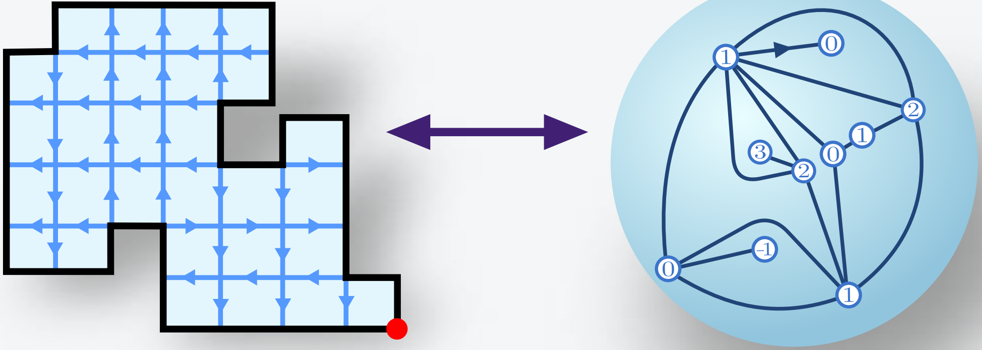
A bijection between rigid and integer-labeled quadrangulations

Timothy Budd



Radboud University

T.Budd@science.ru.nl
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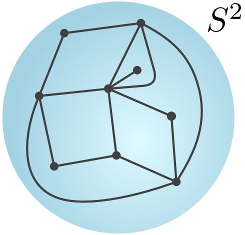
Discrete geometry by planar maps

- Planar map is a connected graph embedded in S^2 viewed up to continuous deformation.



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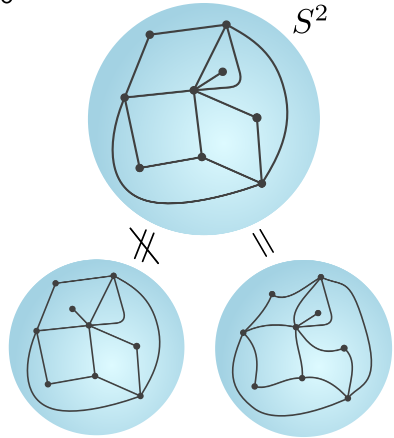

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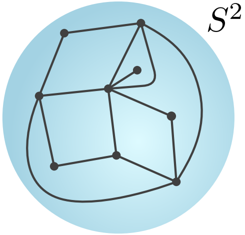

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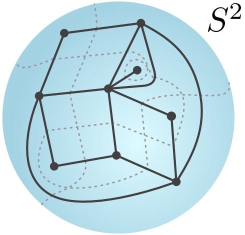

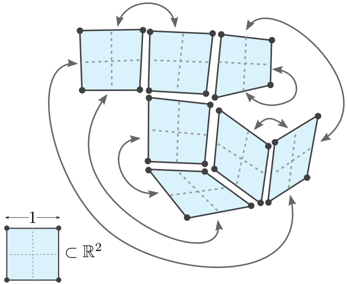

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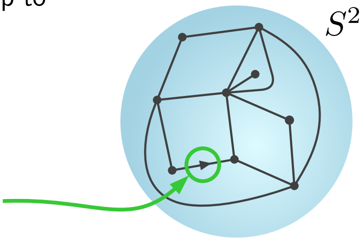

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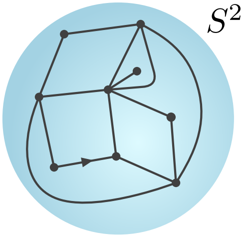

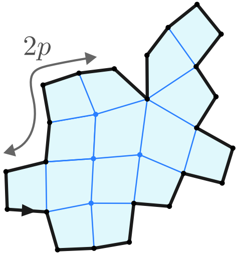

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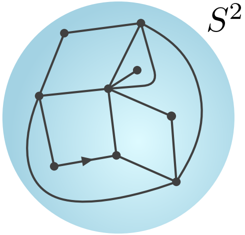

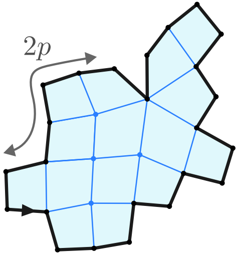

Many map enumeration methods:

- Recursive methods and generating functions [Tutte, '60s] [Brown, Bender, Canfield, Goulden, Jackson, Ambjørn, Bousquet-Mélou, . . .]
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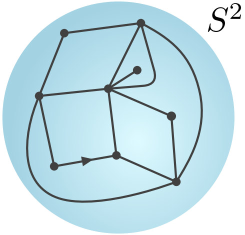

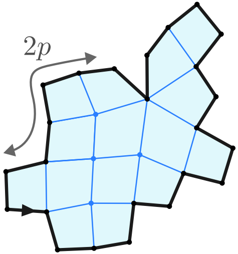

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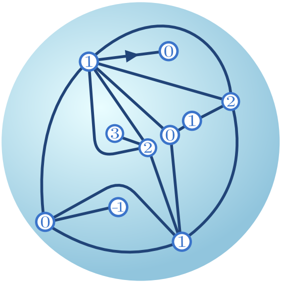

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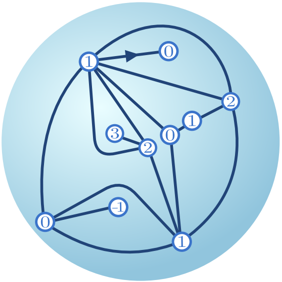

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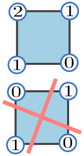

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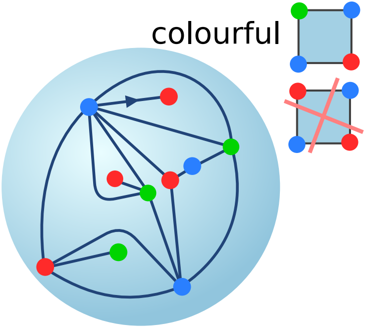

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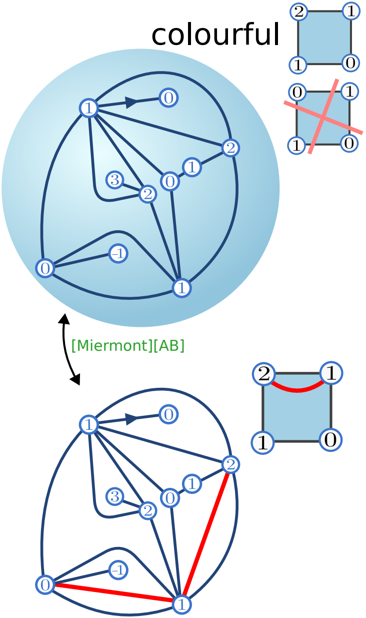

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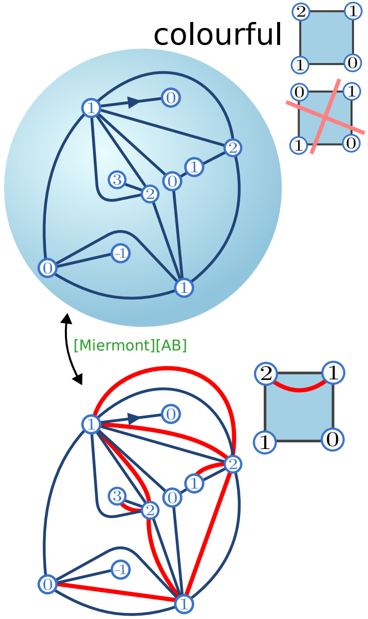

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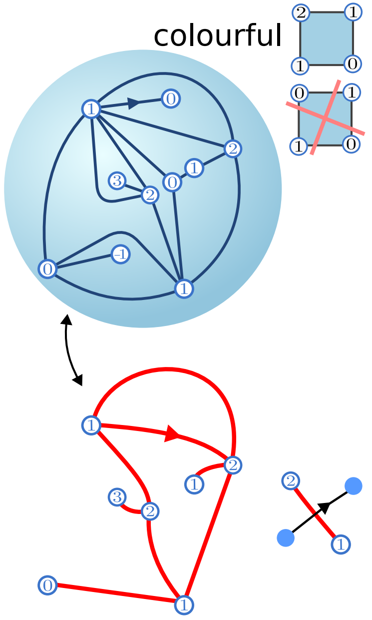

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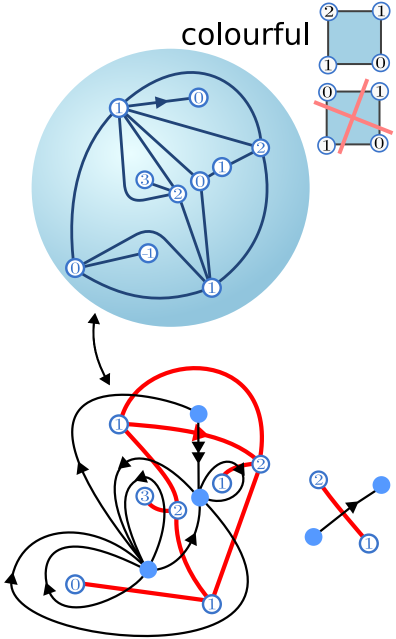

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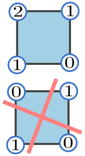

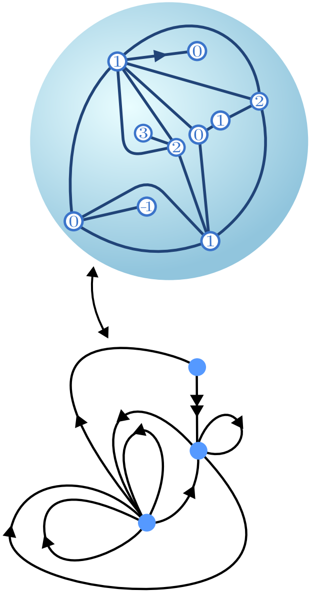

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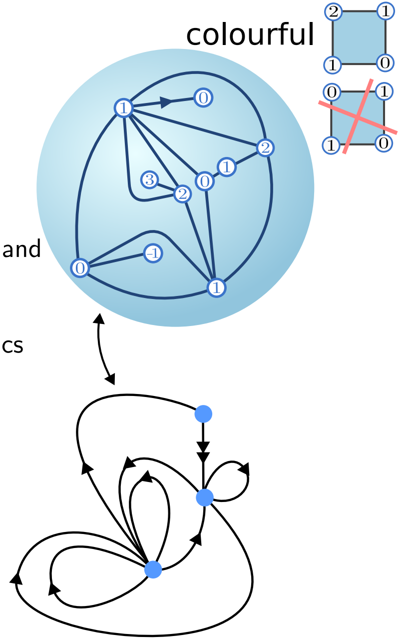

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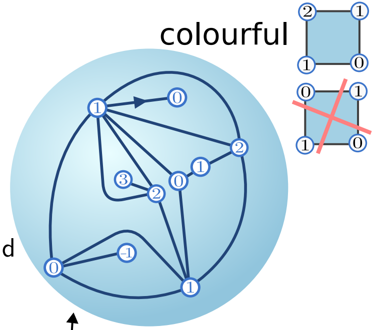

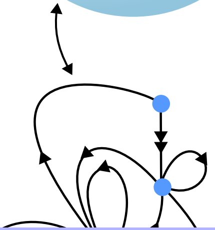

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Generating function is $G(t) = \frac{1}{4t^2}(t - 2t^2 - R(t))$ *where* $\sum_{k\geq 0} \frac{1}{k+1}\binom{2k}{k}^2 R(t)^{k+1} = t$.

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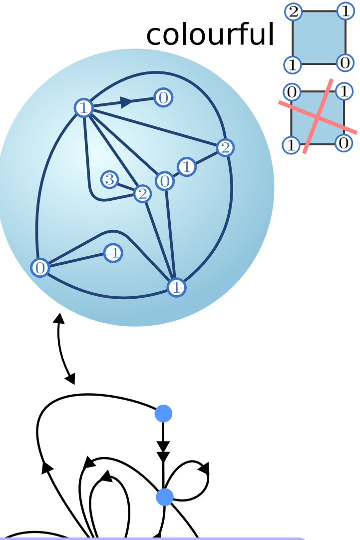

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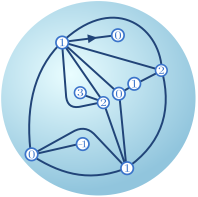

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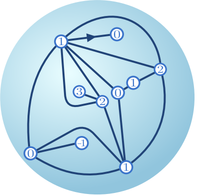

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Bipolar-oriented maps

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Brownian sphere
Uniform planar maps

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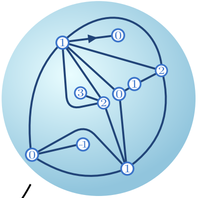

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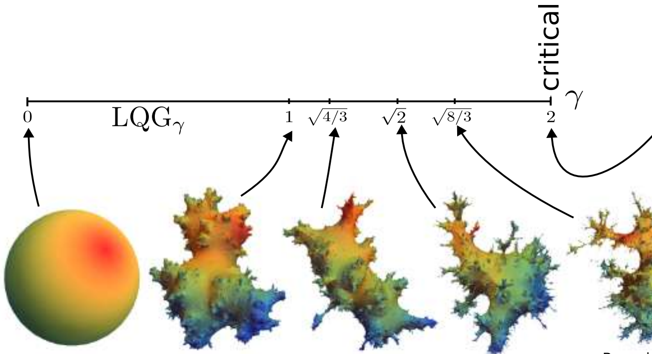

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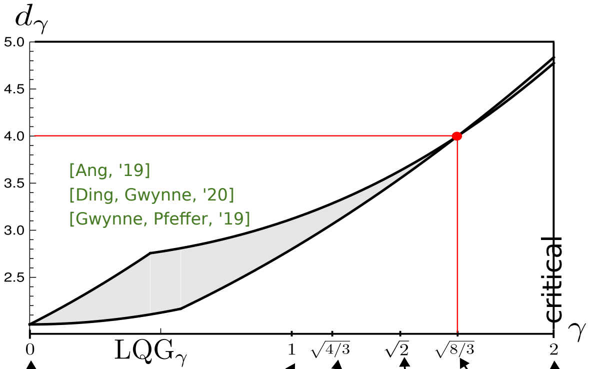

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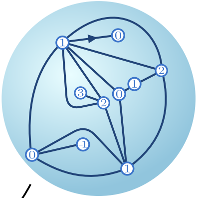

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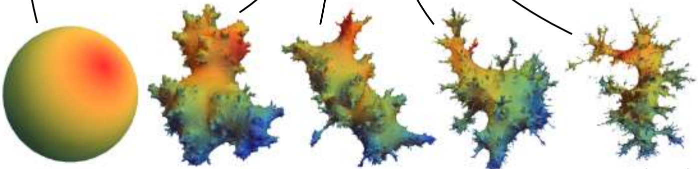

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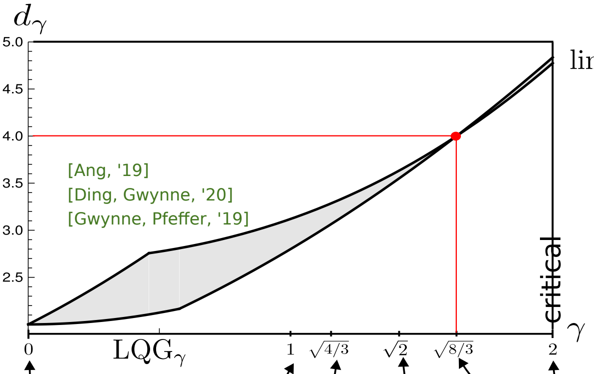

$\lim_{\gamma\to 2} d_\gamma \in (4.77, 4.84)$

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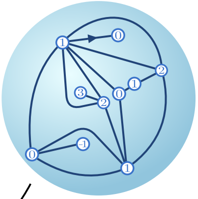

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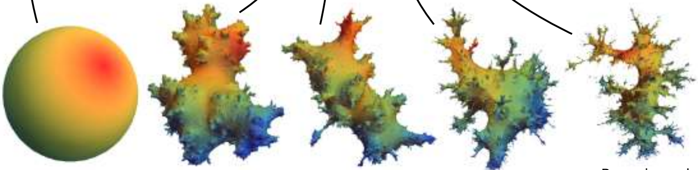

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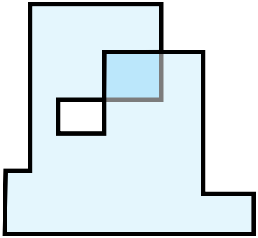

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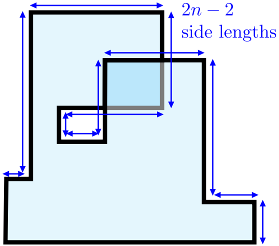

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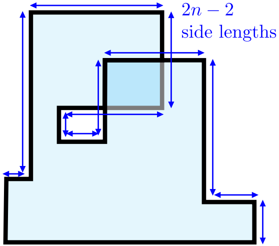

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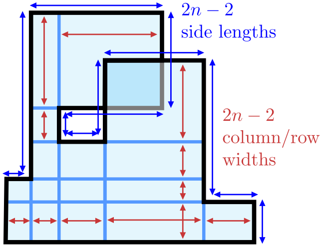

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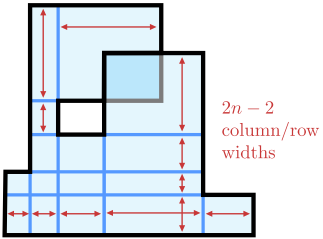

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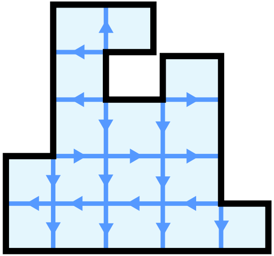
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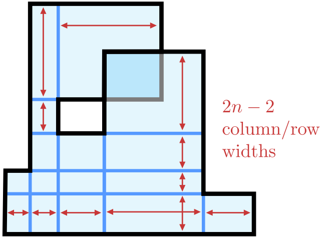
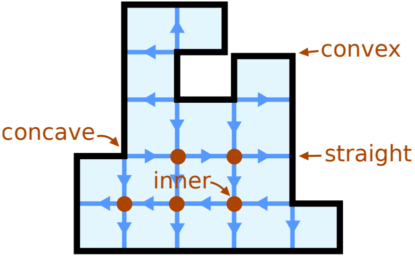
  - Quadrangulation of disk with inner vertices of degree 4, and boundary vertices of angle $\frac{\pi}{2}$ (convex), π (straight), $\frac{3\pi}{2}$ (concave).



rectilinear disk



rigid quadrangulation

Rigid quadrangulations

- *Rectilinear disk*: flat metric on disk with orthogonal piecewise-linear boundary.
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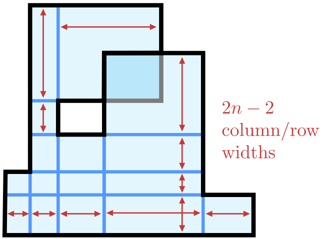
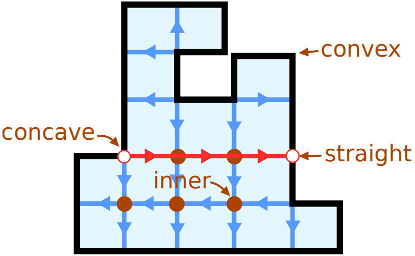
  - Rays start at concave and end at straight vertex.



rectilinear disk



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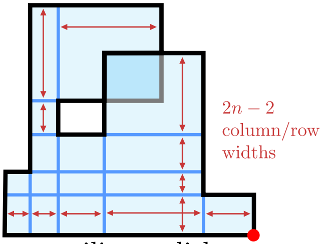
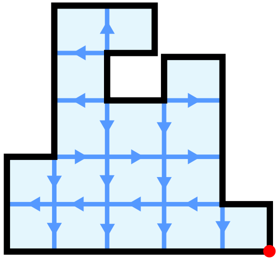
  - Rooted on convex vertex.

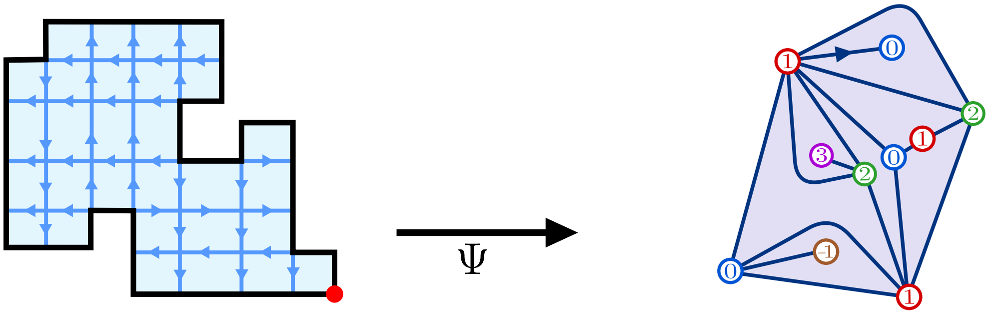
Theorem (TB, '25+)

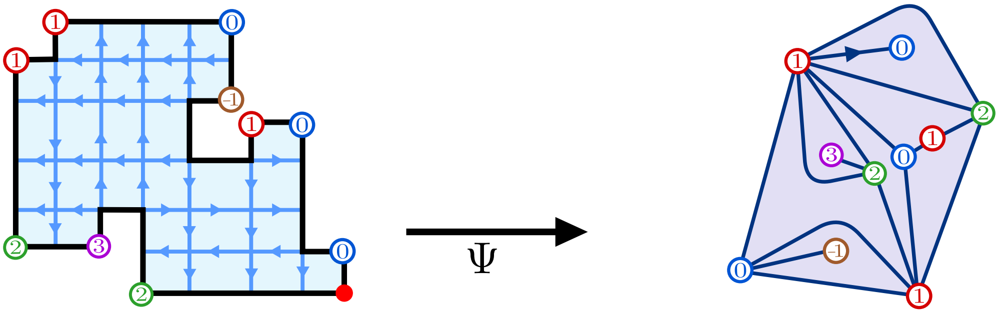
There is a bijection Ψ between rigid quadrangulations with $n+1$ convex vertices and colourful quadrangulations with n vertices (and root face labeled $0, 1, 2, 1$).



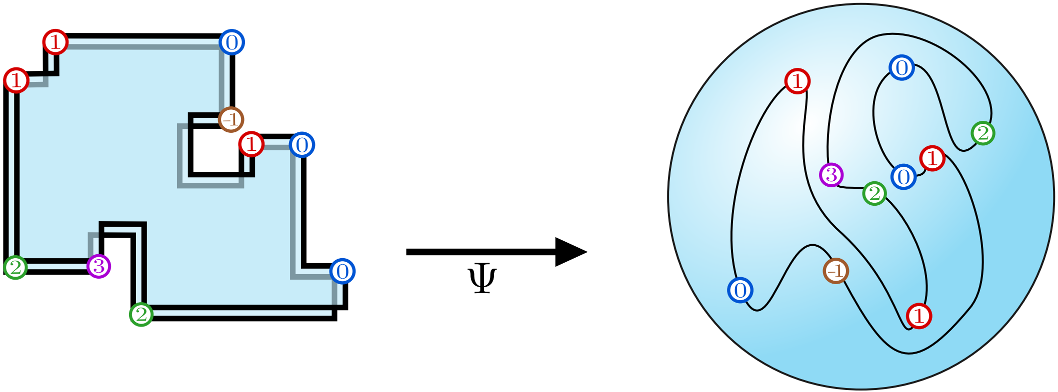
rectilinear disk


rigid quadrangulation

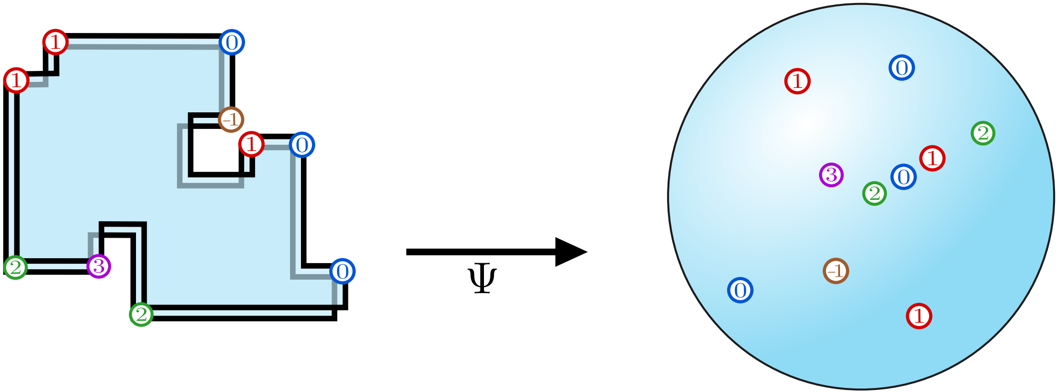

Ψ
1
0
2
1
3
0
2
0
-1
1



- Label convex vertices (except root) by turning number from root (#left − #right).



- Label convex vertices (except root) by turning number from root (#left − #right).
- Double to make it topological S^2.



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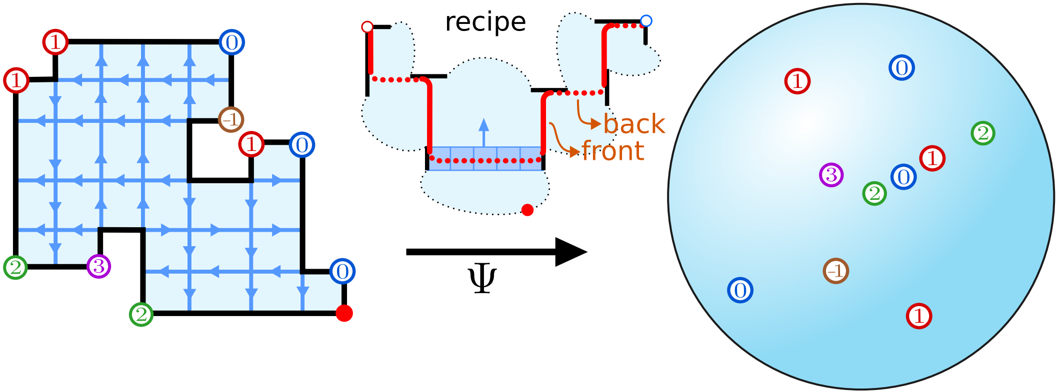
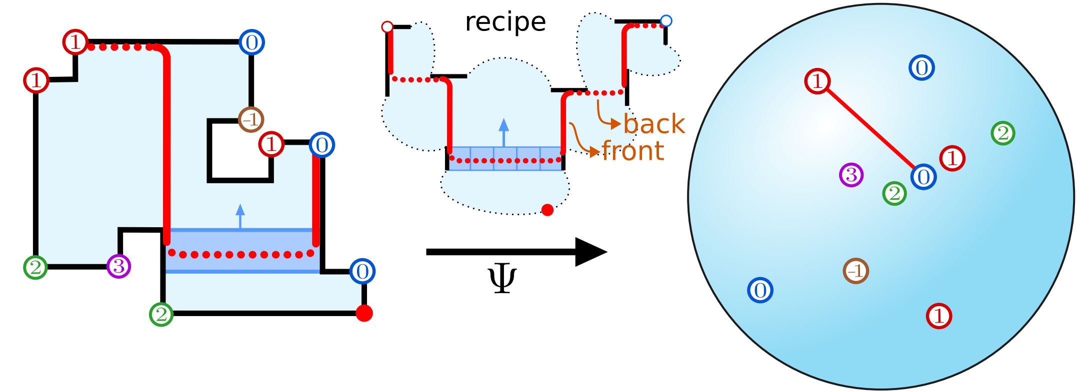


- Label convex vertices (except root) by turning number from root (#left − #right).
- Double to make it topological S^2.
- Draw edge for each row and column: traveling vertically on front, horizontally on back.



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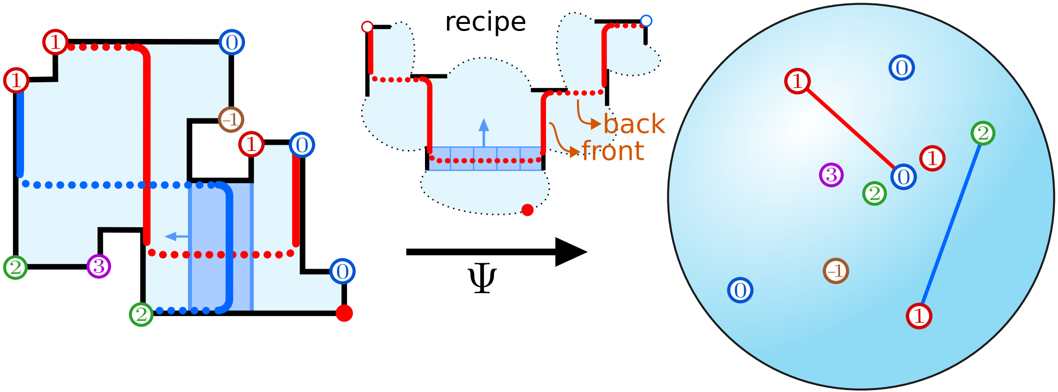

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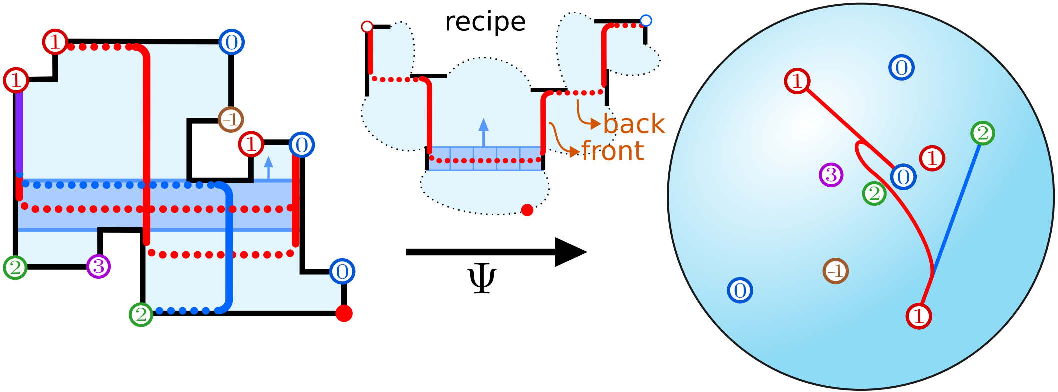

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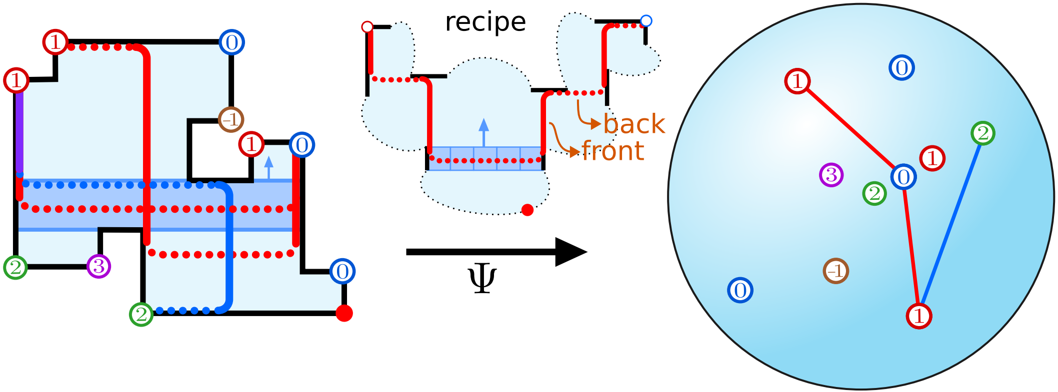

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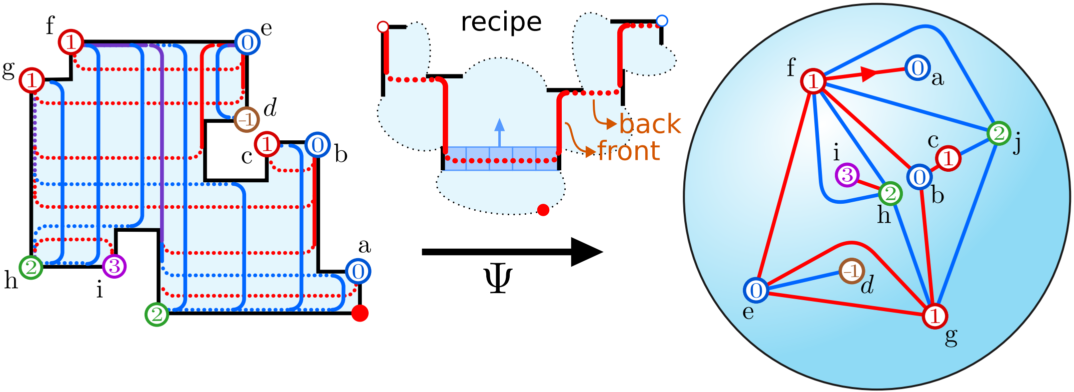

- Label convex vertices (except root) by turning number from root (#left − #right).
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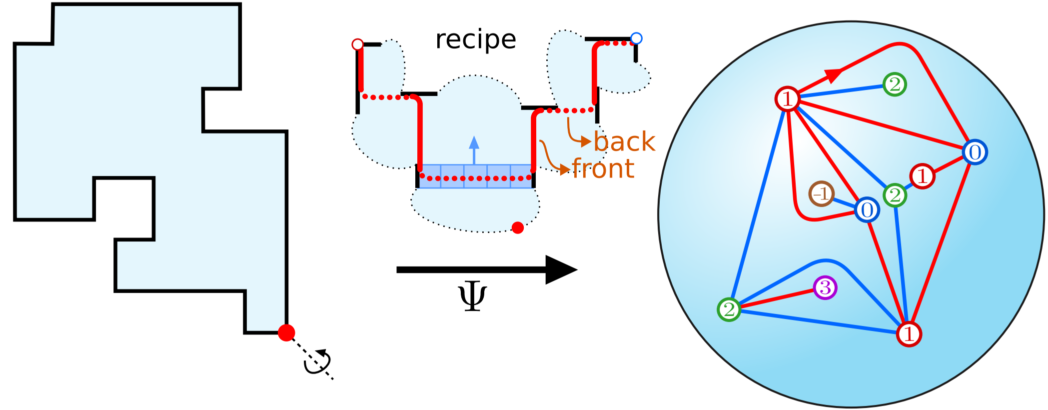
- Result is $\mathbb{Z}$-labeled planar map. Quadrangulation? Colourful? Ψ bijective?



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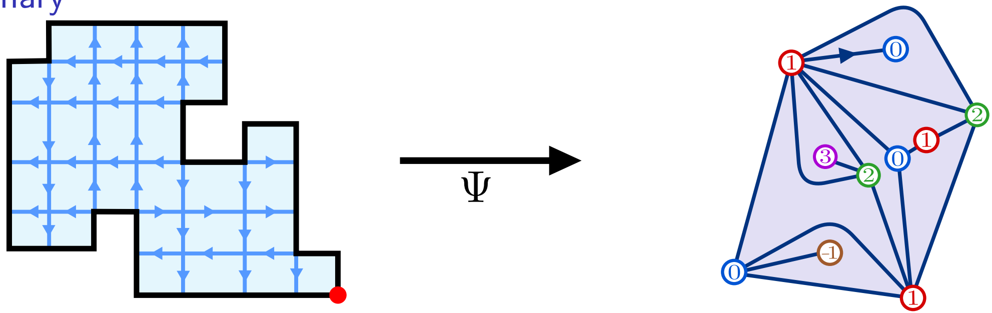
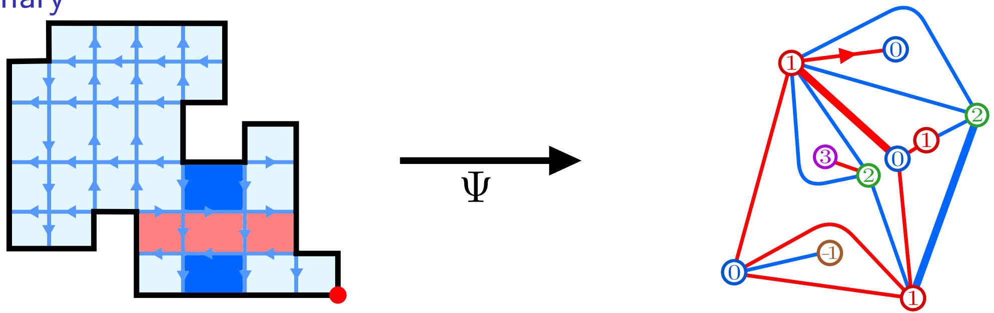
- Result is $\mathbb{Z}$-labeled planar map. Quadrangulation? Colourful? Ψ bijective?
- Useful symmetry: $\Psi \circ$ (Reflection in diagonal) $=$ (Label $\mapsto 2 -$ Label) $\circ \Psi$.

Dictionary



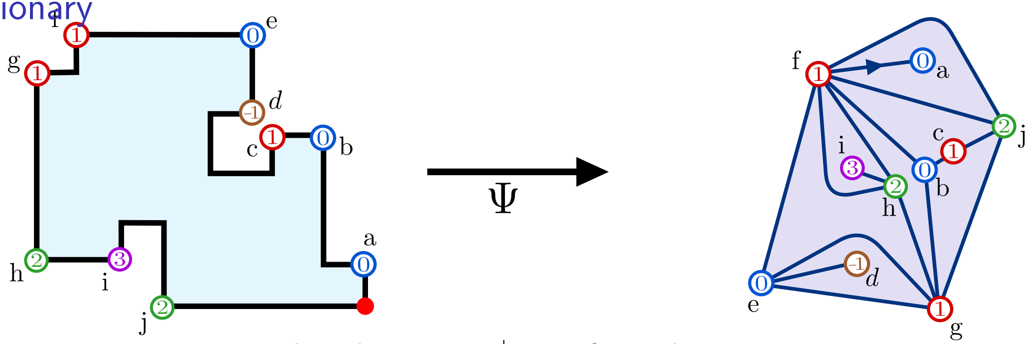
rigid quadrangulation	colourful quadrangulation

Dictionary



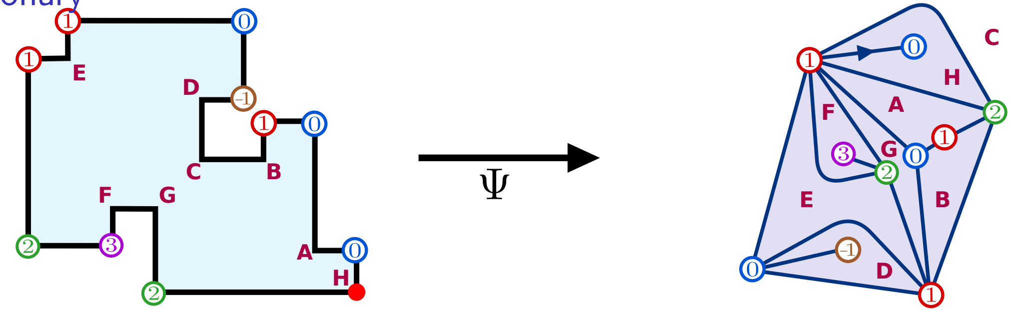
rigid quadrangulation	colourful quadrangulation
row	even edge (minimal label = even)
column	odd edge

Dictionary



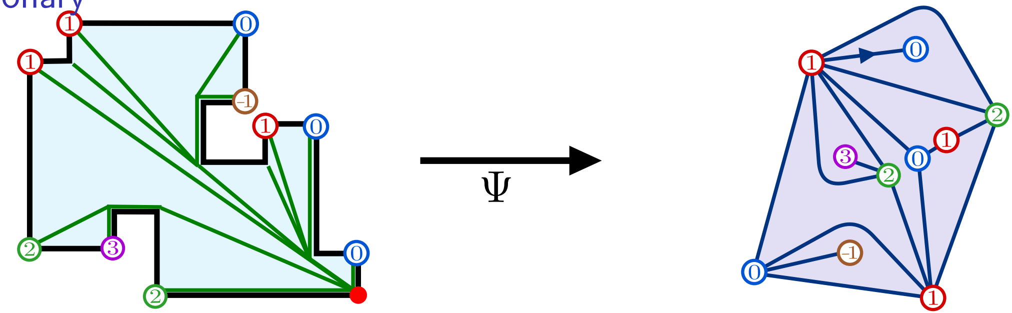
rigid quadrangulation	colourful quadrangulation
row	even edge (minimal label = even)
column	odd edge
convex vertex (≠root) of label ℓ	vertex of label ℓ

Dictionary



rigid quadrangulation	colourful quadrangulation
row	even edge (minimal label $=$ even)
column	odd edge
convex vertex ($\neq$root) of label ℓ	vertex of label ℓ
concave vertex or root	face

Dictionary



rigid quadrangulation	colourful quadrangulation
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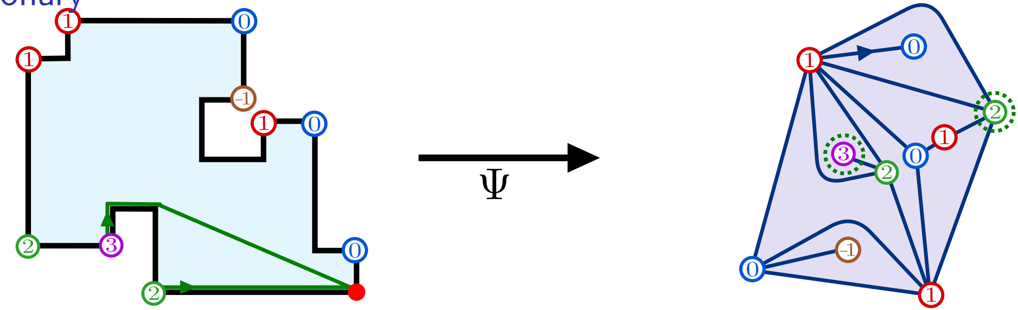
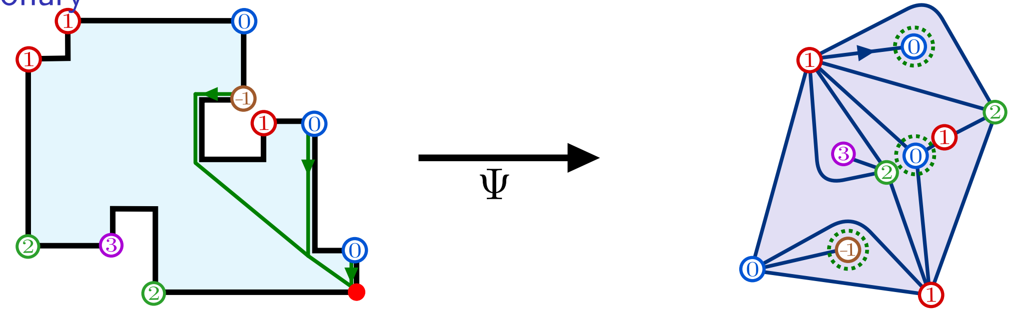
Dictionary



rigid quadrangulation	colourful quadrangulation
row	even edge (minimal label = even)
column	odd edge
convex vertex (≠root) of label ℓ	vertex of label ℓ
concave vertex or root	face
	local maximum

Dictionary



rigid quadrangulation	colourful quadrangulation
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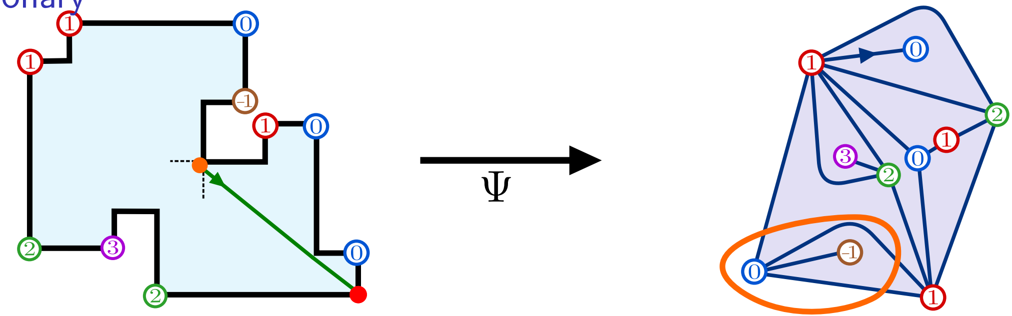
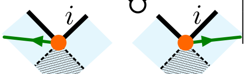
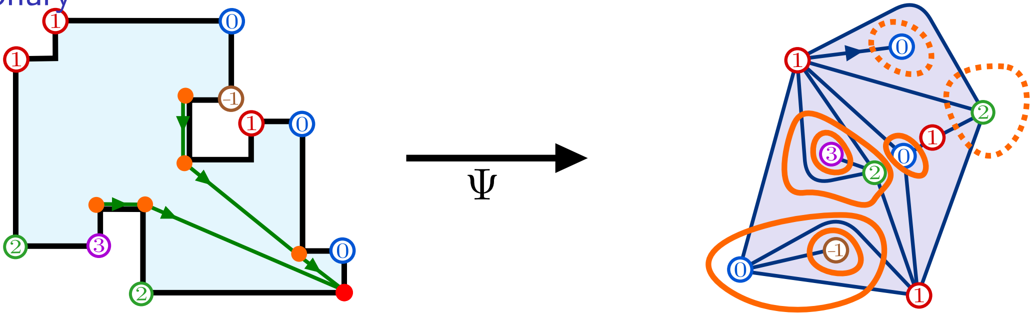
convex vertex (≠root) of label ℓ	vertex of label ℓ
concave vertex or root	face
	local maximum
	local minimum

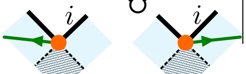
Dictionary



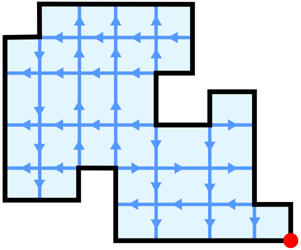
rigid quadrangulation	colourful quadrangulation
row	even edge (minimal label = even)
column	odd edge
convex vertex ($\neq$root) of label ℓ	vertex of label ℓ
concave vertex or root	face
	local maximum
	local minimum
	$(i, i-1)$-level line

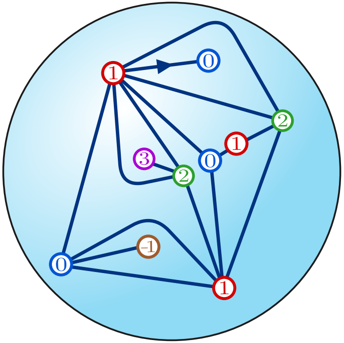
Dictionary



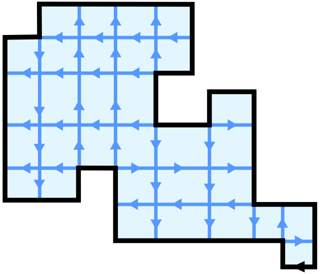
rigid quadrangulation	colourful quadrangulation
row	even edge (minimal label = even)
column	odd edge
convex vertex (≠root) of label ℓ	vertex of label ℓ
concave vertex or root	face
	local maximum
	local minimum
	$(i, i-1)$-level line (not through root face)

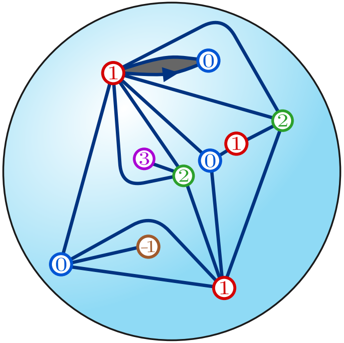
Idea of proof: an extended bijection



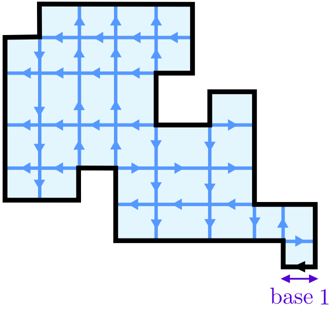


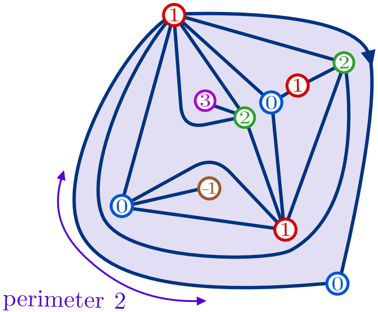
Idea of proof: an extended bijection





Idea of proof: an extended bijection



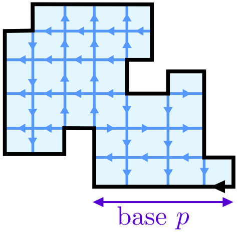


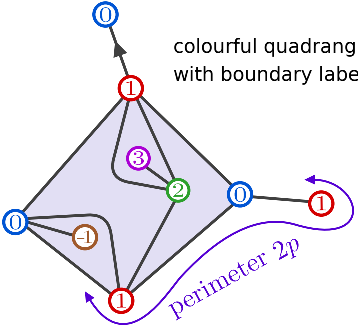
- For $n \geq 2$ and $p \geq 1$ there exists a bijection

$$\left\{ \begin{array}{c} \text{rigid quadrangulations with} \\ n+2 \text{ convex vertices and base } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{colourful quadrangulations of the disk} \\ \text{with } n \text{ vertices and perimeter } 2p \end{array} \right\}$$

Idea of proof: an extended bijection

rigid quadrangulation with base





colourful quadrangulation of disk
with boundary labeled 0101...

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$$\left\{ \begin{array}{c} \text{rigid quadrangulations with} \\ n+2 \text{ convex vertices and base } p \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{colourful quadrangulations of the disk} \\ \text{with } n \text{ vertices and perimeter } 2p \end{array} \right\}$$

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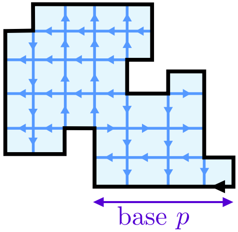

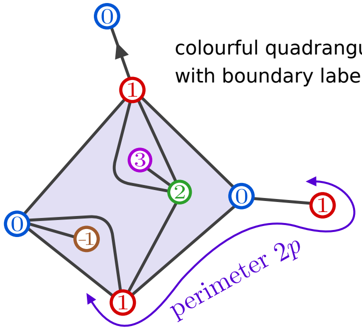

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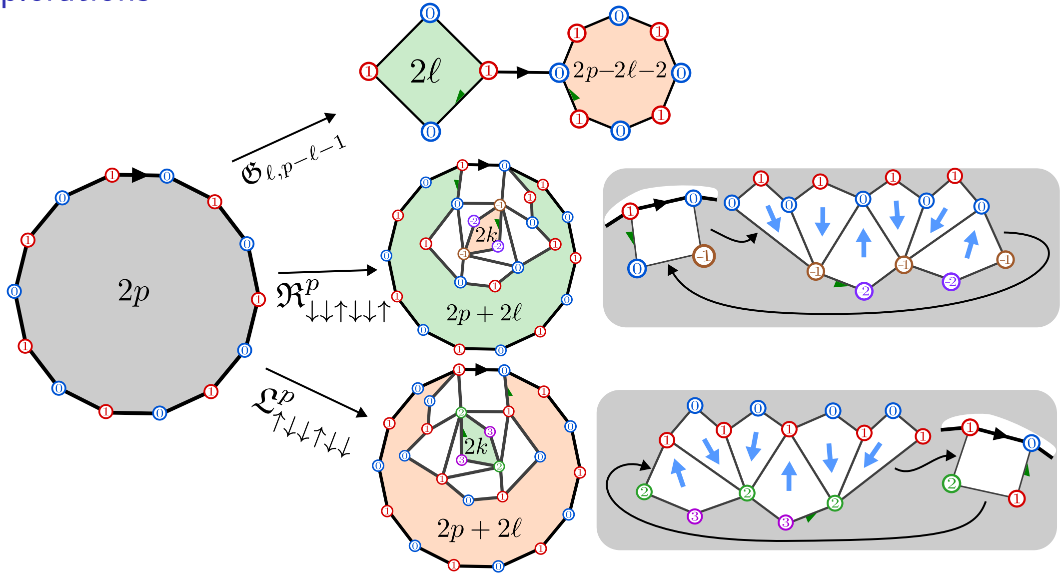
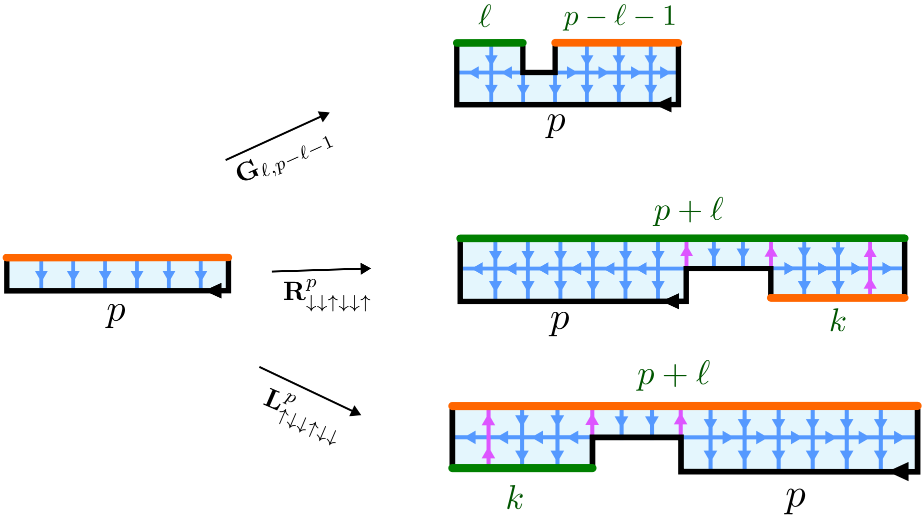
- Proof by relating canonical explorations: row-by-row exploration vs peeling exploration.

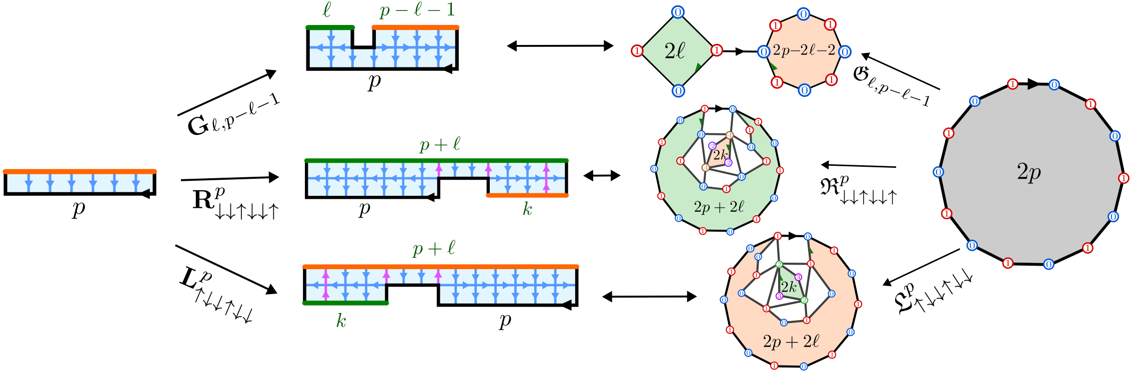
Explorations



Explorations

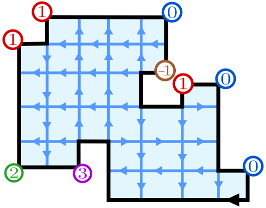


Explorations

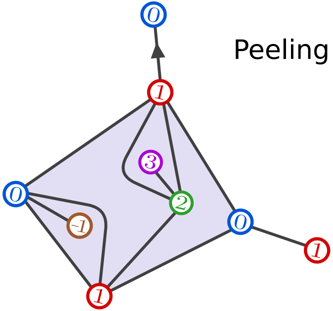


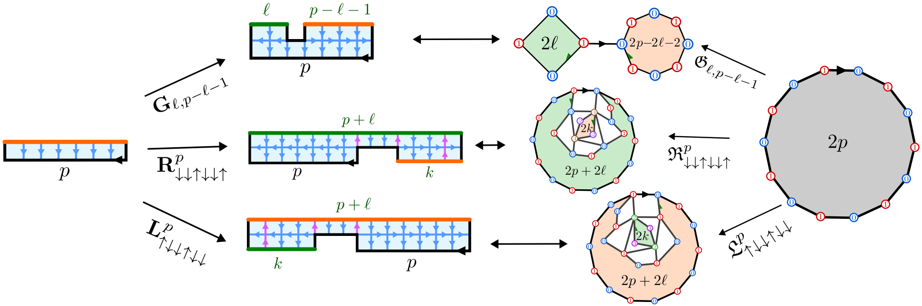
Explorations

Row-by-row exploration



Peeling exploration

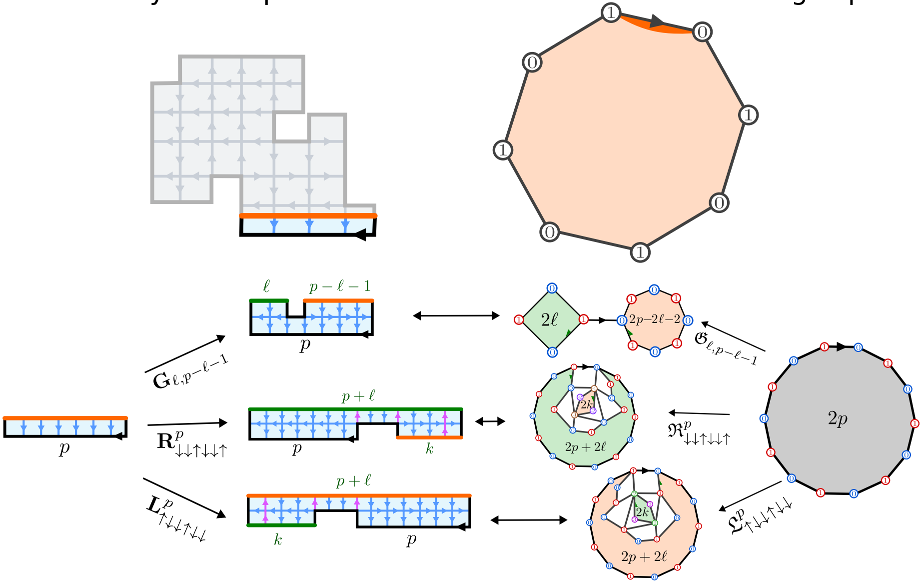




Explorations

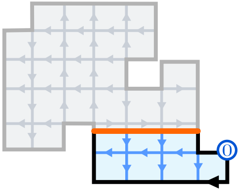
Row-by-row exploration

Peeling exploration

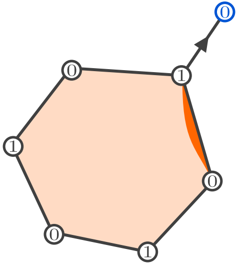


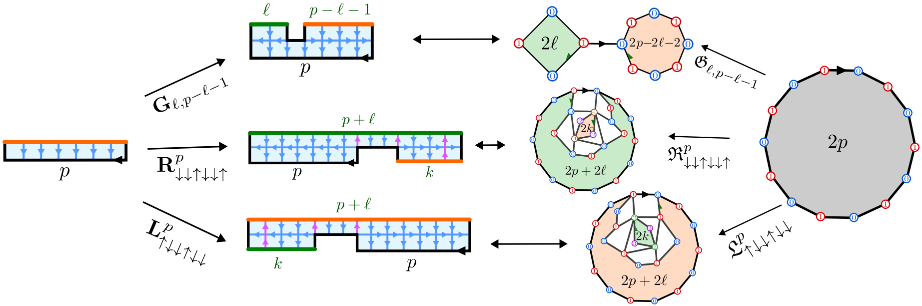
Explorations

Row-by-row exploration



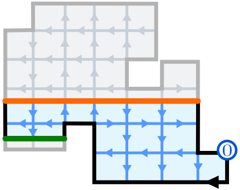
Peeling exploration



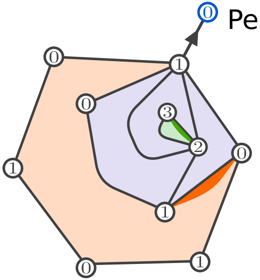


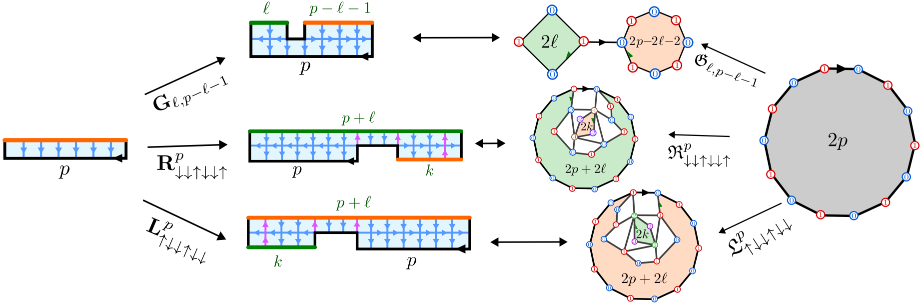
Explorations

Row-by-row exploration



Peeling exploration

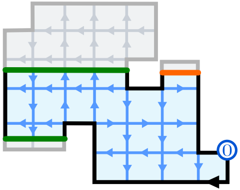


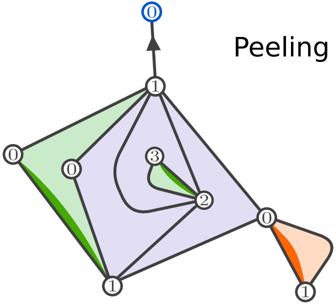


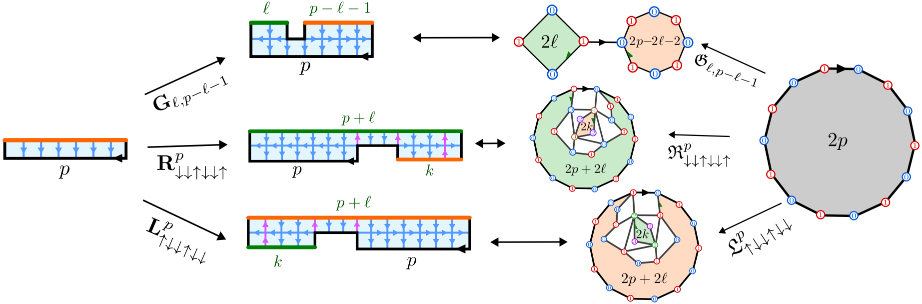
Explorations

Row-by-row exploration

Peeling exploration



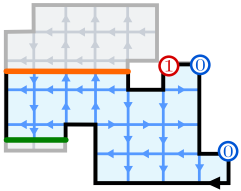


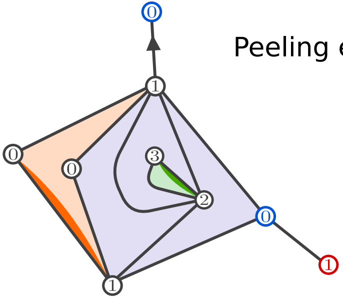


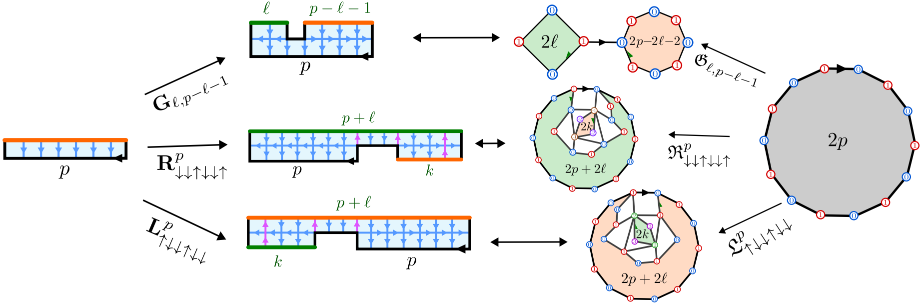
Explorations

Row-by-row exploration

Peeling exploration

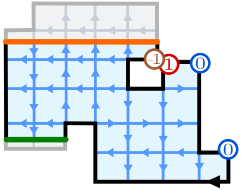




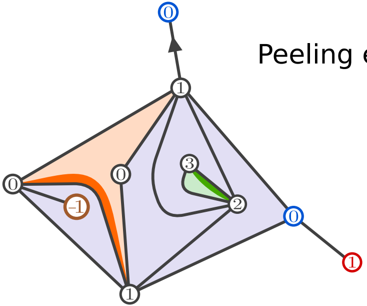


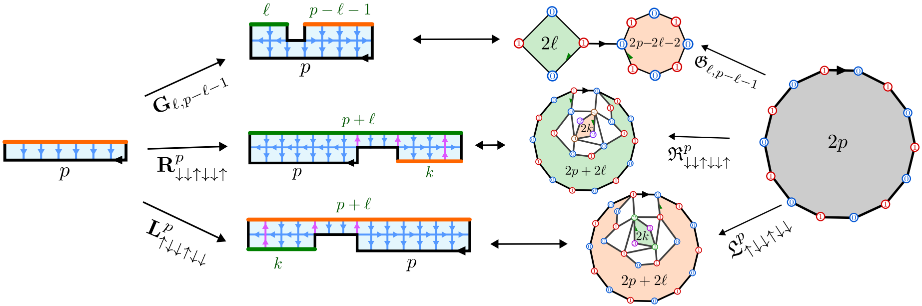
Explorations

Row-by-row exploration



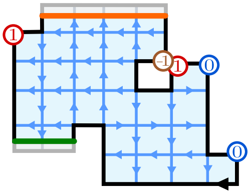
Peeling exploration





Explorations

Row-by-row exploration



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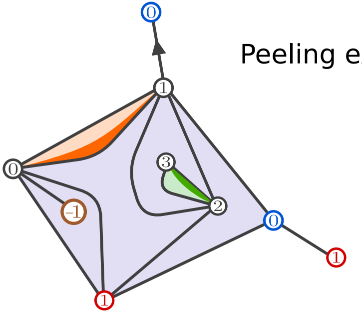

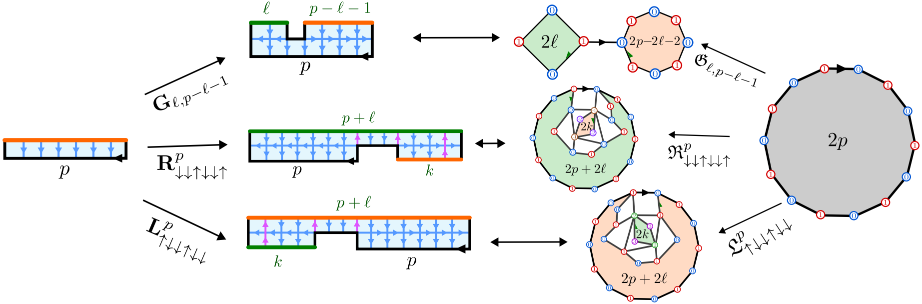

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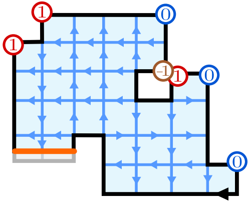

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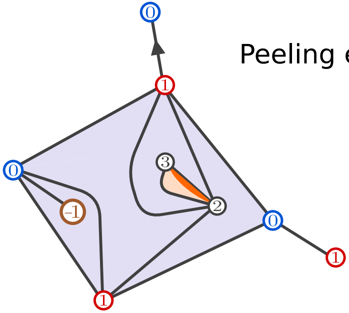

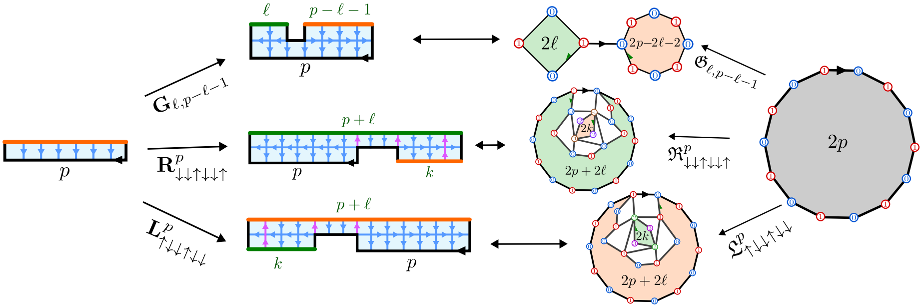

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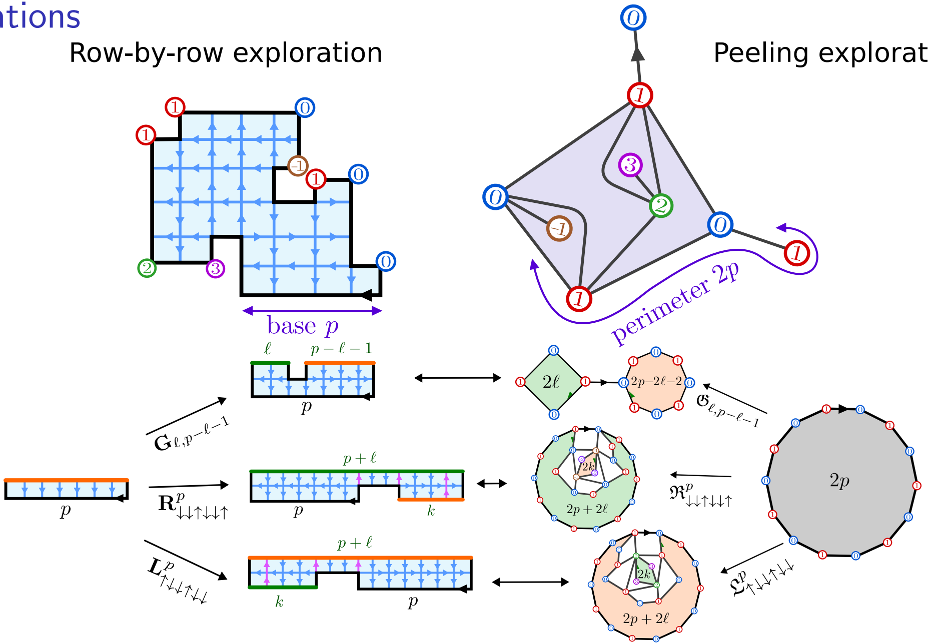

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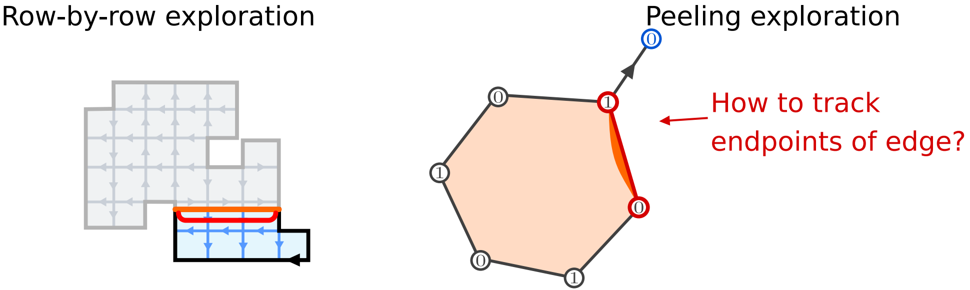

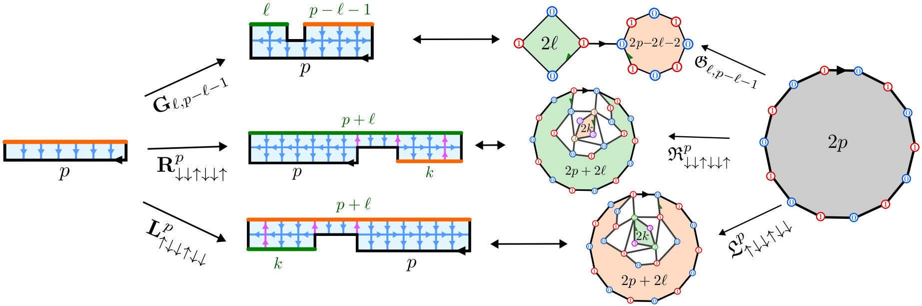

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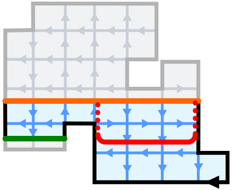

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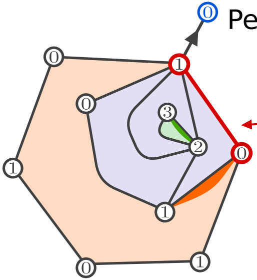

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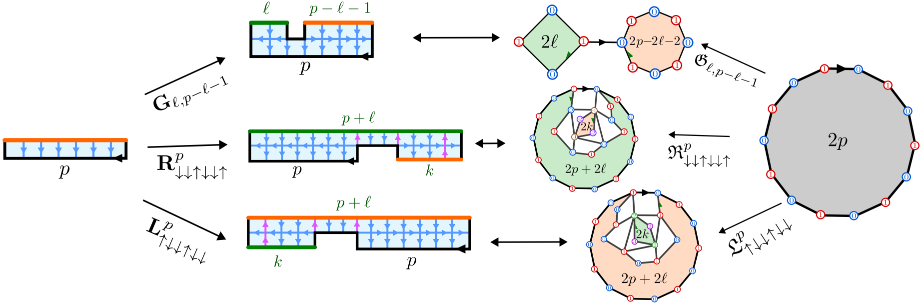

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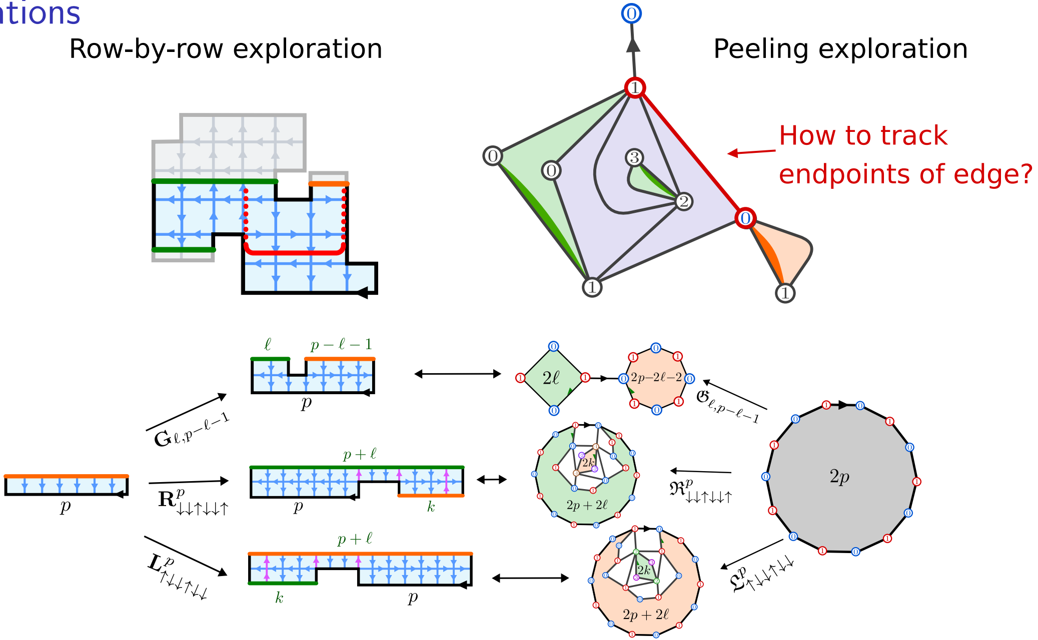

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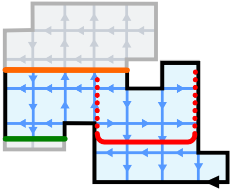

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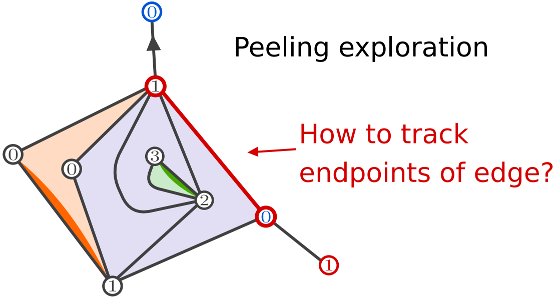

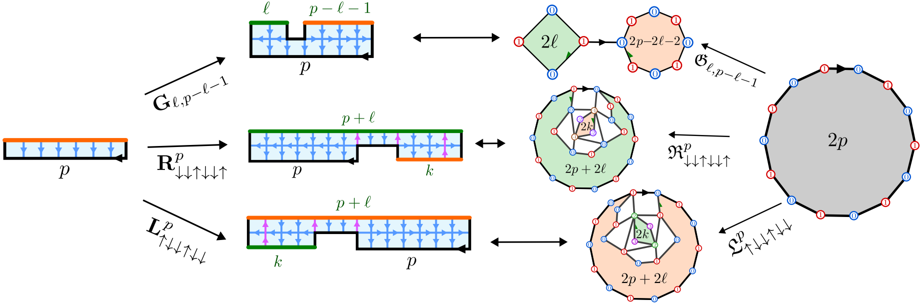

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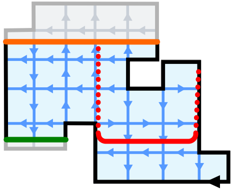

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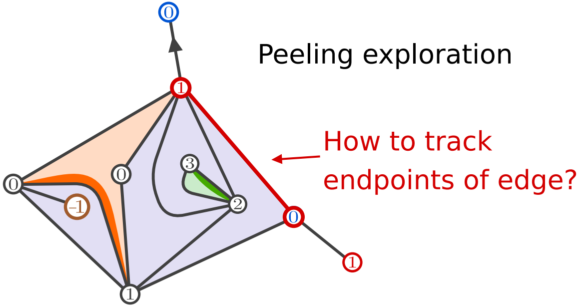

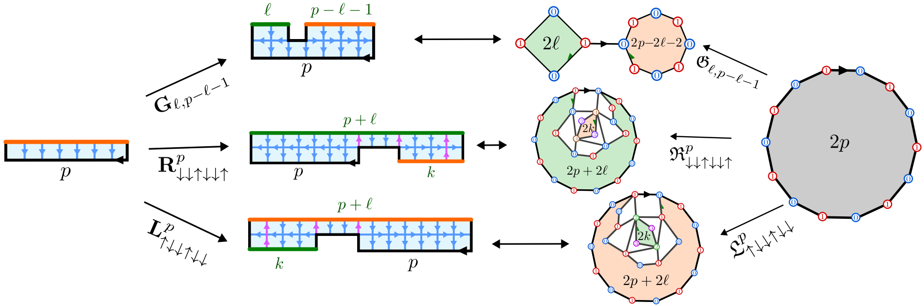

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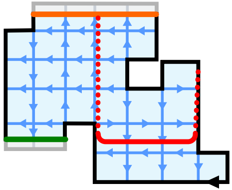

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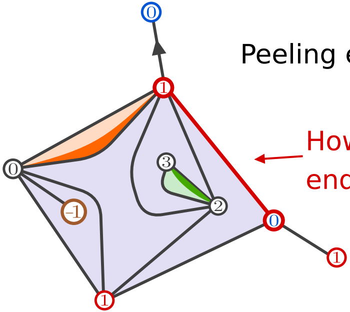

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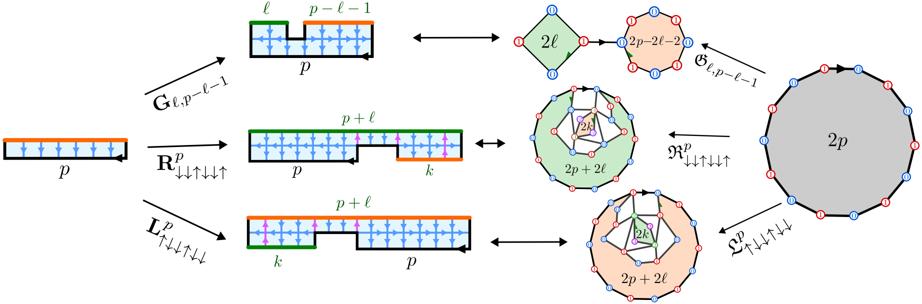

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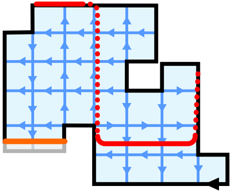

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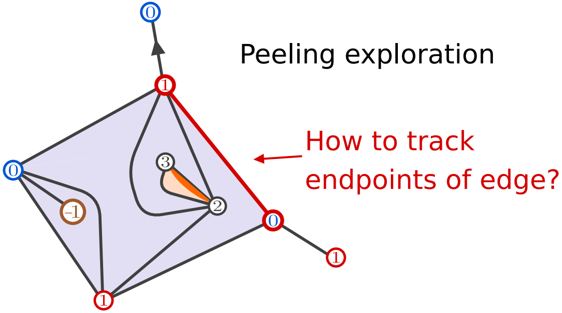

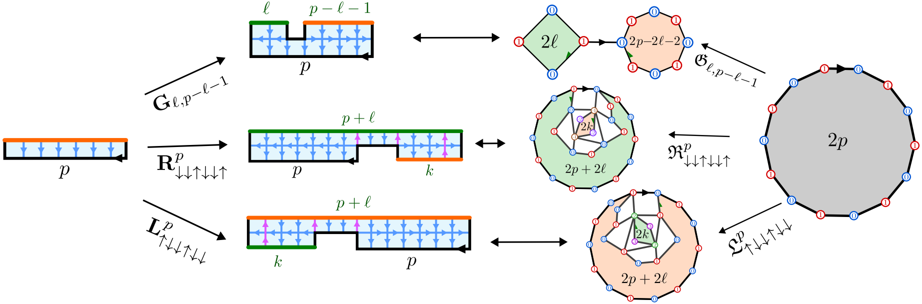

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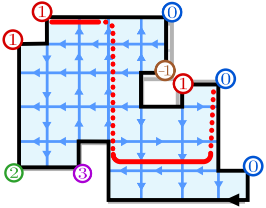

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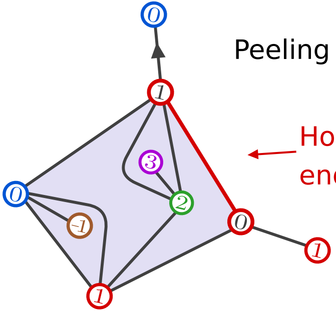

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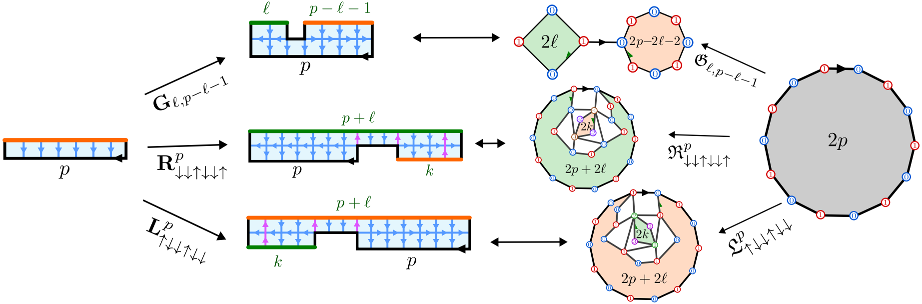

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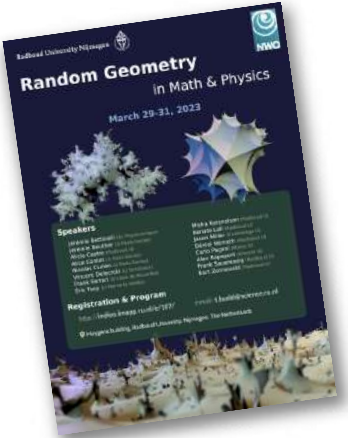

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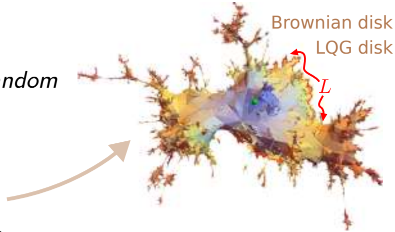

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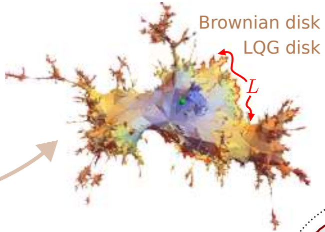

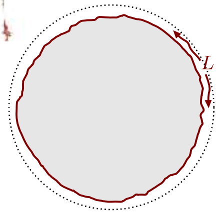

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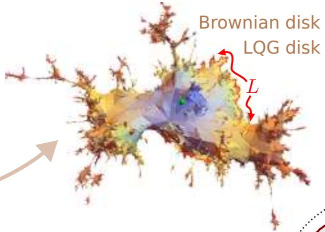

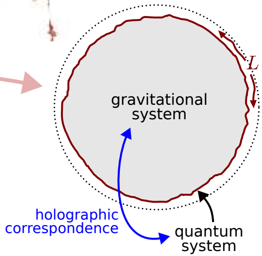

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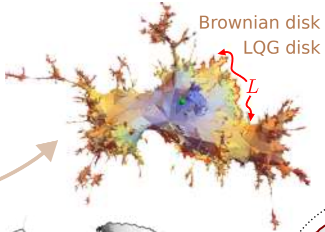

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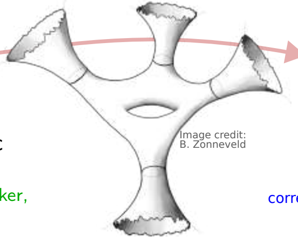

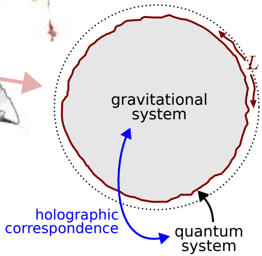

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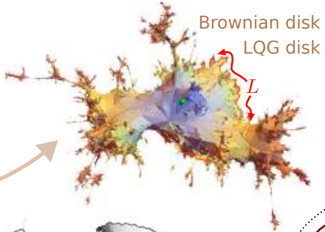

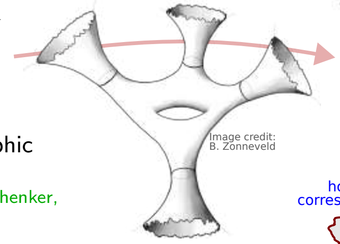

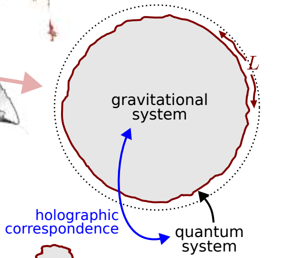

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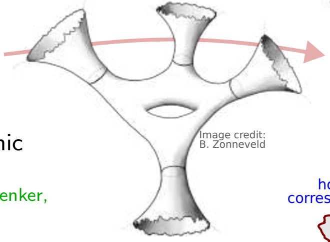

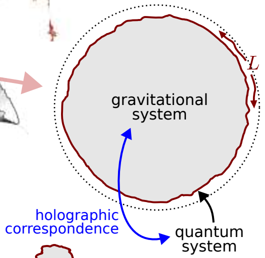

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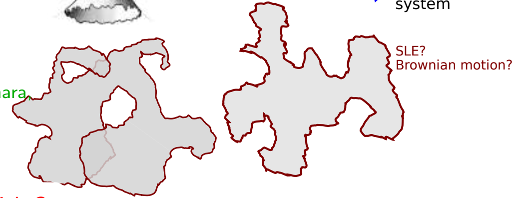
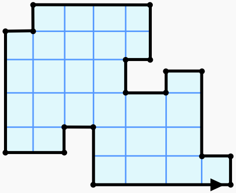
- Ferrari: should allow disks to self-overlap. [Ferrari, '24]



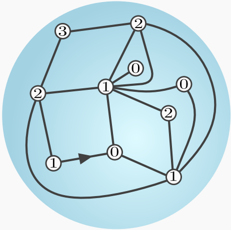
Is there a tractable model of uniform random discrete flat disks?

Rigid quadrangulations



bijection

Colourful quadrangulations



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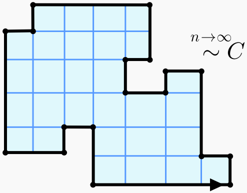

$$\overset{n\to\infty}{\sim} C(4\pi)^n \frac{n^{-2}}{\log^2 n}$$

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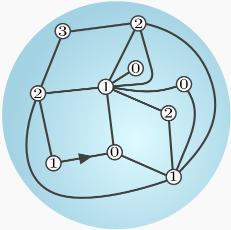

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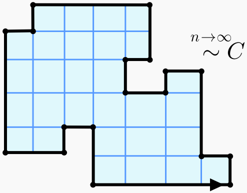

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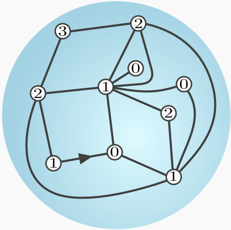

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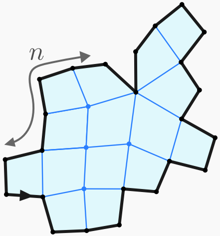

[TB, '25+]

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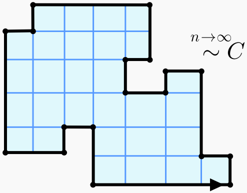

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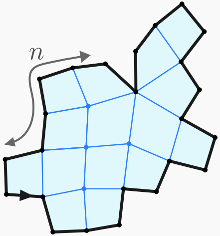

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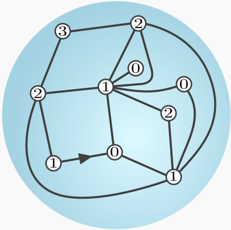

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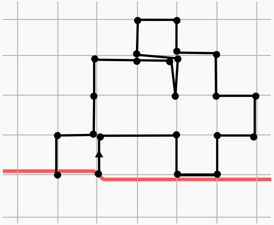
bijection

Walks in half square lattice with no point subexcursions



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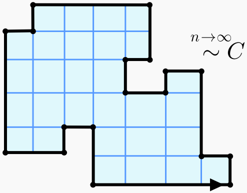

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↕ bijection

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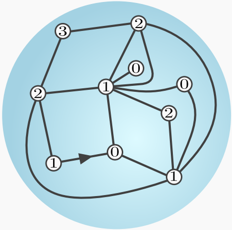

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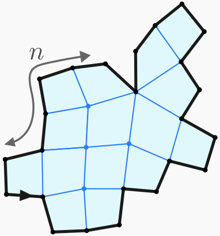

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↕ bijection

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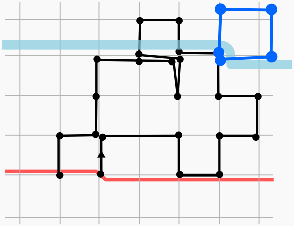
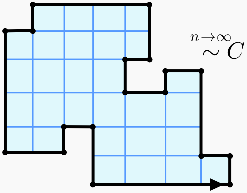


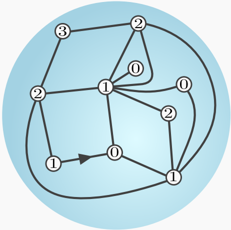
Rigid quadrangulations



$$\overset{n\to\infty}{\sim} C(4\pi)^n \frac{n^{-2}}{\log^2 n}$$

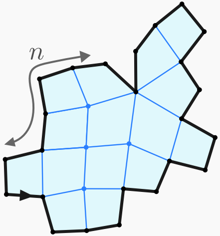
bijection

Colourful quadrangulations



$$\overset{n\to\infty}{\sim} C(4\pi)^n \frac{n^{-2}}{\log^2 n}$$

Flat quadrangulations

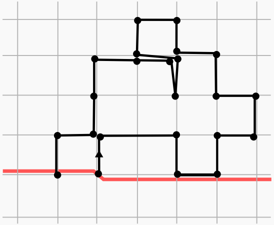


$$\overset{n\to\infty}{\sim} C e^{\frac{4G}{\pi} n} \frac{n^{-2}}{\log^2 n}$$

[TB, '25+]

bijection

Walks in half square lattice with no point subexcursions



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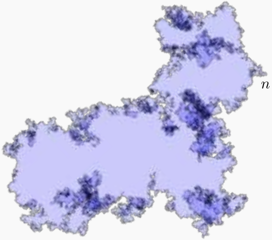
Rigid quadrangulations

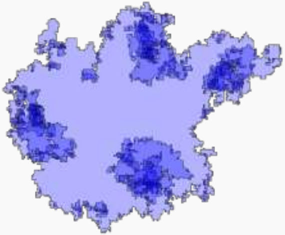


$$\overset{n\to\infty}{\sim} C(4\pi)^n \frac{n^{-2}}{\log^2 n}$$

[simulation: B. Zonneveld]

same universality class?

Flat quadrangulations



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[TB, '25+]

bijection

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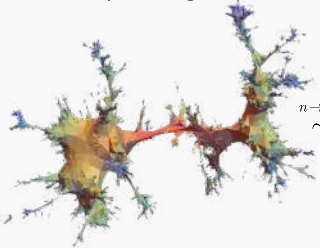

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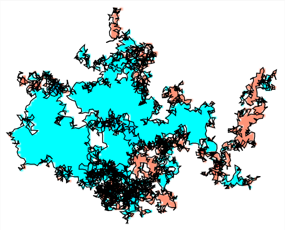
Walks in half square lattice with no point subexcursions



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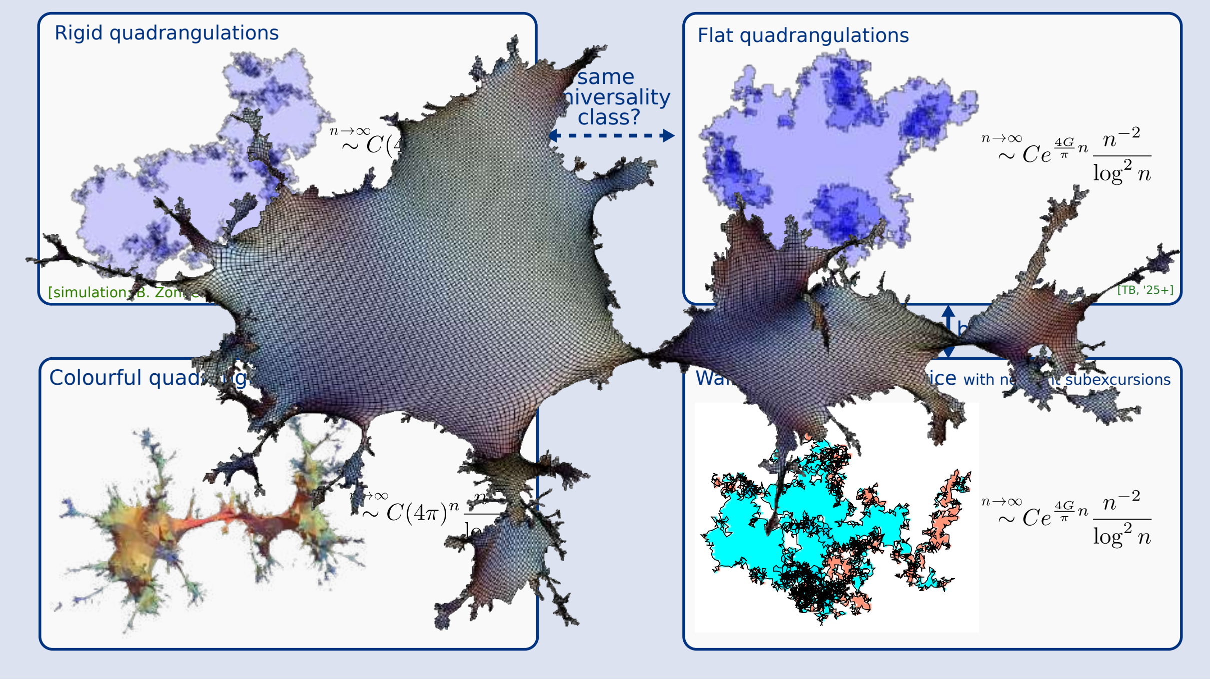
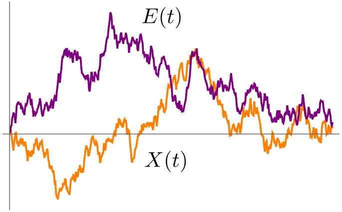
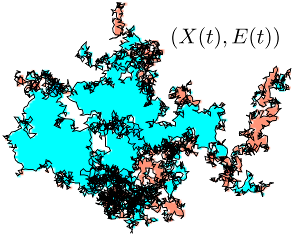


Rigid quadrangulations
same
niversality
class?
Flat quadrangulations
$\overset{n\to\infty}{\sim} C e^{\frac{4G}{\pi}n} \frac{n^{-2}}{\log^2 n}$
[TB, '25+]
Colourful quad
$\overset{n\to\infty}{\sim} C(4\pi)^n$
subexcursions
$\overset{n\to\infty}{\sim} C e^{\frac{4G}{\pi}n} \frac{n^{-2}}{\log^2 n}$

Perspectives: Scaling limit?

Bijection suggests a construction of the limit: let $\begin{cases} (X(t))_{t\in[0,1]} \text{ a Brownian bridge} \\ (E(t))_{t\in[0,1]} \text{ a Brownian excursion} \end{cases}$





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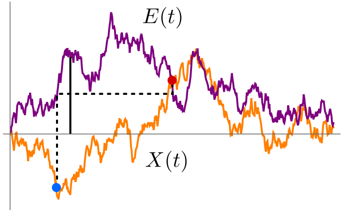

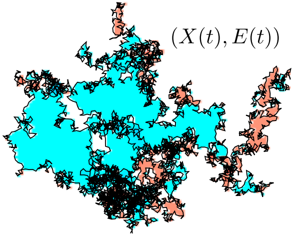

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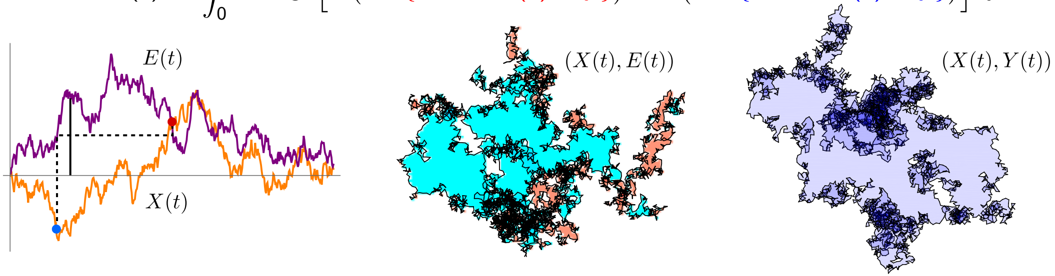

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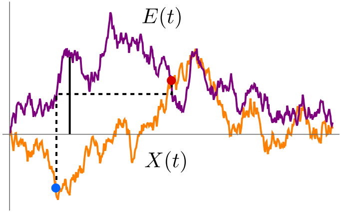

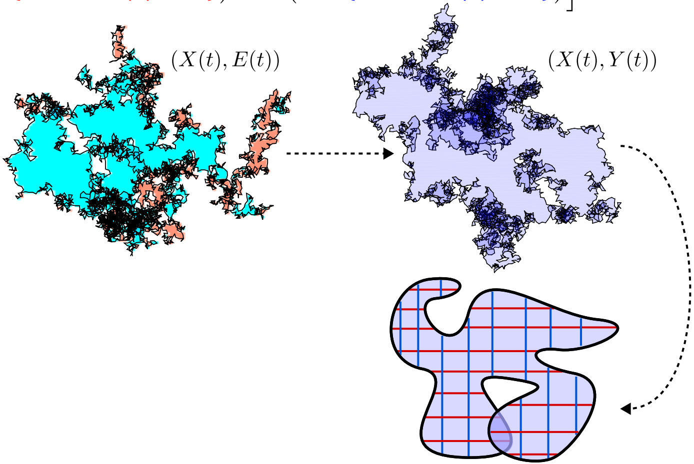

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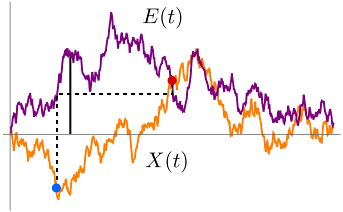

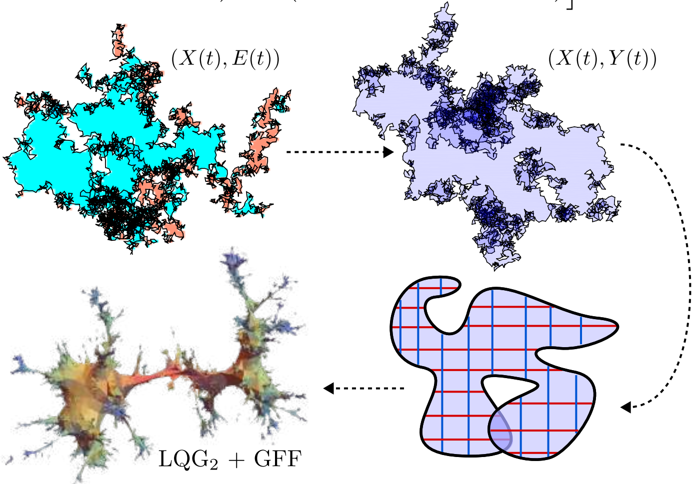

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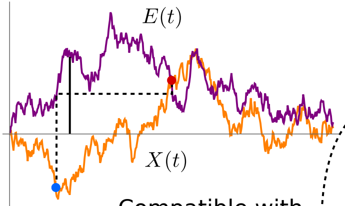
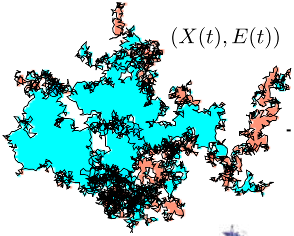
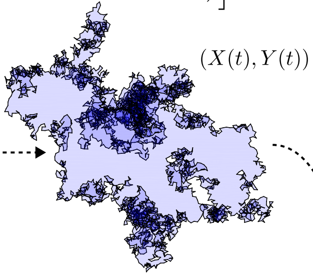






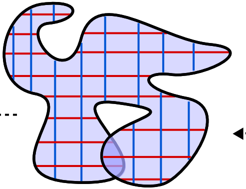
Compatible with critical mating of trees?

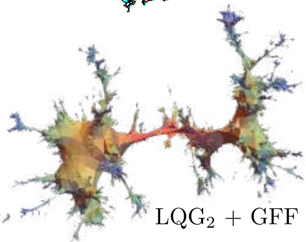
[Duplantier, Miller, Sheffield, '14]

[Aru, Holden, Powell, Sun, '21]

[Aïdékon, Da Silva, '20]

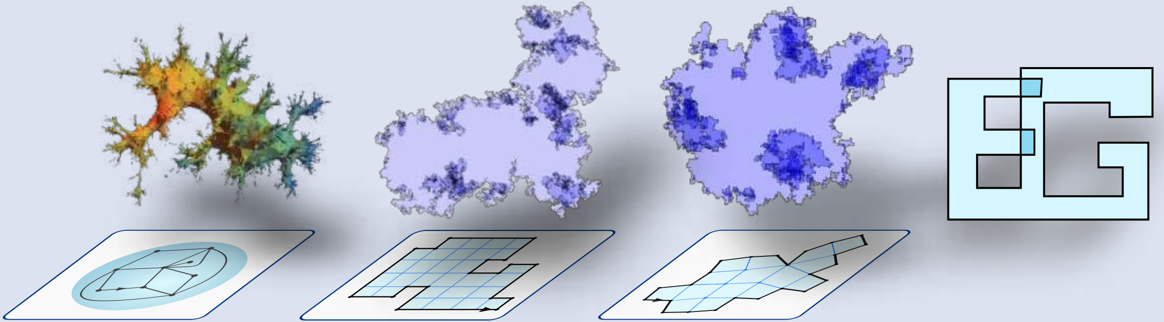
[Lehmkuehler, '23]





Perspectives: questions

- Are there other bijections of this type between flat disks and $\mathbb{Z}$-labeled maps?
- Is there a bijection between rigid quadrangulations and half-plane excursions?
- What is the law of the conformal map to the uniform disk?
- Can the bijection be understood as a discrete version of mating of trees for critical LQG?



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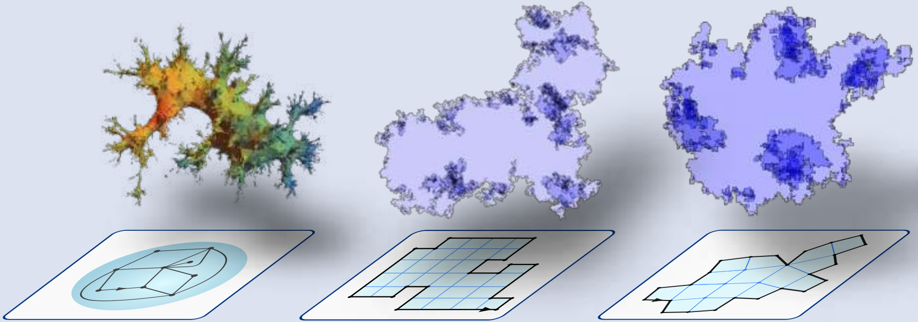

Thanks!