25 ans après, la bijection continue

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Travail en collaboration avec Omer Angel (UBC)
Brett Kolesnik (Oxford),
Emmanuel Jacob (ENS de Lyon)

L'esprit des cartes
Une conférence en l'honneur d'Emmanuel Guitter
IPhT, CEA Paris-Saclay
Jeudi 15 mai 2025

The Cori-Vauquelin-Schaeffer bijection

- Everybody knows the Cori-Vauquelin '81 and Schaeffer '98 bijection, and its extension by Bouttier-Di Francesco-Guitter '04.
- It is often presented as a mapping from well-labeled plane trees to pointed plane quadrangulations, because it is the one that has a nice counterpart in the continuum.

- Link the consecutive corners of a well-labeled tree to their successors: the next available corner of lesser label.
- This yields a pointed quadrangulation $(\mathbf{q}, v_*)$, rooted if the tree was also rooted

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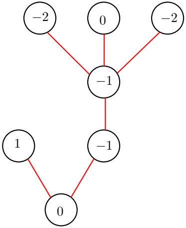

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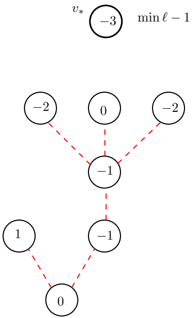

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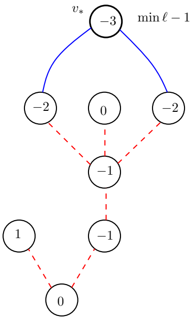

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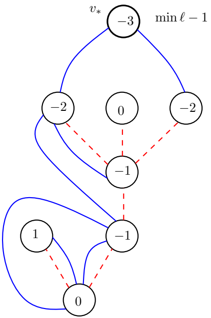

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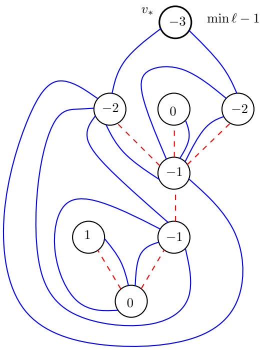

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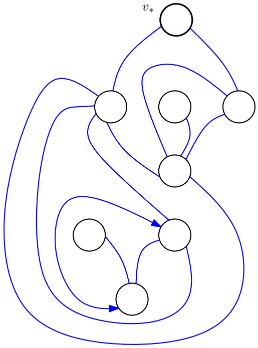

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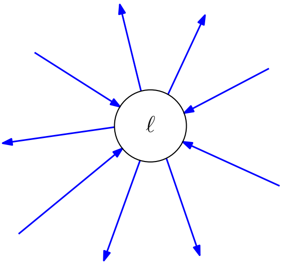
The inverse construction

- The inverse mapping, starting from a pointed (rooted) quadrangulation $(\mathbf{q}, v_*)$, consists essentially in determining a "geodesic spanning tree" by considering the geodesic orientation away from v_*, and drawing dual edges according to the following rule:



- We interpret each vertex $v \neq v_*$ as a British roundabout with label given by $d_{\mathbf{q}}(v, v_*)$ (possibly shifted by $d_{\mathbf{q}}(\mathrm{root}, v_*)$)
- We construct a plane tree whose branches correspond to the cars that must yield.
- In this interpretation, we see that internal vertices of the tree are vertices from which emanate multiple geodesic paths to v_*.

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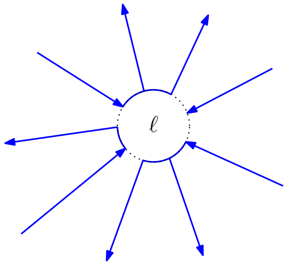

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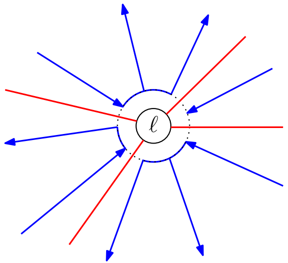

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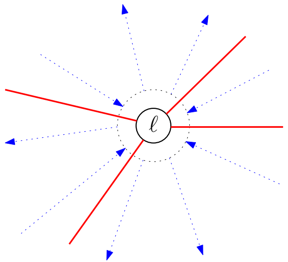

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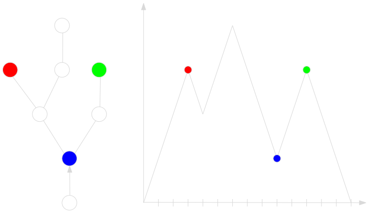
Passing to the limit in a random context

Let (T_n, ℓ_n) be a uniformly random well-labelled rooted plane tree with n edges. We describe this tree through its contour and label process:

$$C_n(i) = \text{height of the } i\text{-th visited corner in contour order}$$

$$L_n(i) = \text{label of that same corner}$$



Then (Chassaing-Schaeffer '04)

$$\left(\frac{C_n(2nt)}{\sqrt{2n}}, \frac{L_n(2nt)}{(8n/9)^{1/4}} \right)$$

converges in distribution to $(\mathbf{e}, Z)$, a standard Brownian motion excursion and a (conditionally) centered Gaussian process with

$$\mathrm{Cov}(Z_s, Z_t \mid \mathbf{e}) = \inf_{[s \wedge t, s \vee t]} \mathbf{e}\,.$$

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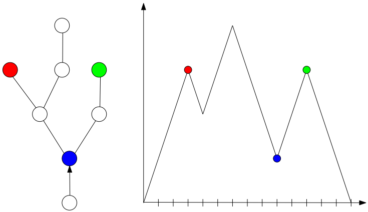

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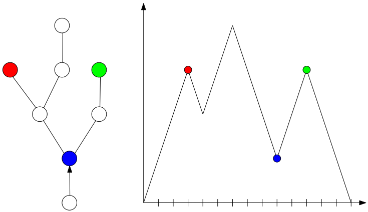

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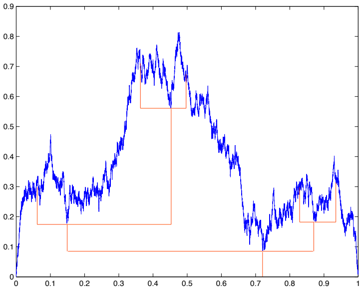

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The continuum CVS mapping

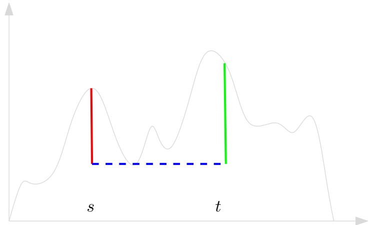
More formally, we define two "tree distance functions" associated with **e** and Z by

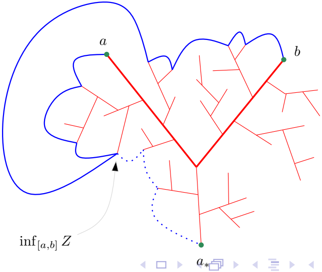
$$d_{\mathbf{e}}(s,t) = \mathbf{e}_s + \mathbf{e}_t - 2 \inf_{[s\wedge t, s\vee t]} \mathbf{e}$$

and

$$d_Z(s,t) = Z_s + Z_t - 2 \max\left(\inf_{[s\wedge t, s\vee t]} Z \,,\, \inf_{[s\vee t,1]\cup[0,s\wedge t]} Z \right)$$

as well as the two quotient metric spaces $\mathcal{T}_{\mathbf{e}} = ([0,1]/\{d_{\mathbf{e}} = 0\})$ and $\mathcal{T}_Z = ([0,1]/\{d_Z = 0\})$, the continuum analogues of the tree T_n and the geodesic tree constructed by the CVS mapping.





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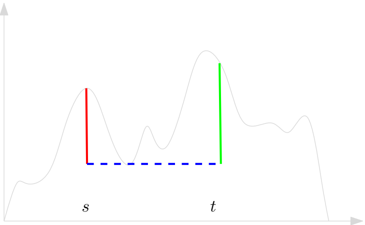

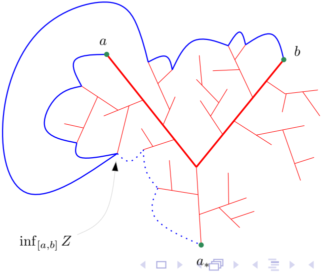

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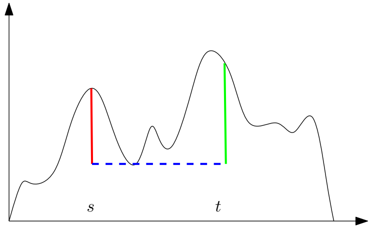

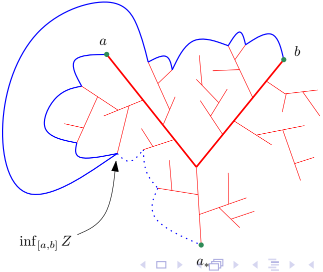

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- The Brownian sphere distance is then defined by

$$D(s,t) = \inf \left\{ \sum_{i=1}^{k} d_Z(s_i, t_i) \right\}$$

  where the infimum is taken over all $k \geq 1$ and $s_i, t_i, 1 \leq i \leq k$ such that $s_1 = s, t_k = t$ and $d_{\mathbf{e}}(t_i, s_{i+1}) = 0$ for $1 \leq i \leq k-1$.
- The Brownian sphere itself is defined by the quotient

$$X = ([0,1]/\{D=0\}, D)\,, \quad \text{with projection } \mathbf{p} : [0,1] \to X$$

  and it can be further decorated by
    - the marked points $x^0 = \mathbf{p}(0)$ and $x^1 = \mathbf{p}(s_*)$ where $s_* = \mathrm{argmin}\, Z$
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The inverse mapping?

- Can one "invert" this construction and recover $(\mathbf{e}, Z)$ from (X, x^0, x^1, μ)? In a sense, this is a folklore result due to
- a theorem by Le Gall ('10) stating that the set

$$\mathrm{Cut}(X, x^1) = \{x \in X : \text{there exists} > 1 \text{ geodesics from } x \text{ to } x^1\}$$

  is the image of

$$\mathrm{Skel}(\mathcal{T}_{\mathbf{e}}) : \{a \in \mathcal{T}_{\mathbf{e}} : \mathcal{T}_{\mathbf{e}} \setminus \{a\} \text{ is disconnected}\}$$

  by a homeomorphism $\theta : \mathrm{Skel}(\mathcal{T}_{\mathbf{e}}) \to \mathrm{Cut}(X, x^1)$, and
- the easily checked fact that for $a \in \mathcal{T}_{\mathbf{e}}$,

$$Z_a = D(\theta(a), x^1) - D(x^0, x^1) .$$

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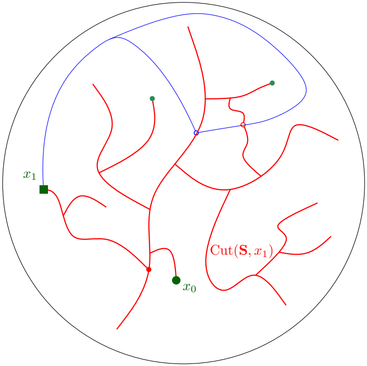
- It remains to perform some steps: order this embedded tree, and recover its metric and its leaves.
- Denoting by $[[x, y]]$ the path in $\mathrm{Cut}(X, x^1)$ between x and y, we can use the fact that $(Z_z, z \in [[x, y]])$ forms a path of a Brownian motion, defined up to parametrization since $d_{\mathbf{e}}(\theta^{-1}(x), \theta^{-1}(y))$ is unknown.
- If $(B_t)_{t \geq 0}$ is a standard Brownian motion and $f : [0, 1] \to \mathbb{R}_+$ is continuous and increasing, then f is a function of $(B_{f(t)})_{0 \leq t \leq 1}$, e.g. $f(1) = \lim_{\varepsilon \downarrow 0} \varepsilon^2 N_\varepsilon$ where N_ε is the number of successive intersections of $(B_{f(t)})$ with $\varepsilon\mathbb{Z}$.



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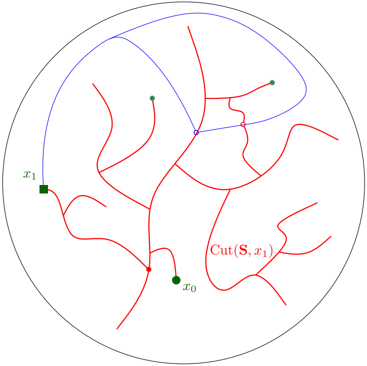

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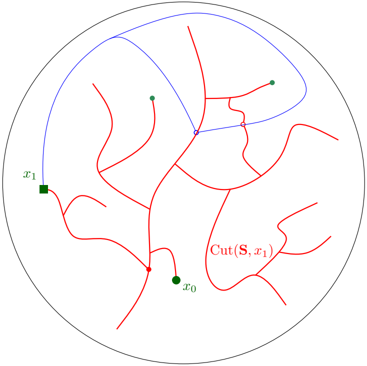

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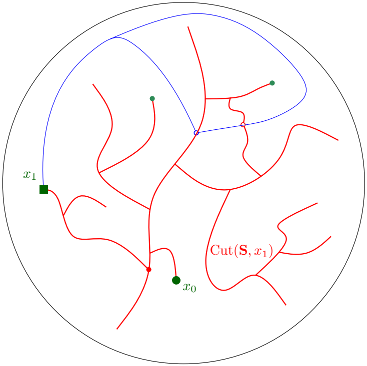

Problem: can this informal reconstruction be performed as a measurable function of X?

The continuum CVS mapping made formal

More formally, let $\mathcal{S}$ be the set of pairs $h = (f, g)$ of continuous functions $[0, 1] \to \mathbb{R}$ with value 0 at 0 and 1. We associate with them:

- the real trees $\mathcal{T}_f$, $\mathcal{T}_g$ as was done for $\mathbf{e}$ and Z,
- the quotient pseudo-distance $D_h = d_g/\{d_f = 0\}$ as was done for D, and the projection $\mathbf{p}_h : [0, 1] \to X_h = [0, 1]/\{D_h = 0\}$,
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Proposition (The formal CVS mapping)

The mapping $\psi : h \mapsto \mathbf{X}_h^{2\bullet} = [X_h, D_h, \mu_h, x_h^0, x_h^1]$ is a Borel mapping from $\mathcal{S}$ the Gromov-Hausdorff-Prokhorov space $m\mathcal{M}^{2\bullet}$ of isometry classes of compact metric measure spaces with two marked points.

In particular, the Brownian sphere is obtained by composing ψ with the random variable $(\mathbf{e}, Z)$.

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There exists a Borel mapping $\phi : m\mathcal{M}^{2\bullet} \times \{-1,1\} \to \mathcal{S}$ *such that*

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Proposition

- *Under $\mathbb{P}_{\text{Snake}}(dh)$, the random variables $\psi(h)$ and ε_h are independent, with ε_h uniform in $\{-1, 1\}$.*
- *Moreover, it holds that, $\mathbb{P}^{2\bullet}_{\text{Sphere}}$-a.s., $\phi(\mathbf{X}^{2\bullet}, -\varepsilon) = R\phi(\mathbf{X}^{2\bullet}, \varepsilon)$.*

- In particular, ε indeed allows to discriminate between h and Rh.
- Interestingly, this orientation of the snake process also corresponds to an orientation of the Brownian sphere. Hence, in a sense, the mapping ϕ allows to make sense of the intuitive fact that conditionally given the Brownian sphere, its orientation is chosen uniformly at random among the two possible orientations.
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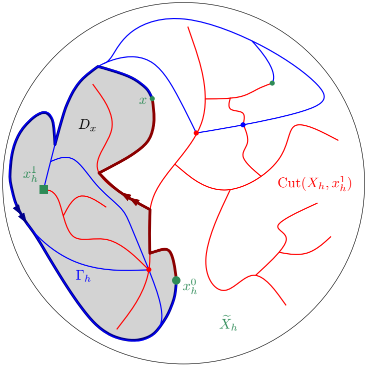
Orienting the Brownian sphere

- We let $C_h = \mathrm{Cut}(X_h, x_h^1)$, Γ_h be the set of points inside some geodesic to x_h^1, and $\widetilde{X}_h$ the complement of $C_h \cup \Gamma_h$.
- For $x \in \widetilde{X}_h$, there is a unique oriented Jordan curve $\gamma(x)$ going from x_h^0 to x in C_h, and then from x to x_h^1 in Γ_h.
- Choosing an orientation amounts to choosing which of the two Jordan domains lies to the left of $\gamma(x)$, and this is independent of x. We let D_x be this domain.
- We fix this choice as follows: take $x = x_h^1$ and let $D_{x_h^1}$ be the domain of smallest μ_h-mass if $\varepsilon_h = 1$, and the domain of largest μ_h-mass if $\varepsilon_h = -1$.



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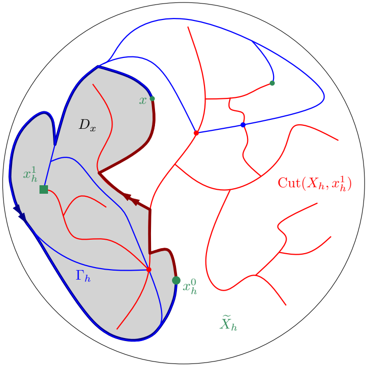

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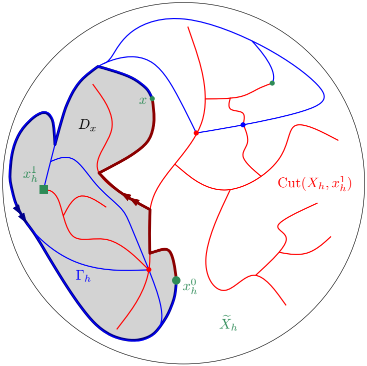

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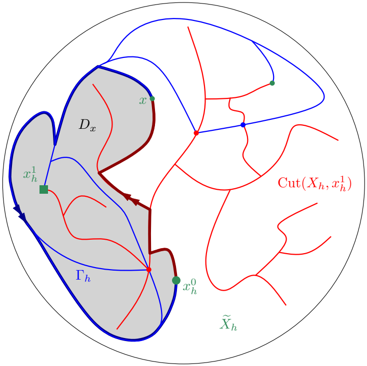

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- As is well-known, the points x_h^1 (the distinguished point) and x_h^0 (the root) play very different roles in the CVS bijection.
- However, given $\mathbf{X}_h = [X_h, D_h, \mu_h]$, these two points are two independent samples from μ_h.

Corollary (Resampling the marked points)

Let $\mathbf{X} = [X, d, \mu]$ have law $\mathbb{P}_{\mathrm{Sphere}}$. Conditionally given $\mathbf{X}$, let x^0, x^1 be two independent random points in $\mathbf{X}$ with law μ, and set

- $\mathbf{X}^{2\bullet} = [X, d, \mu, x^0, x^1]$,
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Then, almost surely, $R(W_+) = W_-$ and $\psi(W_+) = \psi(W_-) = \mathbf{X}^{2\bullet}$. Moreover, if $\sigma \in \{+, -\}$ is itself random, independent of $\mathbf{X}^{2\bullet}$, and uniformly distributed, then W_σ has law $\mathbb{P}_{\mathrm{Snake}}$.

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Then, almost surely, $R(W_+) = W_-$ and $\psi(W_+) = \psi(W_-) = \mathbf{X}^{2\bullet}$. Moreover, if $\sigma \in \{+, -\}$ is itself random, independent of $\mathbf{X}^{2\bullet}$, and uniformly distributed, then W_σ has law $\mathbb{P}_{\mathrm{Snake}}$.

The role of the marked points

- As is well-known, the points x_h^1 (the distinguished point) and x_h^0 (the root) play very different roles in the CVS bijection.
- However, given $\mathbf{X}_h = [X_h, D_h, \mu_h]$, these two points are two independent samples from μ_h.

Corollary (Resampling the marked points)

Let $\mathbf{X} = [X, d, \mu]$ have law $\mathbb{P}_{\mathrm{Sphere}}$. Conditionally given $\mathbf{X}$, let x^0, x^1 be two independent random points in $\mathbf{X}$ with law μ, and set

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The role of the measure

- The measure μ_h is used to recover the time paramatrization of h: for $x \in \widetilde{X}_h$, the point x is explored at time $\mu_h(D_x)$, where D_x is the Jordan domain discussed above.
- However, a result by Le Gall ('22) shows that $\mathbb{P}_{\mathrm{Snake}}(d\mathbf{X})$-a.s., μ is a constant multiple of the Hausdorff measure of (X, d, μ) with gauge function $r^4 \log\log(1/r)$, and hence μ is determined by the metric structure.
- Hence, again by the Lusin-Suslin theorem, there is a Borel mapping $\mathcal{M} \to m\mathcal{M}$, where $\mathcal{M}$ is the Gromov-Hausdorff space, that sends the law of $[X, d]$ to that of $[X, d, \mu]$.

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Conclusion

- The Lusin-Suslin theorem is a theoretical tool, which heuristically says that a measurable map that can be inverted abstractly can also be inverted concretely.
- We described in this context a continuum analogue of the CVS bijection, or BDG with “small faces”. There should also be
  - a related “continuum Chapuy-Marcus-Schaeffer bijection” in the context of Brownian surfaces (Bettinelli-M. ’22)
  - a “continuum BDG bijection” in the context of stable maps with exponent α recently considered in Curien-M.-Riera ’25. A question would be whether a single measurable map works for all α at once.
- The Lusin-Suslin theorem might have interesting things to say on the side of LQG as well.

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Merci de votre attention!