On the combinatorics of one variable catalytic equations

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join work with ENRICA DUCHI

L'esprit des cartes, une conférence
en l'honneur d'Emmanuel Guitter

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1-catalytic equations

Recall from Mireille's talk that a *1-catalytic equation* is an equation of the form

$$P\left(F(u), f_1, f_2, \ldots, f_k, u, t\right) = 0$$

where P is a polynomial with coefficients in some field $\mathbb{F}$
and we seek the unknown formal power series $F(u) \equiv F(t,u) \in \mathbb{F}[[t,u]]$ and $f_i \equiv f_i(t) \in \mathbb{F}[[t]]$.

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$$F(u) = 1 + tu^2 F(u)^2 + t \sum_{i \geq 2} z_i \frac{F(u) - \sum_{j=0}^{i-2} u^j F_j}{u^{i-2}}$$

for the enumeration of rooted planar maps w.r.t. face degree distribution.

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More generally 1-catalytic equations are ubiquitous in map enumeration, and closely related to the *loop equations* of the early matrix integral literature. They are also sometimes known as *discrete differential equations*.

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The celebrated **Bousquet-Mélou – Jehanne theorem** states that 1-catalytic equations of the form

$$F(u) = F_0(u) + tQ(F(u), \Delta F(u), \ldots, \Delta^k F(u), u, t)$$

where $F_0(u)$ and $Q(v, w_1, \ldots, w_k, u)$ are polynomials with coefficients in $\mathbb{F}$, and

$$\Delta^k F(u) = \frac{F(u) - f_1 - uf_2 - \ldots - u^{k-1} f_k}{u^k},$$

have unique solutions, and it provides a non degenerated system of algebraic equations that they satisfy.

1-catalytic equations have algebraic solutions

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BMJ theorem yields a parametrization that can be then rewritten as

$$F_2(t) = S_1^2 + S_2 - 2S_1[v^{-2}]W - [v^{-3}]W$$

with

$$W = \sum_{i\geq 1} z_i(v + S_1 + S_2/v)^{i-1}, \quad S_1 = t[u^0]W, \quad \text{and} \quad S_2 = t + t[v^{-1}]W.$$

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For instance for triangulations, $z_i = 0$ for all $i \neq 3$, and we get:

$$F_2(t) = S_1^2 + S_2 - 2S_1 S_2^2$$

with

$$S_1 = t(S_1^2 + 2S_2), \quad \text{and} \quad S_2 = t + 2t(S_1 S_2).$$

Context free languages and algebraic specifications/decompositions

Algebraic equations are closely related to well funded **context-free specifications**:

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e.g. $\mathcal{A} \equiv \mathbf{z} + \mathcal{A} \times \mathcal{A}$

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Conversely when the gf of a combinatorial familly $\mathcal{A}$ is known to be $\mathbb{N}$-algebraic, one would like to explain it via a **context-free specification** of $\mathcal{A}$.

(Schützenberger's methodology for algebraic gf)

Context-free specifications and multitype simply generated trees

Context-free decompositions are naturally associated with multitype simply generated trees:

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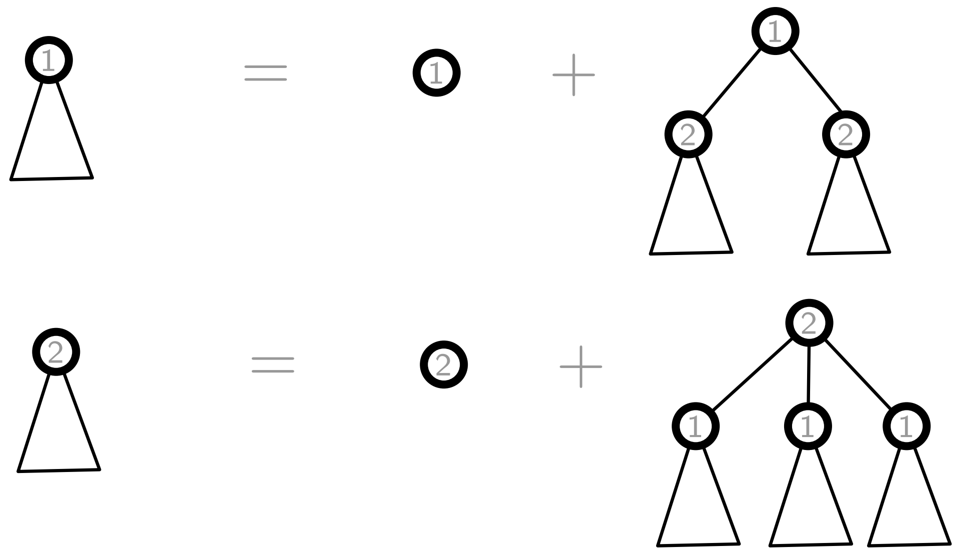

The *derivation trees of a context-free specification* are multitype simply generated trees, *i.e.* trees specified by the allowed node progeny for each color, with independent subtrees.

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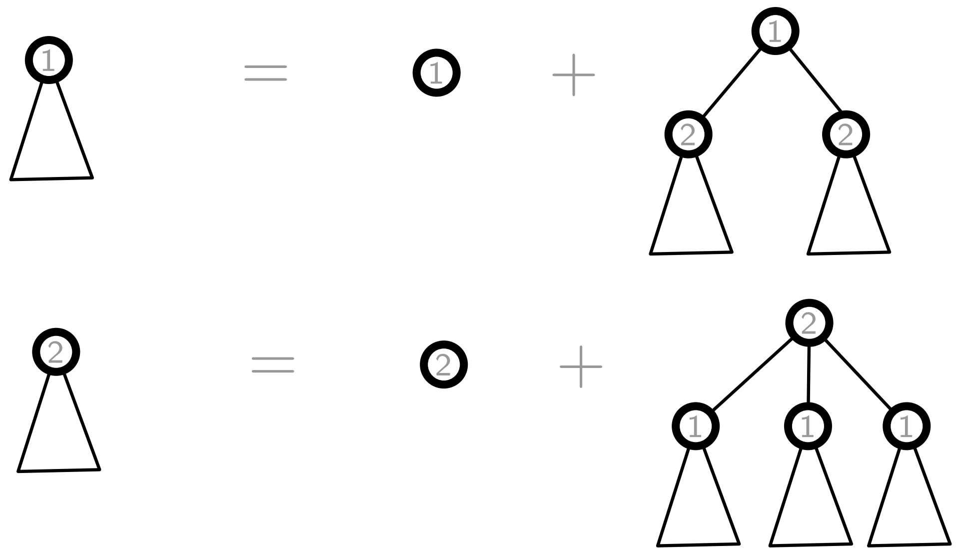

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Context-free specification for maps

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This algebraic parametrization was given **two** beautiful combinatorial interpretations by **Emmanuel Guitter** with Jérémie Bouttier and Philippe Di Francesco

- first in terms of *blossoming trees*,

[Census of Planar Maps: From the One-Matrix Model Solution to a Combinatorial Proof, *Nuclear physics*, 2002]

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Later Emmanuel and Jérémie also gave a direct *context-free specification* of maps

Planar maps as pizza slices, *aka* [Planar Maps and continous fractions, *Comm. Math. Phys.*, 2012]

Catalytic specifications as a source of bijective problems

In summary:

- Maps admit "easy" catalytic specifications
- Catalytic equations have *nice* algebraic solutions

$\Rightarrow$ combinatorial interpretation problem !

These ideas have been generalized for a huge variety of map families...

and have led to many developments in Combinatorics, Probability or Algorithmics

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What about a systematic approach?

BMJ theorem, for order one 1-catalytic equations

Let $Q(v, w, u)$ be a polynomial with $Q(0, 0, u) \neq 0$

and $F(u) \equiv F(t, u)$ the unique fps solution of the catalytic equation

$$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right), \qquad \text{where } f \equiv f(t) = F(t, 0).$$

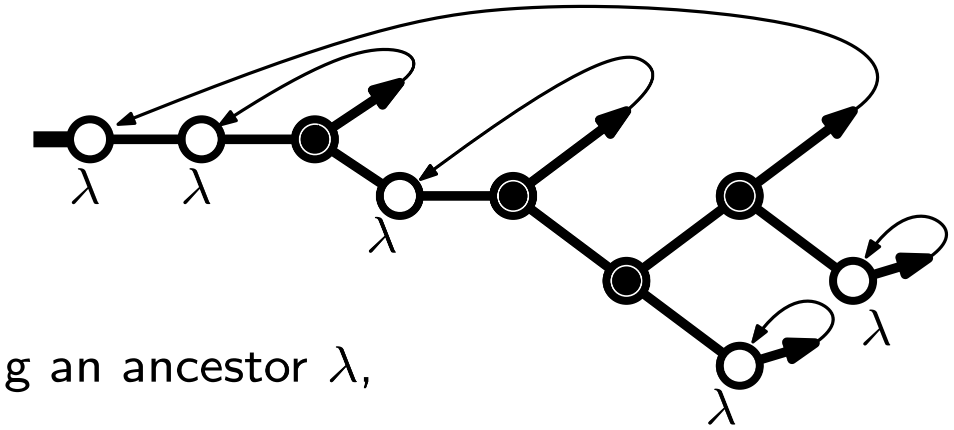
Let U, V, W and R be the unique fps satisfying the system

$$\begin{cases} V &= t \cdot Q(V, W, U) \\ R &= t \cdot (1 + R) \cdot Q'_v(V, W, U) \\ U &= t \cdot (1 + R) \cdot Q'_w(V, W, U) \\ W &= t \cdot (1 + R) \cdot Q'_u(V, W, U) \end{cases}$$

Then f is given by $\quad f = V - UW \quad$ or $\quad t f'_t = (1 + R) \cdot V$

$\Rightarrow$ The particularly simple form of this parametrization calls for a combinatorial lifting.

The example of planar λ-terms



Planar λ-terms can be presented as trees with

- **applications**: binary nodes
- λ-**abstractions**: unary nodes
- **variables**: leaves, represented as arrows , each matching an ancestor λ,

with condition that each λ is binded to exactly one variable in a planar way...

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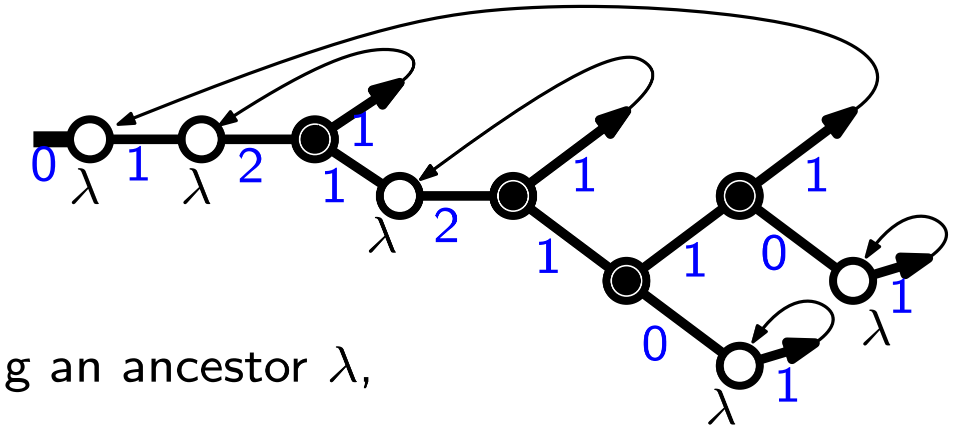

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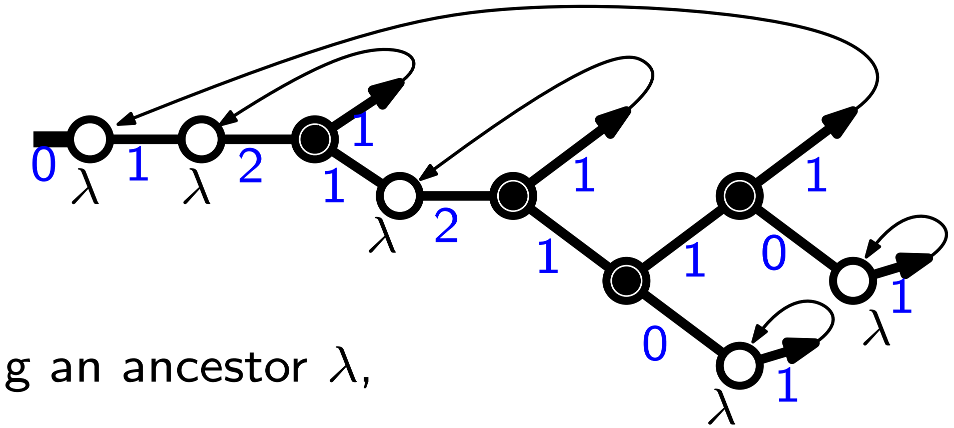

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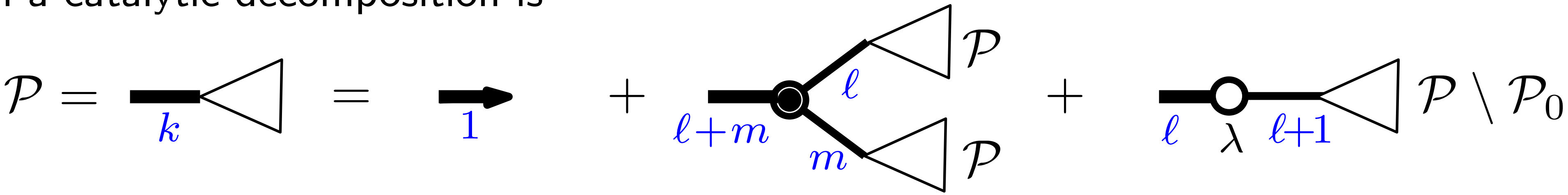

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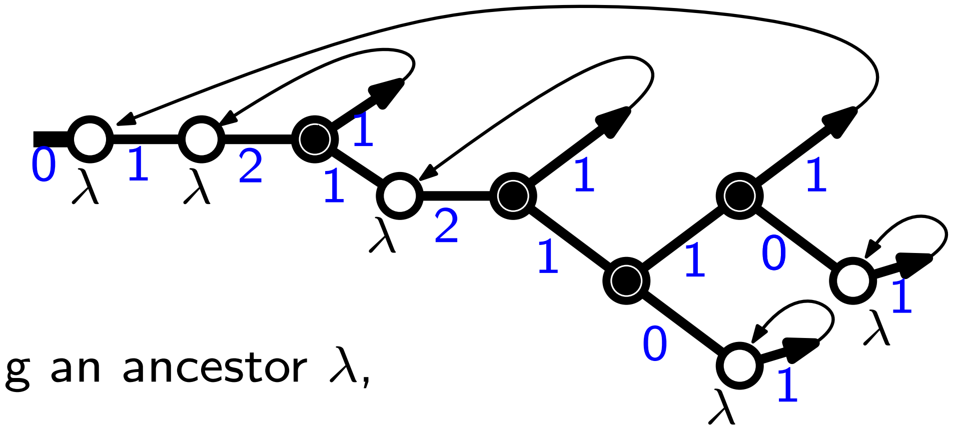

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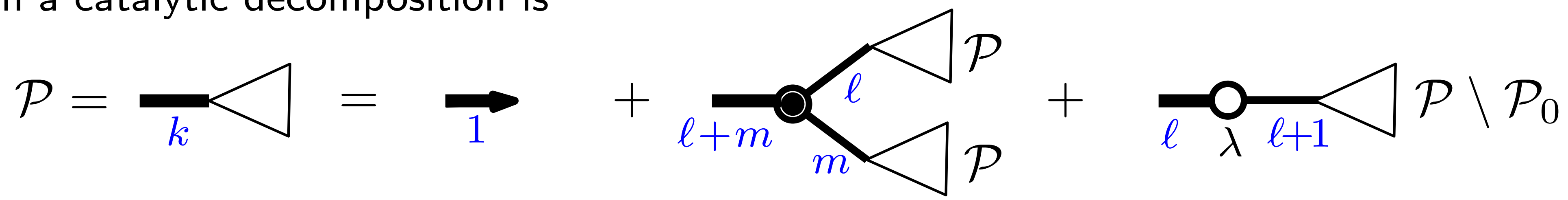


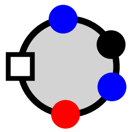
and the catalytic equation for the gf $P(u) = \sum_{\tau \in \mathcal{P}} t^{|\tau|} u^{excess(\tau)}$ is

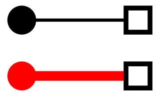
$$P(u) \quad = \quad tu \quad + \quad tP(u)^2 \quad + \quad \frac{t}{u}(P(u) - P(0))$$

Decorated trees and non negative trees

non-negative $\mathcal{Q}$-tree $=$ necklace tree s.t.

the excess at each pearl is non negative.

$\mathcal{Q} = \{$  $, \ldots \}$



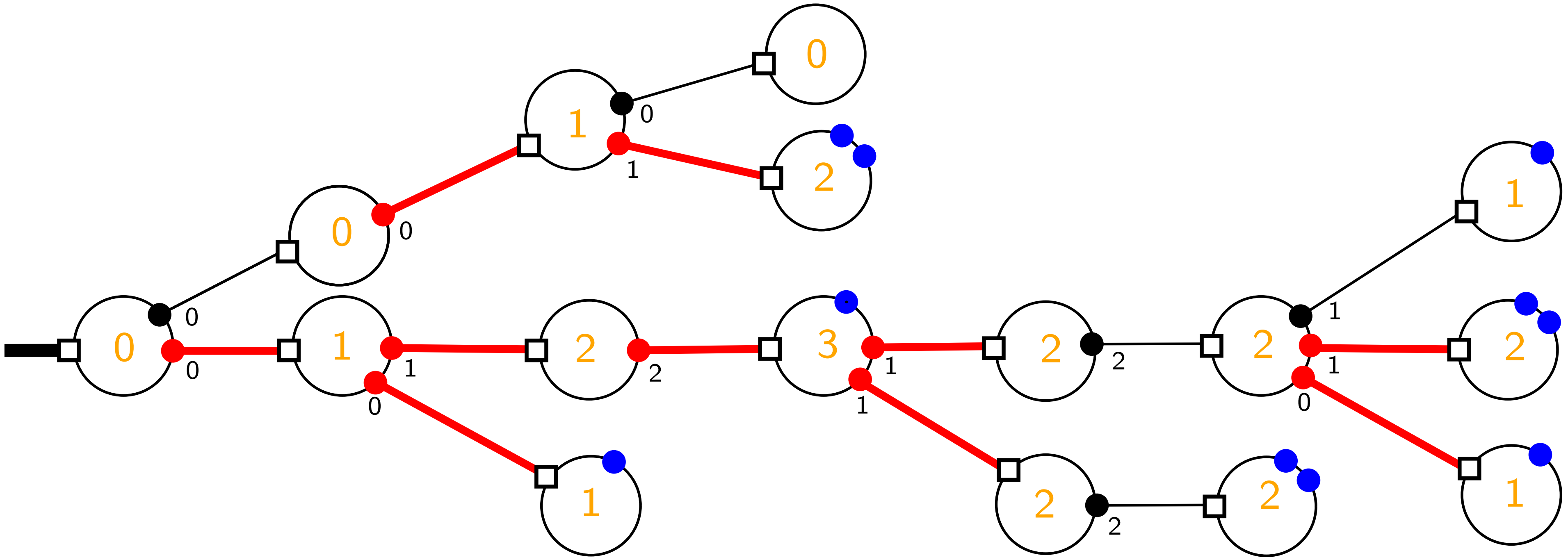
$\bullet$, $\bullet$ are all matched.

$\#\{\bullet\} \geq \#\{\bullet\}$
in planted subtrees

Observe:

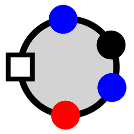
slightly stronger condition than
just asking non negative excess on vertices

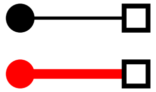
$$\boxed{\text{excess} = \#\{\bullet\} - \#\{\bullet\}}$$



Non negative $\mathcal{Q}$-trees and catalytic equations

Let $\mathcal{F} = \{$ **non-negative $\mathcal{Q}$-trees** $\}$,

$\mathcal{Q} = \Big\{$  $, \dots \Big\}$



$\bullet$, ${\color{red}\bullet}$ are all matched.

$\#\{{\color{blue}\bullet}\} \geq \#\{{\color{red}\bullet}\}$
in planted subtrees

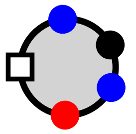
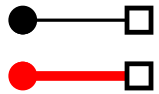
$Q(v, w, u) = \sum_{s\in\mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$ the vertex type gf, where q_s are weights

and $F(u) \equiv F(t, u) = \sum_{\tau\in\mathcal{F}} q_\tau t^{|\tau|} u^{\mathrm{excess}(\tau)}$, where $q_\tau = \prod_{s\in\tau} q_s$

Proposition. The gf $F(u)$ of non negative $\mathcal{Q}$-trees satisfies a catalytic equation of order one:

$$F(u) = tQ\Big(F(u), \tfrac{1}{u}(F(u) - F(0)), u\Big)$$

Non negative $\mathcal{Q}$-trees and catalytic equations

Let $\mathcal{F} = \{$ **non-negative $\mathcal{Q}$-trees** $\}$, $\mathcal{Q} = \Big\{$  $, \ldots \Big\}$  $\bullet$, ${\color{red}\bullet}$ are all matched.

$\#\{{\color{blue}\bullet}\} \geq \#\{{\color{red}\bullet}\}$
in planted subtrees

$Q(v, w, u) = \sum_{s \in \mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$ the vertex type gf, where q_s are weights

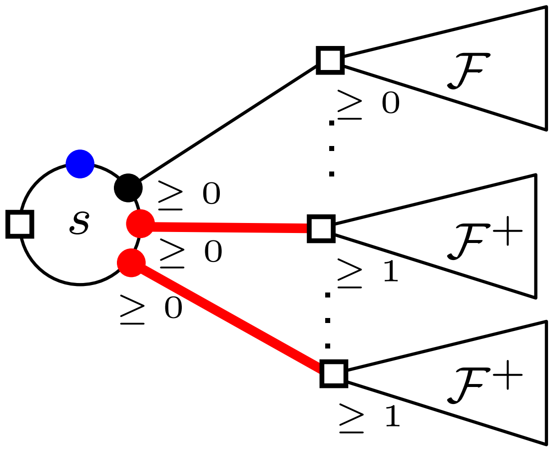
and $F(u) \equiv F(t, u) = \sum_{\tau \in \mathcal{F}} q_\tau t^{|\tau|} u^{\mathrm{excess}(\tau)}$, where $q_\tau = \prod_{s \in \tau} q_s$

Proposition. The gf $F(u)$ of non negative $\mathcal{Q}$-trees satisfies a catalytic equation of order one:

$$F(u) = tQ\Big(F(u), \tfrac{1}{u}(F(u) - F(0)), u\Big)$$

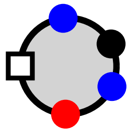
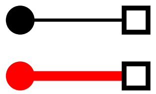
Indeed the equation

$$F(u) = t \sum_{s \in \mathcal{Q}} q_s F(u)^{\bullet(s)} \left(\tfrac{1}{u}(F(u) - F(0))\right)^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$$

follows from a decomposition at the root: $\mathcal{F} \equiv \sum_{s \in \mathcal{Q}} q_s \cdot$ 

where $\mathcal{F}^+ = \mathcal{F} \setminus f$

Non negative $\mathcal{Q}$-trees and catalytic equations

Let $\mathcal{F} = \{$ **non-negative $\mathcal{Q}$-trees** $\}$, $\mathcal{Q} = \Big\{$  $, \ldots \Big\}$ 

$\bullet$, ${\color{red}\bullet}$ are all matched.

$\#\{{\color{blue}\bullet}\} \geq \#\{{\color{red}\bullet}\}$
in planted subtrees

$Q(v, w, u) = \sum_{s \in \mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$ the vertex type gf, where q_s are weights

and $F(u) \equiv F(t, u) = \sum_{\tau \in \mathcal{F}} q_\tau t^{|\tau|} u^{\mathrm{excess}(\tau)}$, where $q_\tau = \prod_{s \in \tau} q_s$

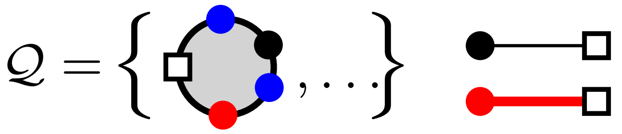
Proposition. The gf $F(u)$ of non negative $\mathcal{Q}$-trees satisfies a catalytic equation of order one:

$$F(u) = tQ\Big(F(u), \tfrac{1}{u}(F(u) - F(0)), u\Big)$$

$\Rightarrow$ non-negative $\mathcal{Q}$-trees give a generic combinatorial interpretation
for catalytic equations of order one with non negative coefficients.

Non negative $\mathcal{Q}$-trees and companion $\mathcal{Q}$-trees

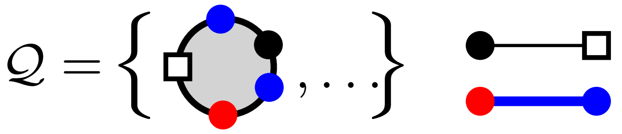
non-negative $\mathcal{Q}$-tree = necklace tree s.t.
- necklaces are in $\mathcal{Q}$
- the excess at each pearl is non negative.



•, • are all matched.

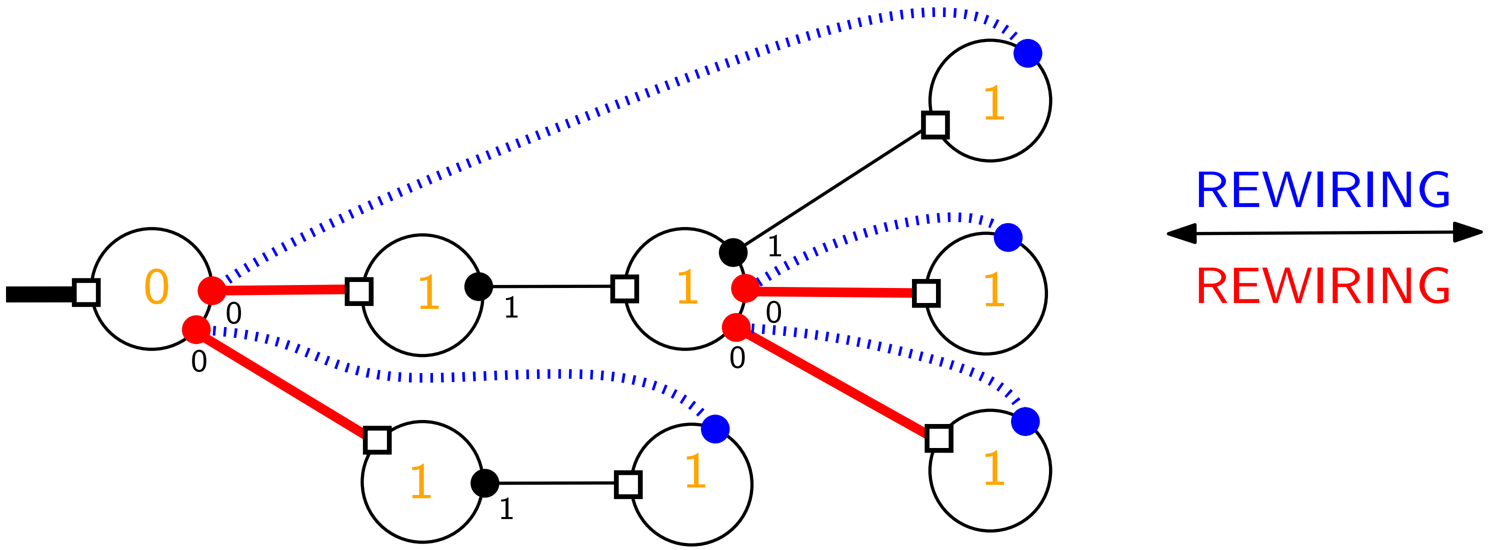
$\#\{\bullet\} \geq \#\{\bullet\}$ in planted subtrees

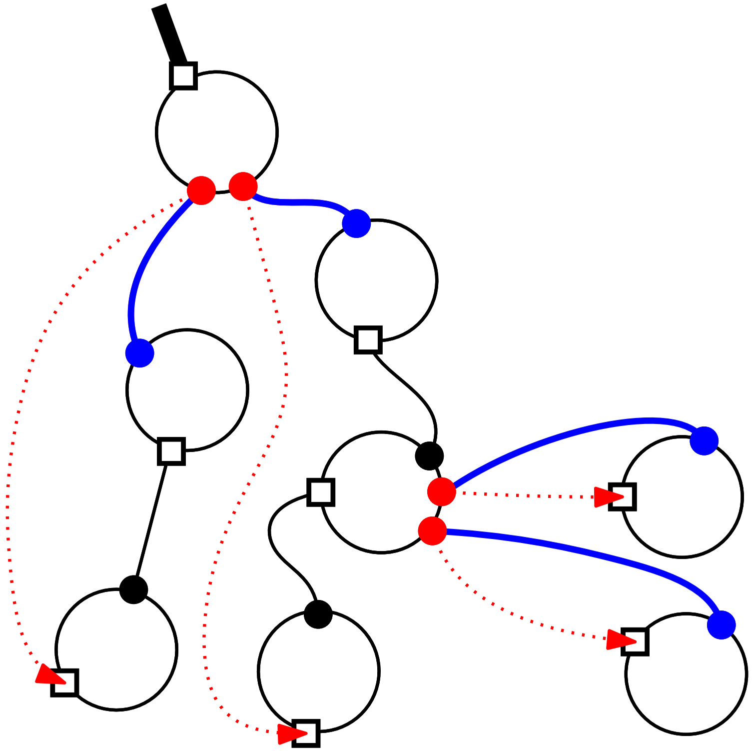
companion $\mathcal{Q}$- tree = necklace tree s.t.
- necklaces are in $\mathcal{Q}$



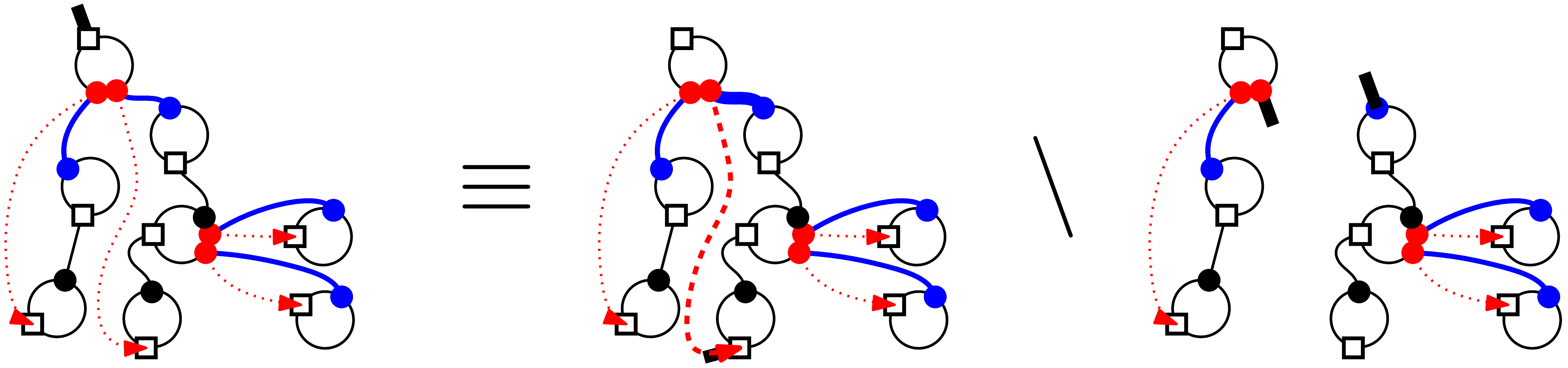
•, •, • are all matched.

THEOREM (Duchi-S. 23). **Rewiring** is a vertex type preserving bijection between non negative $\mathcal{Q}$-trees and balanced companion $\mathcal{Q}$-trees



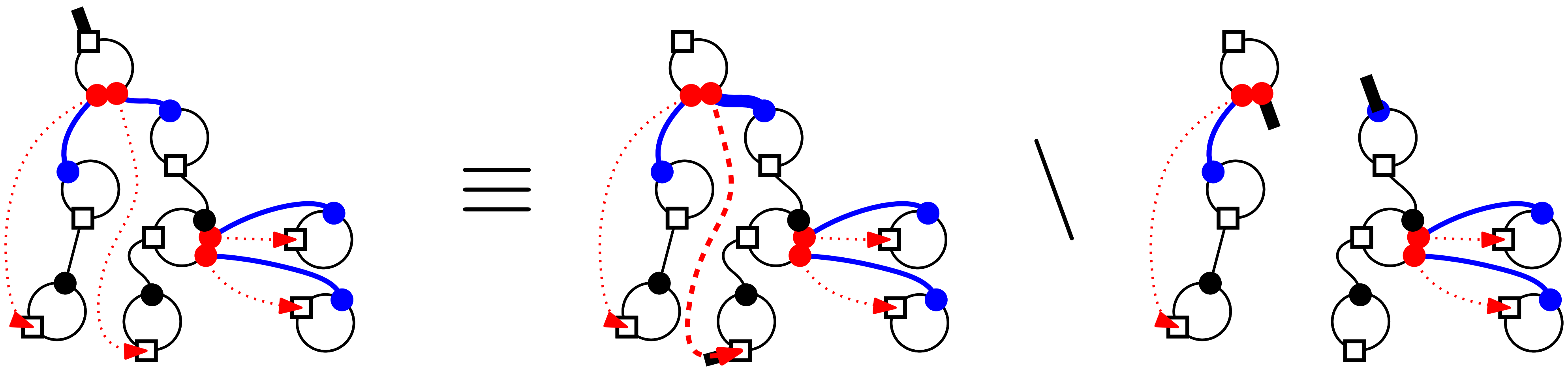


Balanced companion $\mathcal{Q}$-trees VS rooted companion $\mathcal{Q}$-trees

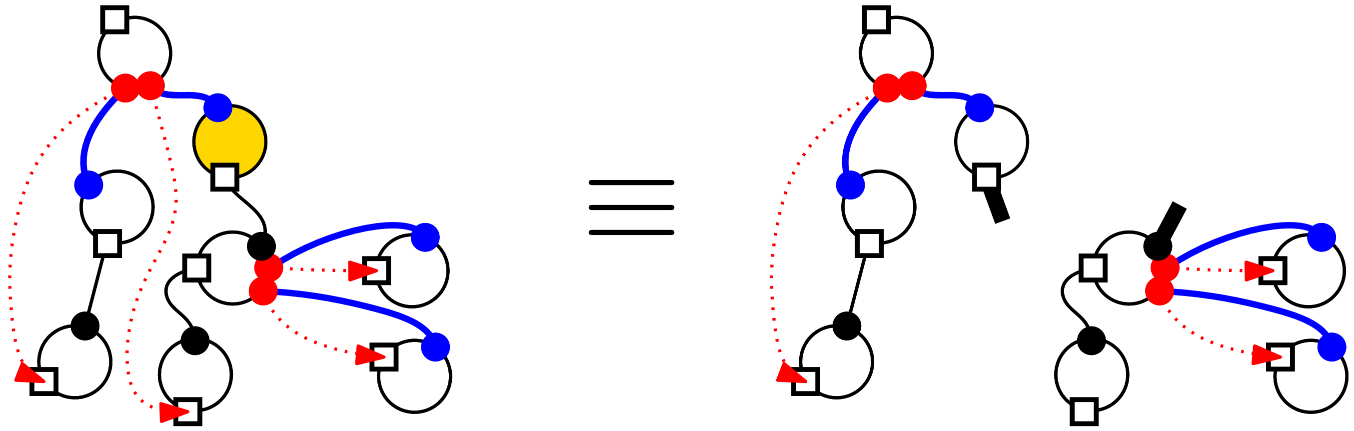


$$\boxed{\mathcal{BC}_{\square} \equiv \mathcal{C}_{\square} - \mathcal{C}_{\bullet} \times \mathcal{C}_{\bullet}}$$

Balanced companion $\mathcal{Q}$-trees VS rooted companion $\mathcal{Q}$-trees



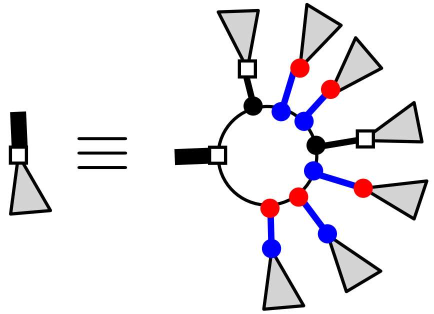
$$\mathcal{BC}_{\square} \equiv \mathcal{C}_{\square} - \mathcal{C}_{\color{red}\bullet} \times \mathcal{C}_{\color{blue}\bullet}$$

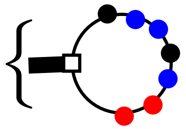


$$\mathcal{C}^{\color{yellow}\circ} \equiv \mathcal{C}_{\square} \times (\varepsilon + \mathcal{C}_{\bullet})$$

Context-free specifications for companion $\mathcal{Q}$-trees

$$C_{\square} = \mathcal{Z} \times Q(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$

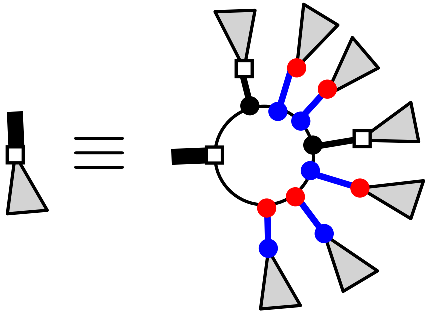


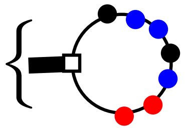
$\mathcal{Q} = \{$  $, \ldots \}$

$$Q(v, w, u) = \sum_{s \in \mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$$

Context-free specifications for companion $\mathcal{Q}$-trees

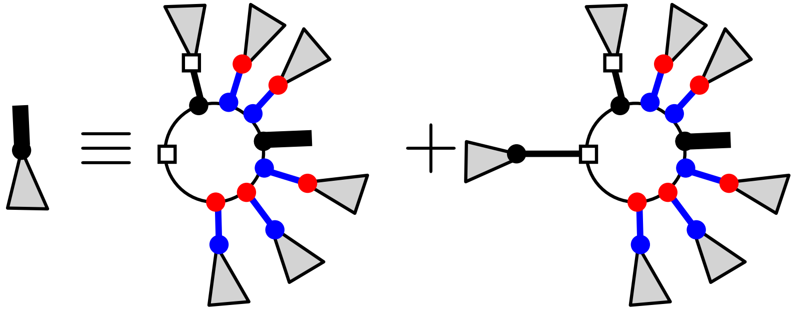
$$C_{\square} = \mathcal{Z} \times Q(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$

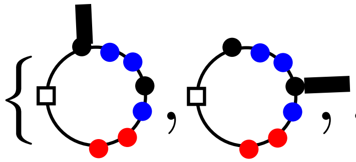


$\mathcal{Q} = \{$  $, \dots \}$

$$Q(v,w,u) = \sum_{s\in\mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$$

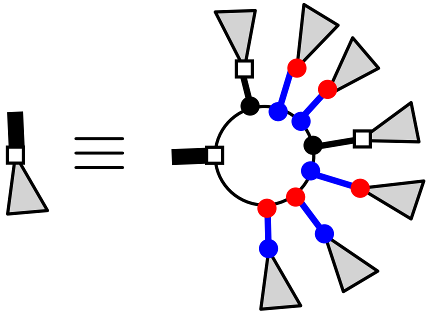
$$C_{\bullet} = \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$

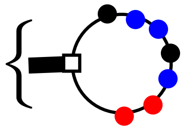


$Q'_{\bullet} = \{$  $, \dots \}$

Context-free specifications for companion $\mathcal{Q}$-trees

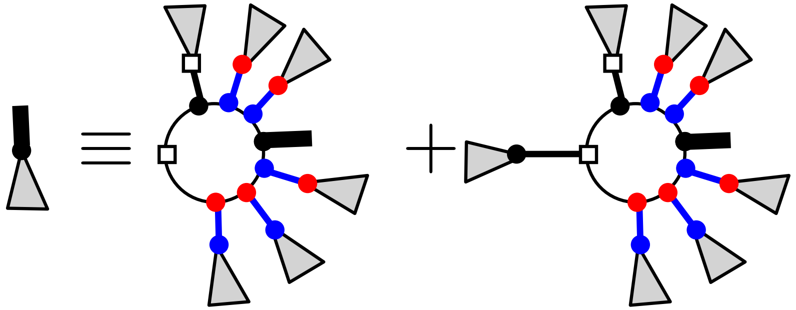
$$C_\square = \mathcal{Z} \times Q(C_\square, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$

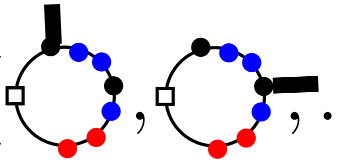


$\mathcal{Q} = \{$  $, \ldots \}$

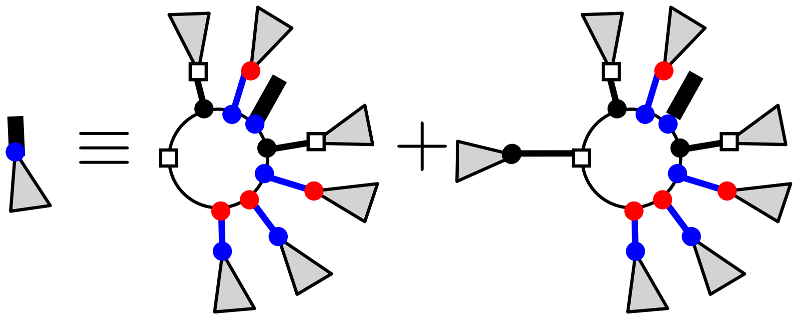
$$Q(v, w, u) = \sum_{s \in \mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$$

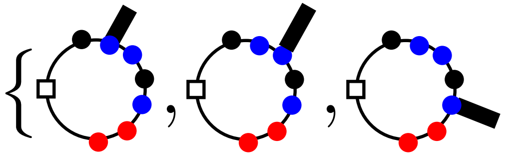
$$C_\bullet = \mathcal{Z} \times (1 + C_\bullet) \times Q'_\bullet(C_\square, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$



$Q'_\bullet = \{$  $, \ldots \}$

$$C_{\color{blue}\bullet} = \mathcal{Z} \times (1 + C_\bullet) \times Q'_{\color{blue}\bullet}(C_\square, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$

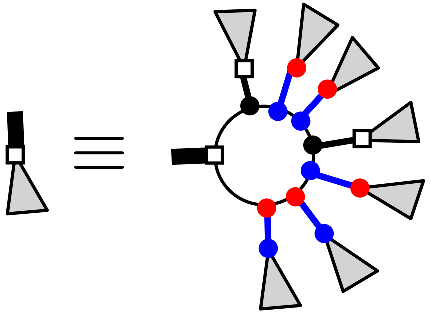


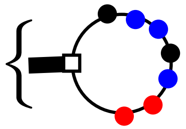
$Q'_{\color{blue}\bullet} = \{$ 

Context-free specifications for companion $\mathcal{Q}$-trees

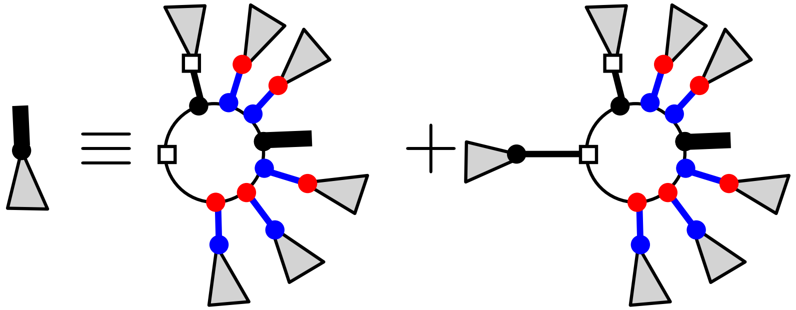
$$C_{\square} \quad = \quad \mathcal{Z} \times Q(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$

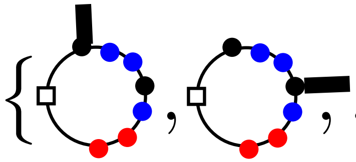
$$Q(v, w, u) = \sum_{s \in \mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$$



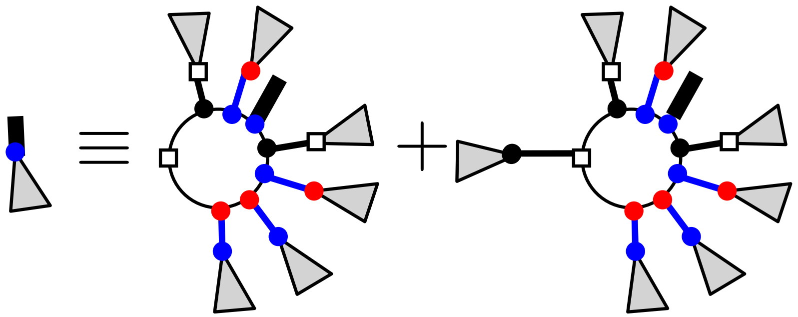
$\mathcal{Q} = \{$  $, \ldots \}$

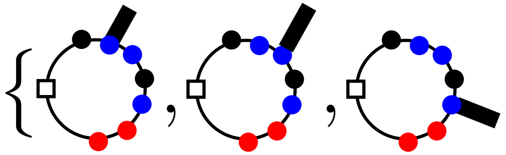
$$C_{\bullet} \quad = \quad \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$



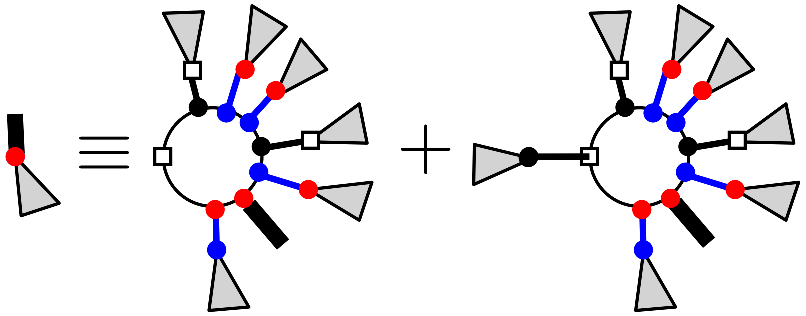
$Q'_{\bullet} = \{$  $, \ldots \}$

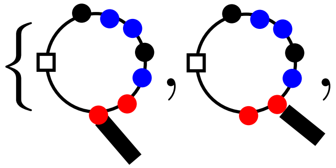
$$C_{\color{blue}\bullet} \quad = \quad \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\color{blue}\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$



$Q'_{\color{blue}\bullet} = \{$ 

$$C_{\color{red}\bullet} \quad = \quad \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\color{red}\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet})$$



$Q'_{\color{blue}\bullet} = \{$  $, \ldots \}$

The combinatorial lifting of BMJ theorem

THEOREM (Duchi-S. 23)

Let $\mathcal{F} \equiv \mathcal{Z} \times Q\left(\mathcal{F}, \frac{1}{u}(\mathcal{F} \setminus f), u\right)$ be a catalytic decomposition of order one

where $Q(v,w,u) = \sum_{s \in \mathcal{Q}} q_s v^{\bullet(s)} w^{{\color{red}\bullet}(s)} u^{{\color{blue}\bullet}(s)}$ is the node gf of the associated non negative derivation $\mathcal{Q}$-trees

then

$$\begin{aligned} f &\overset{\text{rewiring}}{\equiv} C &&= C_{\square} - C_{\color{blue}\bullet} \times C_{\color{red}\bullet} \\ f'_t &\overset{\text{rewiring}}{\equiv} C^{\circ} &&= (1 + C_{\bullet}) \times Q(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet}) \end{aligned}$$

where the companion trees satisfy:

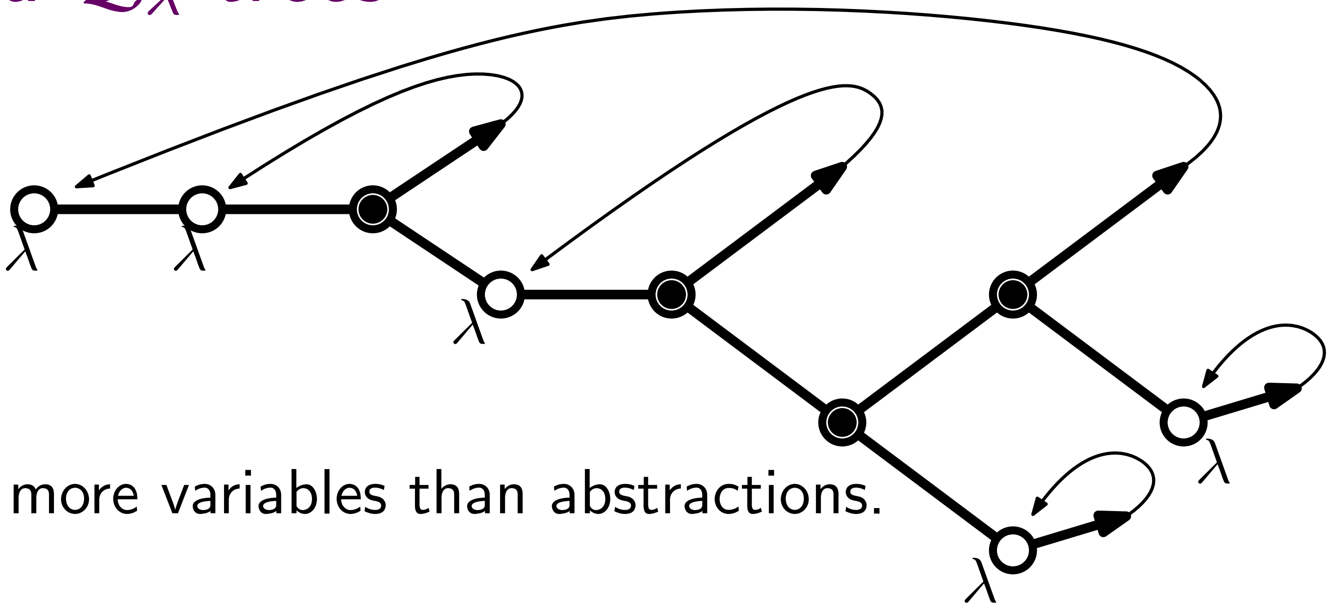
$$\left\{ \begin{array}{lcl} C_{\square} &=& \mathcal{Z} \times Q(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet}) \\ C_{\bullet} &=& \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet}) \\ C_{\color{red}\bullet} &=& \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\color{red}\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet}) \\ C_{\color{blue}\bullet} &=& \mathcal{Z} \times (1 + C_{\bullet}) \times Q'_{\color{blue}\bullet}(C_{\square}, C_{\color{blue}\bullet}, C_{\color{red}\bullet}) \end{array} \right.$$

Planar λ-terms and $\mathcal{Q}_\lambda$-trees

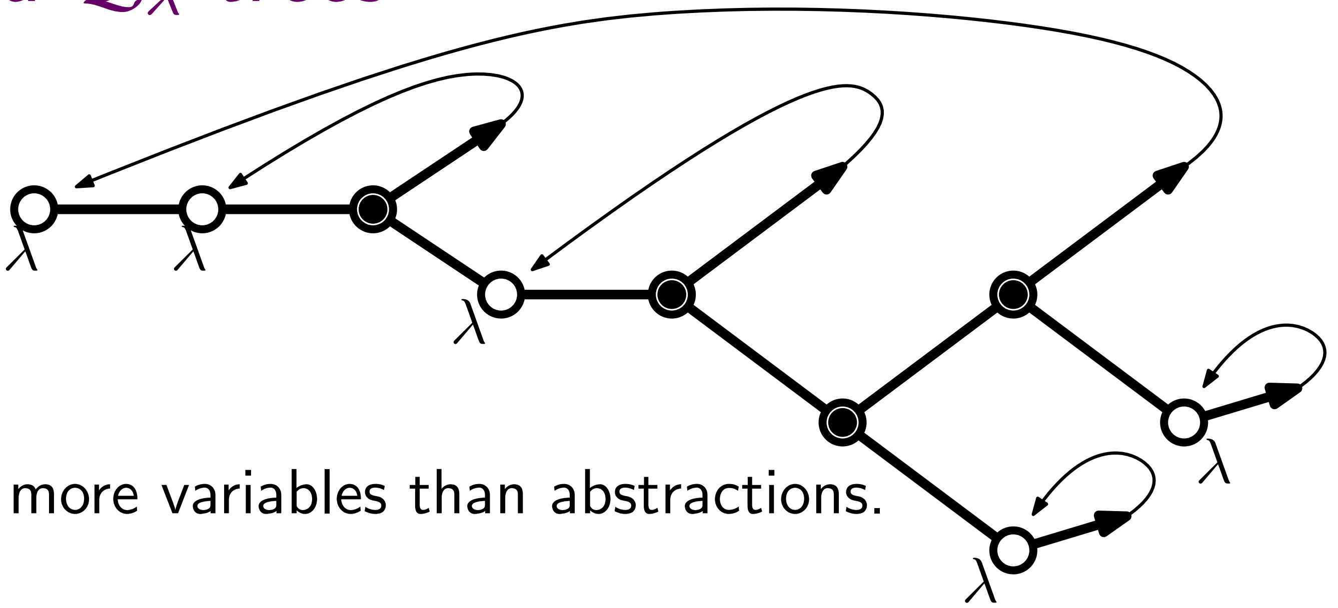
Open planar λ-term are to plane trees with

- applications: binary nodes
- abstractions: unary nodes
- variables: leaves, represented as arrow.

with condition that in each subterm there are more variables than abstractions.



Planar λ-terms and $\mathcal{Q}_\lambda$-trees

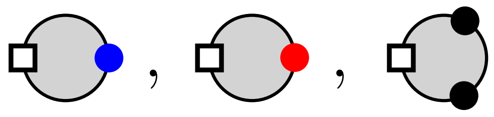


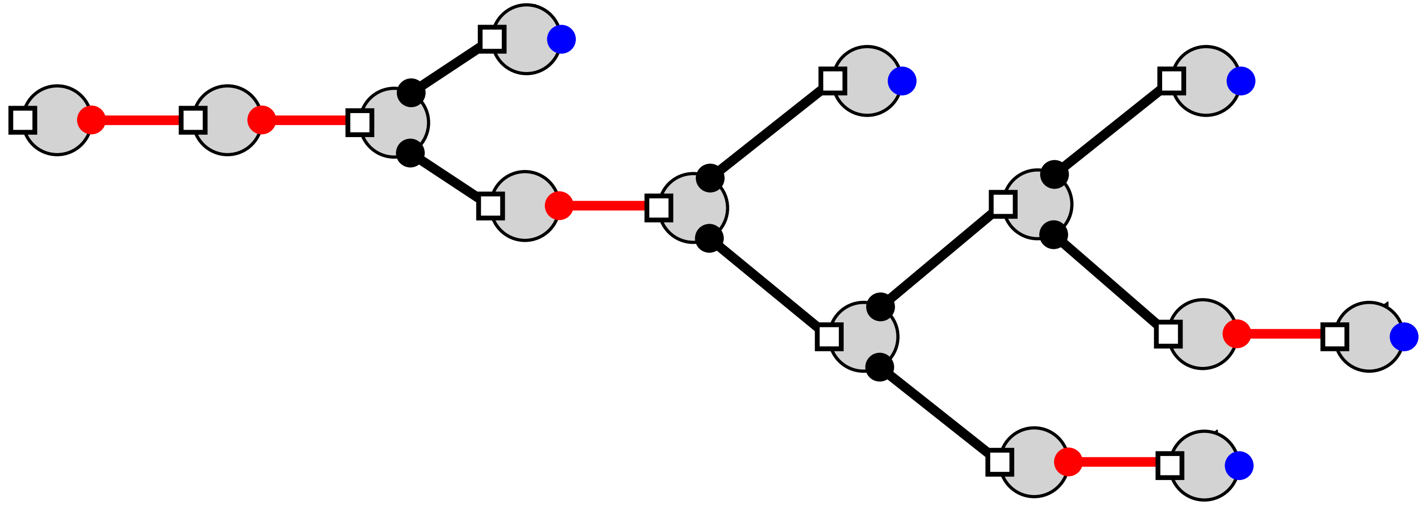
Open planar λ-term are to plane trees with

- applications: binary nodes
- abstractions: unary nodes
- variables: leaves, represented as arrow.

with condition that in each subterm there are more variables than abstractions.

Mark variables with $\color{blue}{\bullet}$ and abstractions λ with $\color{red}{\bullet}$,
then the set of vertex types is

$\mathcal{Q}_\lambda = \{$  $\}$



Then **non negative $\mathcal{Q}_\lambda$-trees =**
open planar λ-terms

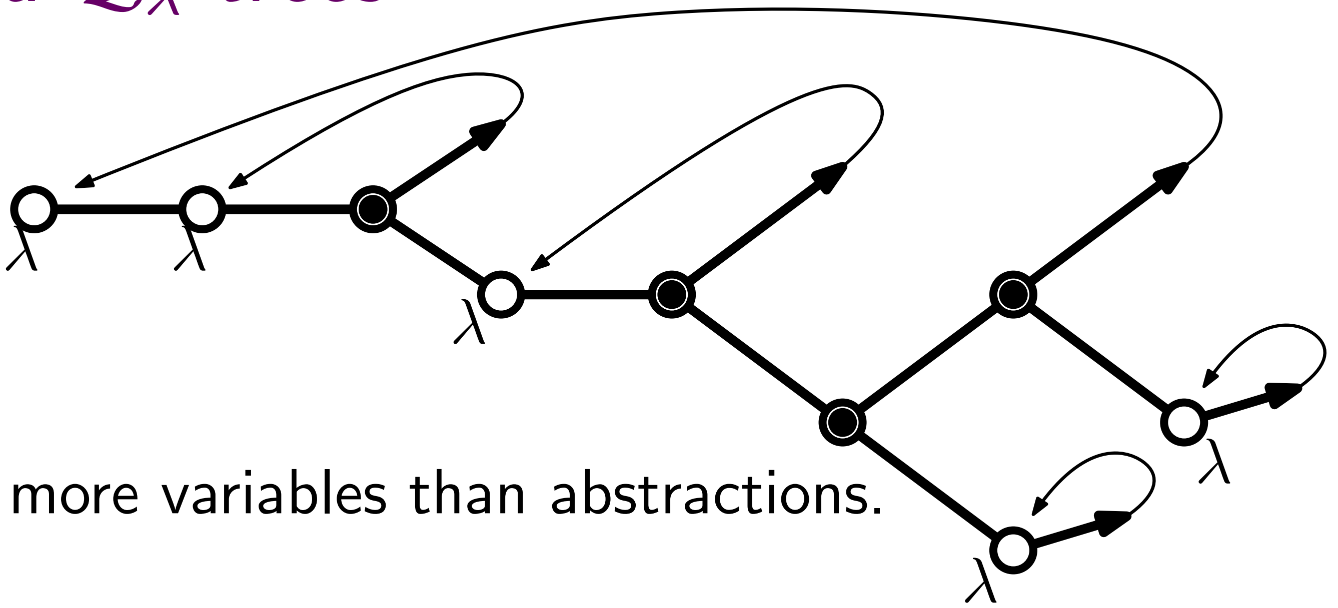
non negative $\mathcal{Q}_\lambda$-trees with excess 0 =
closed planar λ-terms

Planar λ-terms and $\mathcal{Q}_\lambda$-trees

Open planar λ-term are to plane trees with

- applications: binary nodes
- abstractions: unary nodes
- variables: leaves, represented as arrow.

with condition that in each subterm there are more variables than abstractions.



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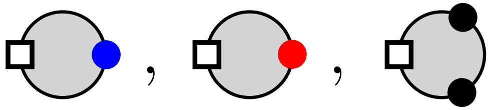
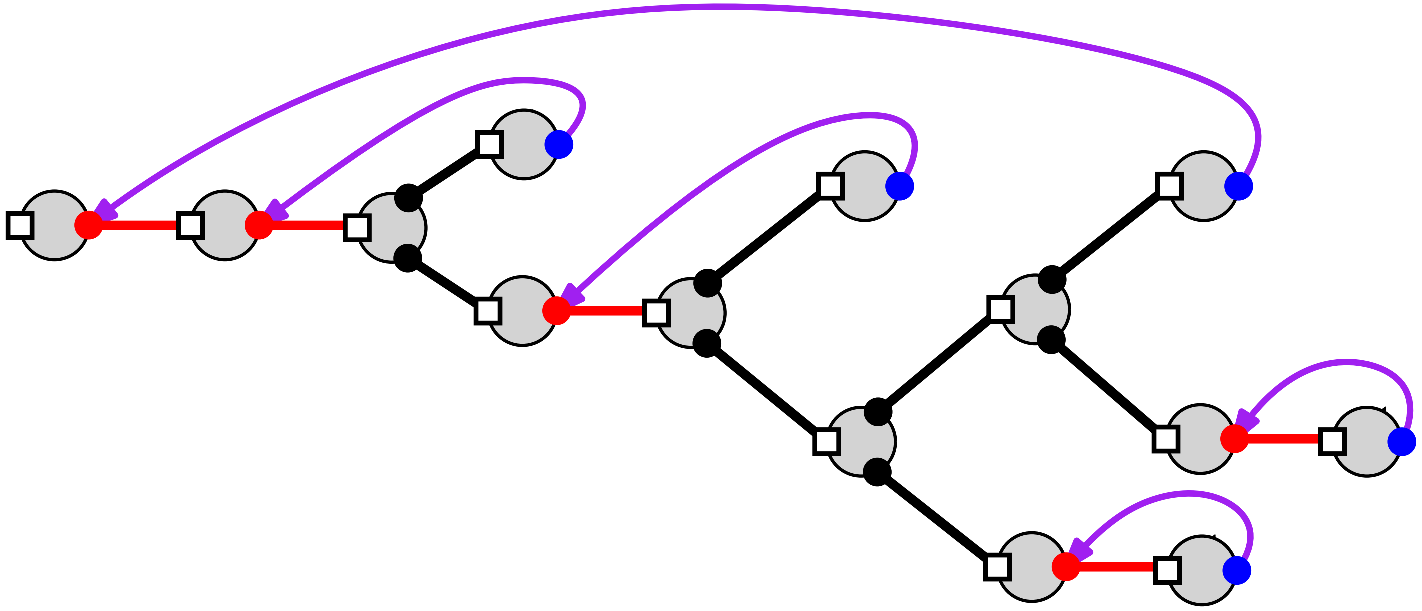
$\mathcal{Q}_\lambda = \{$  $\}$

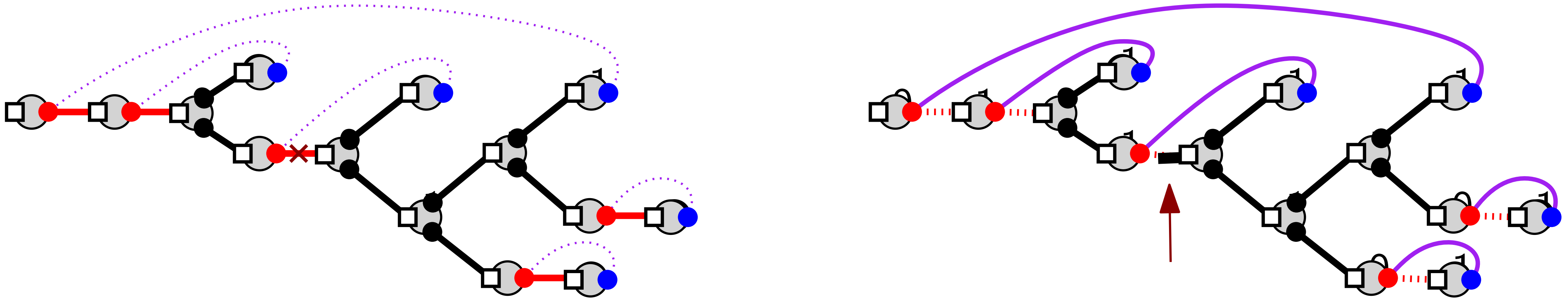


Then **non negative $\mathcal{Q}_\lambda$-trees =**
open planar λ-terms

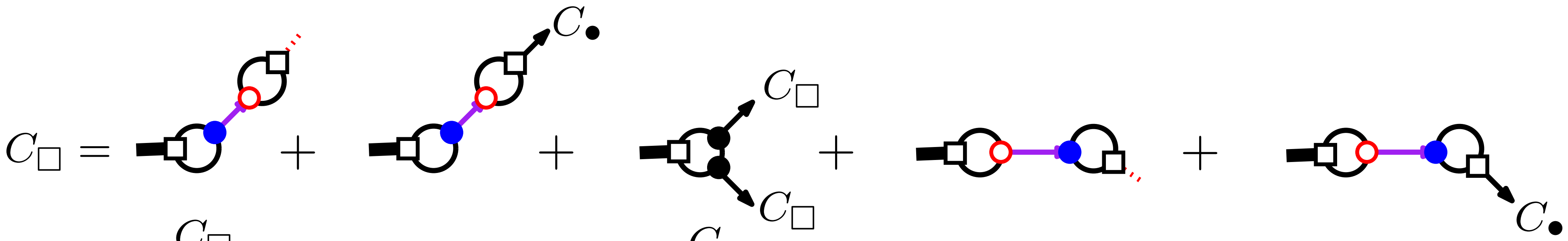
non negative $\mathcal{Q}_\lambda$-trees with excess 0 =
closed planar λ-terms

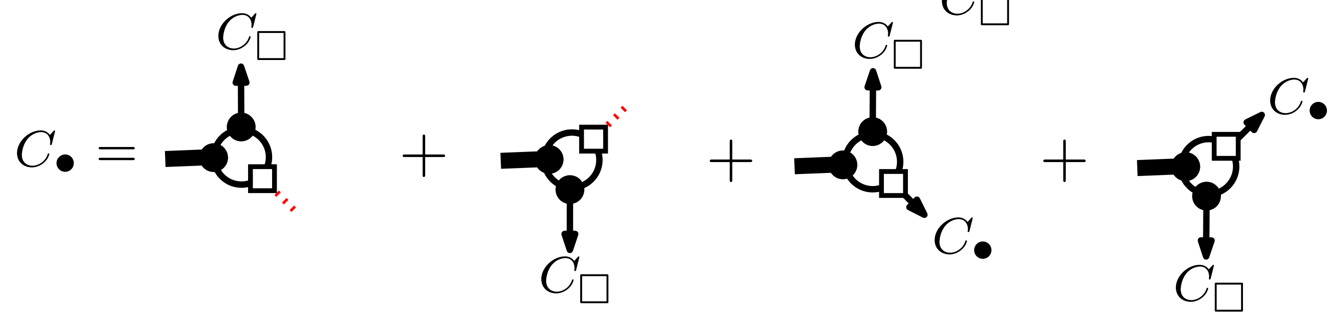
The closure corresponds to the planar abstraction-variable binding.

Planar λ-terms, closure and rewiring



Corollary.
Rewiring yields a size-preserving bijection between marked planar λ-terms and companion trees with context-free specification:





$$C_\square = \frac{2t^2}{1-2tC_\square} + tC_\square^2$$

What's next?

Catalytic equations also surface in various other enumeration problems, for instance for

- Families of pattern avoiding permutations (Zeilberger 92, Bona, Bousquet-Mélou, late 90's)
- Families of Tamari intervals (Chapoton, 2000's, Bousquet-Mélou-Chapoton 2022)
- Families of Planar (normal) λ-terms (Zeilberger and Giorgietti, 2015)
- Fighting fish and variants (Duchi et al, 2016)
- Fully parked trees (Chen 2021, Contat et al 2023)
- ...

Rewiring gives bijections with trees for these models...

$\Rightarrow$ but what are the *pizza slices* for these structures ?

Bijections allow to tackle new parameters...

$\Rightarrow$ so what is the equivalent of *distances in maps* for these structures ?

What's next? Higher order 1-catalytic equations

For order 1 we started from

$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u)-f), u\right),$ where $f \equiv f(t) = F(t,0)$.

and the $\mathbb{N}$-algebraic system
$$\begin{cases} V &= t \cdot Q(V,W,U) \\ R &= t \cdot (1+R) \cdot Q'_v(V,W,U) \\ U &= t \cdot (1+R) \cdot Q'_w(V,W,U) \\ W &= t \cdot (1+R) \cdot Q'_u(V,W,U) \end{cases}$$

For order k we need to deal with

$P\,(F(u), f_1, f_2, \ldots, f_k, u, t) = 0$ or $P(u) = Q(F(u), \Delta F(u), \ldots \Delta^k F(u), u, t)$

What's next? Higher order 1-catalytic equations

For order 1 we started from

$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u) - f), u\right)$, where $f \equiv f(t) = F(t, 0)$.

and the $\mathbb{N}$-algebraic system
$$\begin{cases} V & = & t \cdot Q(V, W, U) \\ R & = & t \cdot (1 + R) \cdot Q'_v(V, W, U) \\ U & = & t \cdot (1 + R) \cdot Q'_w(V, W, U) \\ W & = & t \cdot (1 + R) \cdot Q'_u(V, W, U) \end{cases}$$

For order k we need to deal with

$P\left(F(u), f_1, f_2, \ldots, f_k, u, t\right) = 0$ or $P(u) = Q(F(u), \Delta F(u), \ldots \Delta^k F(u), u, t)$

BMJ Theorem for order k equations lead to a system of $3k$ equations for $3k$ unknowns: the analogs $u_1, \ldots, u_k$ of the series u, the $F(u_1), \ldots, F(u_k)$ by F and the $f_1, \ldots, f_k$.

What's next? Higher order 1-catalytic equations

For order 1 we started from

$F(u) = t\,Q\left(F(u), \frac{1}{u}(F(u)-f), u\right),$ where $f \equiv f(t) = F(t,0)$.

and the $\mathbb{N}$-algebraic system
$$\begin{cases} V &= t \cdot Q(V,W,U) \\ R &= t \cdot (1+R) \cdot Q'_v(V,W,U) \\ U &= t \cdot (1+R) \cdot Q'_w(V,W,U) \\ W &= t \cdot (1+R) \cdot Q'_u(V,W,U) \end{cases}$$

For order k we need to deal with

$P\left(F(u), f_1, f_2, \ldots, f_k, u, t\right) = 0$ or $P(u) = Q(F(u), \Delta F(u), \ldots \Delta^k F(u), u, t)$

BMJ Theorem for order k equations lead to a system of $3k$ equations for $3k$ unknowns: the analogs $u_1, \ldots, u_k$ of the series u, the $F(u_1), \ldots, F(u_k)$ by F and the $f_1, \ldots, f_k$.

However this system is not immediately $\mathbb{N}$-algebraic

in fact the series u_i do not have non negative coefficients in general...

What's next? Higher order 1-catalytic equations

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This is making things harder: I think Emmanuel will indeed agree that

bijections are easier to find if one has a nice **and** complicated formula to interpret!

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So here is the plan...

The linear case: essentially the kernel method for 1d walks with arbitray up and down steps

$\rightarrow$ the kernel method works systematically for finite sets of steps (Bousquet-Mélou, around 2000)

$\rightarrow$ the corresponding generalized Dyck path admit a context-free specification (Duchon 1998)

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The non linear case: the resulting heuristic is to rewrite the BMJ systems in terms of the elementary functions in the u_i instead, and to avoid the $F(u_i)$, use the discriminant form of the sytem.

in progress: apply the combinatorial specification of the linear case along a branch and sort out the ugly details to see what comes out !

Thank you,

happy anniversary,

and long life to combinatorial physics !