$$d = 5 \rightarrow d_s = 1.34 \pm 0.02, \quad d = 4 \rightarrow d_s = 1.30 \pm 0.04,$$
  
 $d = 3 \rightarrow d_s = 1.32 \pm 0.01, \quad d = 2 \rightarrow d_s = 1.318 \pm 0.001$ 

(Source: D. Ben-Avraham and S. Havlin. Diffusion and reactions in fractals and disordered systems. Cambridge University Press, Cambridge, 2000.)



# Happy Birthday, Emmanuel!

**Recall:** *D* domain whose boundary is an  $\text{SLE}_{16/\kappa}$  loop,  $\Gamma \sim \text{CLE}_{\kappa}$  in *D*,

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega), \quad \mathfrak{N}_{\epsilon}(\omega) = \mathsf{Leb}(\epsilon \text{ neighborhood of } \omega)$$

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w and

$$\mathfrak{m}_{\epsilon} = \operatorname{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$$

**Tightness theorem:** tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$ ; subsequential limits are geodesic metrics

**Recall:** *D* domain whose boundary is an  $\text{SLE}_{16/\kappa}$  loop,  $\Gamma \sim \text{CLE}_{\kappa}$  in *D*,

 $\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega), \quad \mathfrak{N}_{\epsilon}(\omega) = \mathsf{Leb}(\epsilon \text{ neighborhood of } \omega)$ 

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w and

 $\mathfrak{m}_{\epsilon} = \operatorname{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$ 

**Tightness theorem:** tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$ ; subsequential limits are geodesic metrics

Proof has three main steps:

**Recall:** *D* domain whose boundary is an  $\text{SLE}_{16/\kappa}$  loop,  $\Gamma \sim \text{CLE}_{\kappa}$  in *D*,

 $\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega), \quad \mathfrak{N}_{\epsilon}(\omega) = \mathsf{Leb}(\epsilon \text{ neighborhood of } \omega)$ 

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w and

 $\mathfrak{m}_{\epsilon} = \operatorname{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$ 

**Tightness theorem:** tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$ ; subsequential limits are geodesic metrics

Proof has three main steps:

I. Prove tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$  restricted to the boundary

**Recall:** *D* domain whose boundary is an  $\text{SLE}_{16/\kappa}$  loop,  $\Gamma \sim \text{CLE}_{\kappa}$  in *D*,

 $\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega), \quad \mathfrak{N}_{\epsilon}(\omega) = \mathsf{Leb}(\epsilon \text{ neighborhood of } \omega)$ 

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w and

 $\mathfrak{m}_{\epsilon} = \operatorname{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$ 

**Tightness theorem:** tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$ ; subsequential limits are geodesic metrics

Proof has three main steps:

- I. Prove tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$  restricted to the boundary
- II. Extend tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  to the interior

**Recall:** *D* domain whose boundary is an  $\text{SLE}_{16/\kappa}$  loop,  $\Gamma \sim \text{CLE}_{\kappa}$  in *D*,

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega), \quad \mathfrak{N}_{\epsilon}(\omega) = \operatorname{Leb}(\epsilon \text{ neighborhood of } \omega)$$

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w and

$$\mathfrak{m}_{\epsilon} = \operatorname{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$$

**Tightness theorem:** tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$ ; subsequential limits are geodesic metrics

Proof has three main steps:

- I. Prove tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$  restricted to the boundary
- II. Extend tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  to the interior
- III. Show that the subsequential limits are positive definite and geodesic

**Recall:** *D* domain whose boundary is an  $\text{SLE}_{16/\kappa}$  loop,  $\Gamma \sim \text{CLE}_{\kappa}$  in *D*,

$$\mathfrak{d}_{\epsilon}(z,w) = \inf_{\omega} \mathfrak{N}_{\epsilon}(\omega), \quad \mathfrak{N}_{\epsilon}(\omega) = \mathsf{Leb}(\epsilon \text{ neighborhood of } \omega)$$

where the infimum is over all paths  $\omega$  in  $\Upsilon$  from z to w and

$$\mathfrak{m}_{\epsilon} = \operatorname{median}(\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)).$$

**Tightness theorem:** tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$ ; subsequential limits are geodesic metrics

Proof has three main steps:

- I. Prove tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot,\cdot)$  restricted to the boundary
- II. Extend tightness of  $\mathfrak{m}_{\epsilon}^{-1}\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  to the interior
- III. Show that the subsequential limits are positive definite and geodesic

Tricky because the metric depends heavily on the boundary conditions.



Work in the infinite volume setup;



Work in the infinite volume setup; m<sub>e</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>e</sub>(0, z)).



- Work in the infinite volume setup; m<sub>e</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>e</sub>(0, z)).
- ▶ Want  $\mathbf{P}[\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)) \ge x\mathfrak{m}_{\epsilon}] \rightarrow 0$  superpolynomially as  $x \rightarrow \infty$



- Work in the infinite volume setup; m<sub>e</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>e</sub>(0, z)).
- ▶ Want  $\mathbf{P}[\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)) \ge x\mathfrak{m}_{\epsilon}] \to 0$  superpolynomially as  $x \to \infty$
- As the geodesic from  $\eta(0) = 0$  to  $\partial B(0, 1)$  passes through pairs of intersecting loops, it traverses  $\gtrsim \delta^{-d_{\text{double}}}$  bubbles of diameter  $\asymp \delta$  where  $d_{\text{double}}$  is the double point dimension of  $\text{SLE}_{\kappa}$



- Work in the infinite volume setup; m<sub>ϵ</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>ϵ</sub>(0, z)).
- ▶ Want  $\mathbf{P}[\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)) \ge x\mathfrak{m}_{\epsilon}] \to 0$  superpolynomially as  $x \to \infty$
- As the geodesic from  $\eta(0) = 0$  to  $\partial B(0, 1)$  passes through pairs of intersecting loops, it traverses  $\gtrsim \delta^{-d_{\text{double}}}$  bubbles of diameter  $\asymp \delta$  where  $d_{\text{double}}$  is the double point dimension of  $\text{SLE}_{\kappa}$
- A priori bound: bubbles are approximately independent, so the probability that the  $\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  distance across any of them is at least  $\mathfrak{m}_{\epsilon}$  is  $O(\delta^{d_{\text{double}}})$ .



- Work in the infinite volume setup; m<sub>ϵ</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>ϵ</sub>(0, z)).
- ▶ Want  $\mathbf{P}[\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)) \ge x\mathfrak{m}_{\epsilon}] \rightarrow 0$  superpolynomially as  $x \rightarrow \infty$
- As the geodesic from  $\eta(0) = 0$  to  $\partial B(0, 1)$  passes through pairs of intersecting loops, it traverses  $\gtrsim \delta^{-d_{\text{double}}}$  bubbles of diameter  $\asymp \delta$  where  $d_{\text{double}}$  is the double point dimension of  $\text{SLE}_{\kappa}$
- A priori bound: bubbles are approximately independent, so the probability that the  $\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  distance across any of them is at least  $\mathfrak{m}_{\epsilon}$  is  $O(\delta^{d_{\text{double}}})$ .
- Bootstrap the bubble estimate to get that the probability that the ∂<sub>ε</sub>(·, ·) distance across any of them is at least m<sub>ε</sub> is O(δ<sup>p</sup>) for all p > 0.



- Work in the infinite volume setup; m<sub>ϵ</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>ϵ</sub>(0, z)).
- ▶ Want  $\mathbf{P}[\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)) \ge x\mathfrak{m}_{\epsilon}] \rightarrow 0$  superpolynomially as  $x \rightarrow \infty$
- As the geodesic from  $\eta(0) = 0$  to  $\partial B(0, 1)$  passes through pairs of intersecting loops, it traverses  $\gtrsim \delta^{-d_{\text{double}}}$  bubbles of diameter  $\asymp \delta$  where  $d_{\text{double}}$  is the double point dimension of  $\text{SLE}_{\kappa}$
- A priori bound: bubbles are approximately independent, so the probability that the  $\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  distance across any of them is at least  $\mathfrak{m}_{\epsilon}$  is  $O(\delta^{d_{\text{double}}})$ .
- Bootstrap the bubble estimate to get that the probability that the ∂<sub>ε</sub>(·, ·) distance across any of them is at least m<sub>ε</sub> is O(δ<sup>p</sup>) for all p > 0.



- Work in the infinite volume setup; m<sub>ϵ</sub> = median(inf<sub>z∈∂B(0,1)</sub> 0<sub>ϵ</sub>(0, z)).
- ▶ Want  $\mathbf{P}[\inf_{z \in \partial B(0,1)} \mathfrak{d}_{\epsilon}(0,z)) \ge x\mathfrak{m}_{\epsilon}] \rightarrow 0$  superpolynomially as  $x \rightarrow \infty$
- As the geodesic from  $\eta(0) = 0$  to  $\partial B(0, 1)$  passes through pairs of intersecting loops, it traverses  $\gtrsim \delta^{-d_{\text{double}}}$  bubbles of diameter  $\asymp \delta$  where  $d_{\text{double}}$  is the double point dimension of  $\text{SLE}_{\kappa}$
- A priori bound: bubbles are approximately independent, so the probability that the  $\mathfrak{d}_{\epsilon}(\cdot, \cdot)$  distance across any of them is at least  $\mathfrak{m}_{\epsilon}$  is  $O(\delta^{d_{\text{double}}})$ .
- Bootstrap the bubble estimate to get that the probability that the ∂<sub>ε</sub>(·, ·) distance across any of them is at least m<sub>ε</sub> is O(δ<sup>p</sup>) for all p > 0.