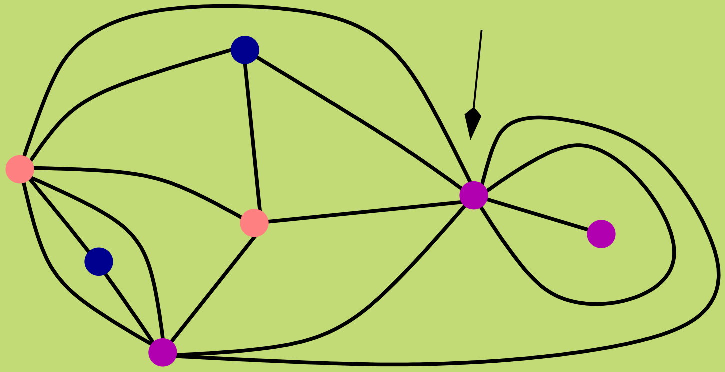
The 3-state Potts model on planar maps



Mireille Bousquet-Mélou
CNRS, LaBRI, Université de Bordeaux

Hadrien Notarantonio
IRIF, Université Paris Cité


slices
matrix
integrals
continued
fractions
KPZ
calculations
bijections

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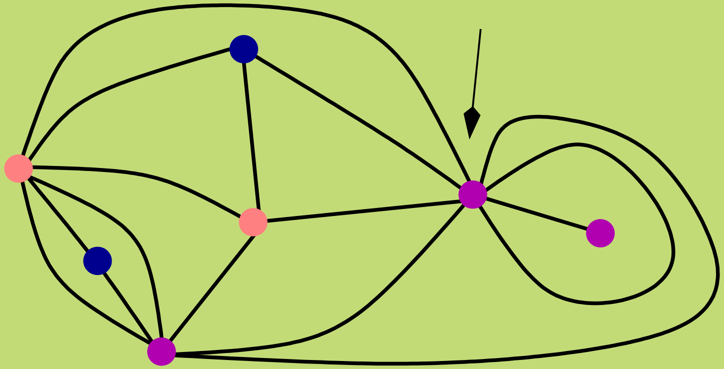

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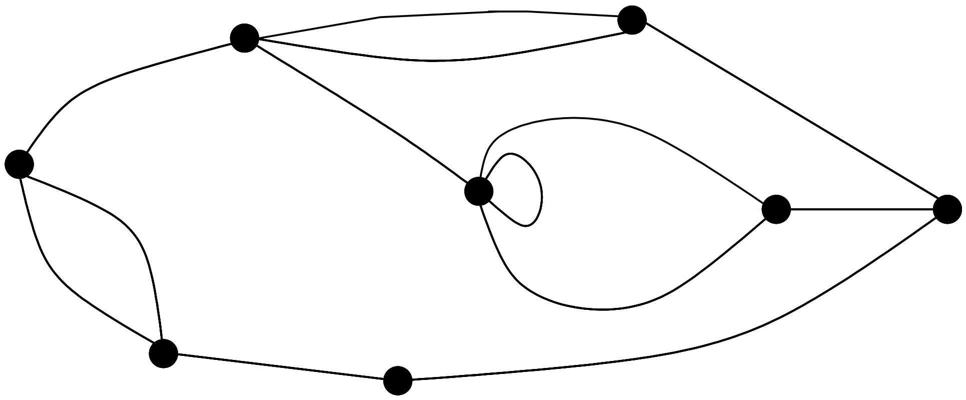
Planar maps

Def. A connected planar (multi)graph, given with a proper embedding in the plane, taken up to continuous deformation.

Components:

- vertices
- edges
- faces



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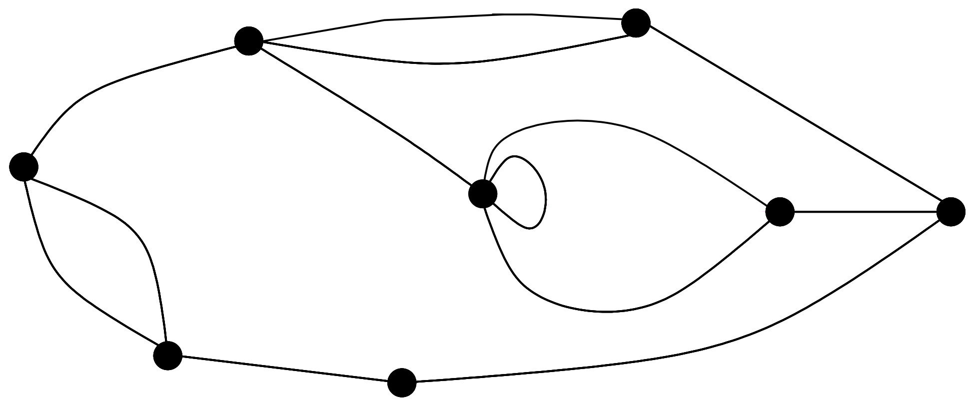

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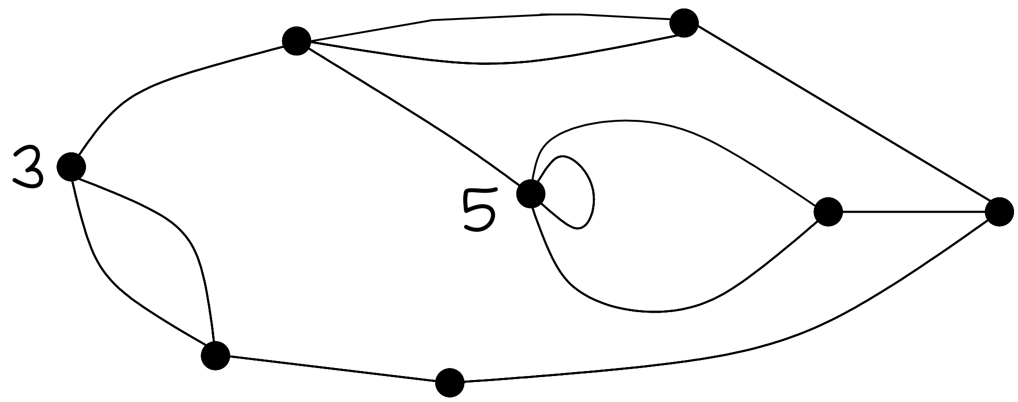

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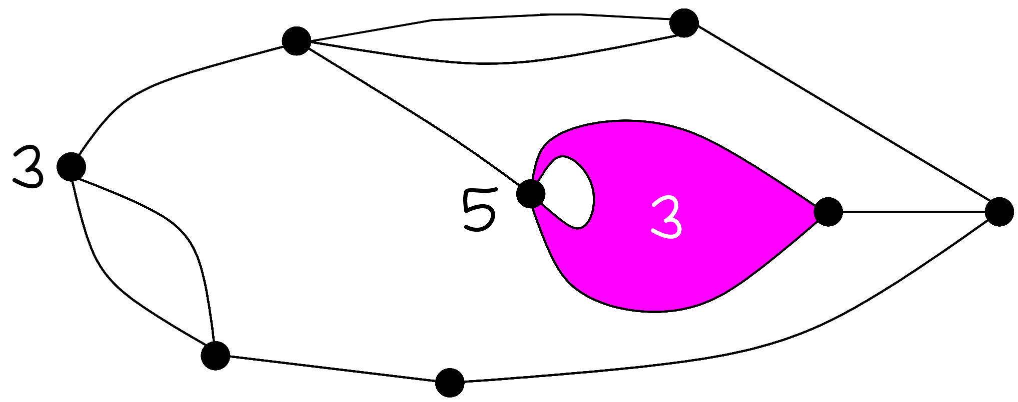

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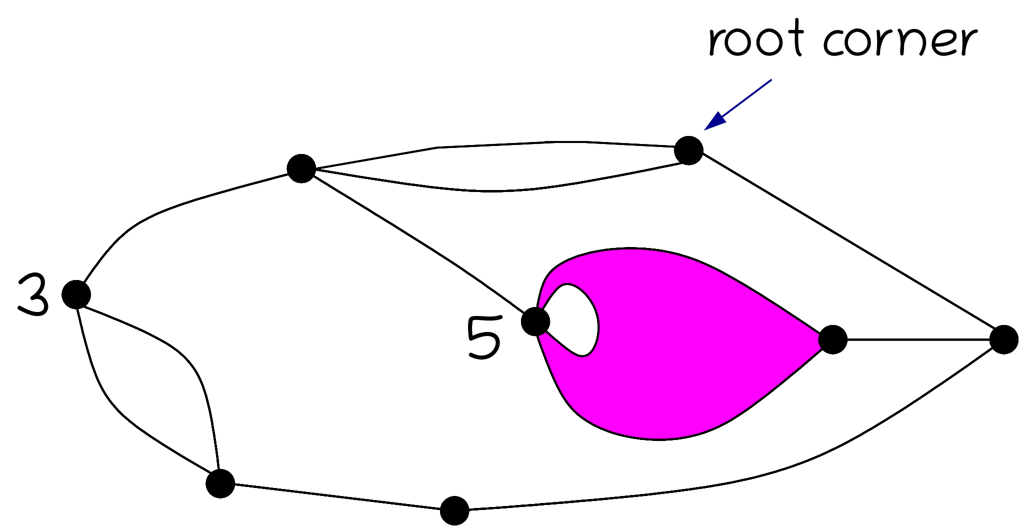

Rooted map: a distinguished corner in the outer face

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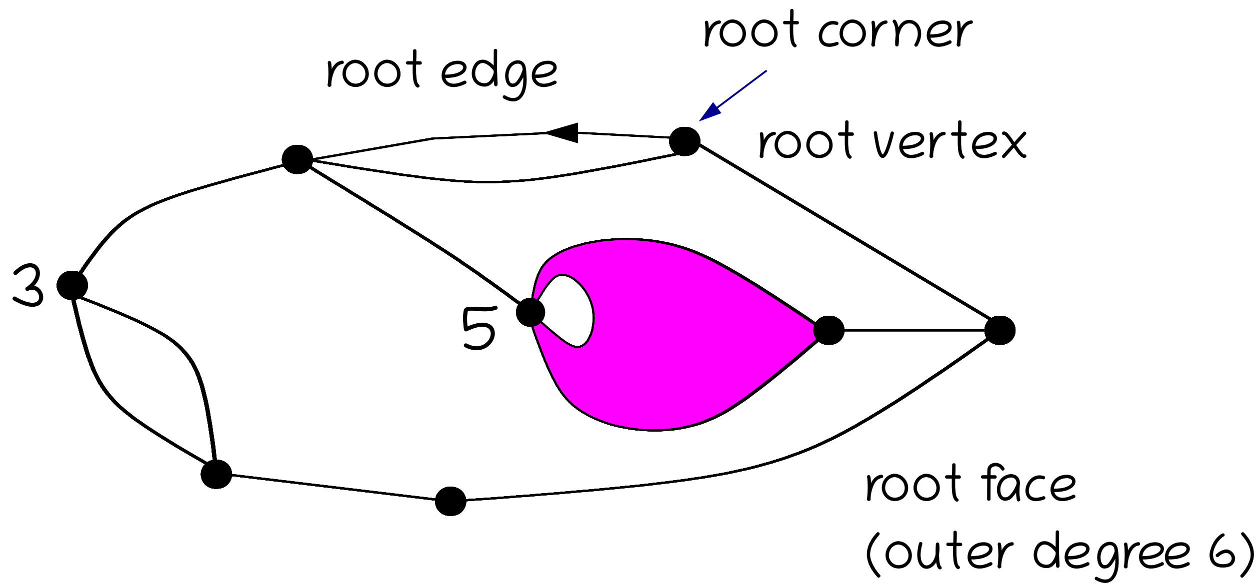

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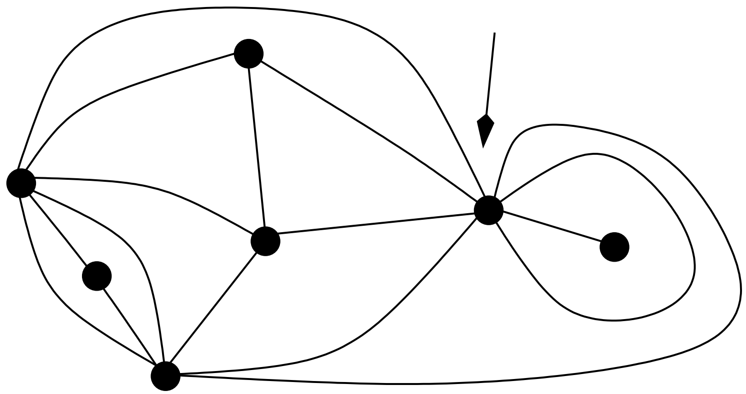

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Triangulation: all faces have degree 3

Near-triangulation: all **finite** faces have degree 3

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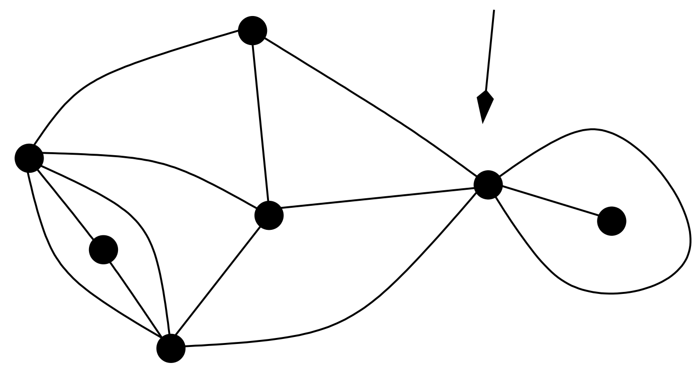

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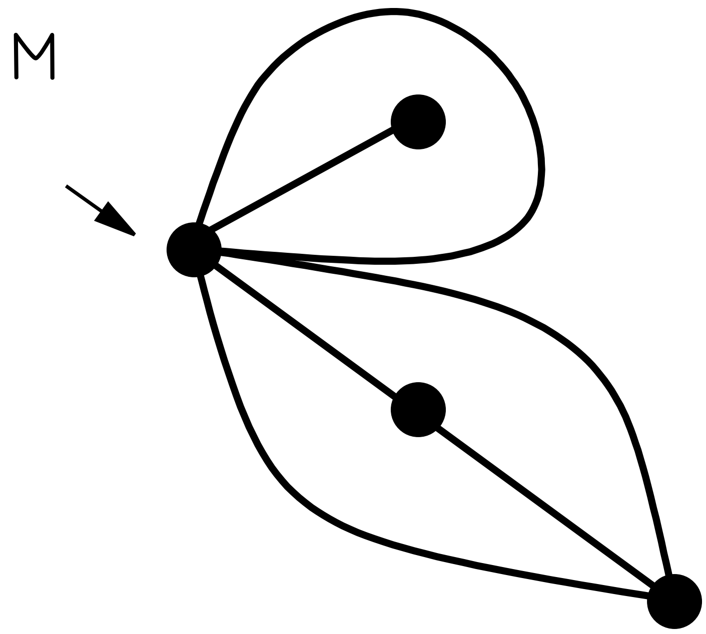
Duality

Exchange faces and vertices



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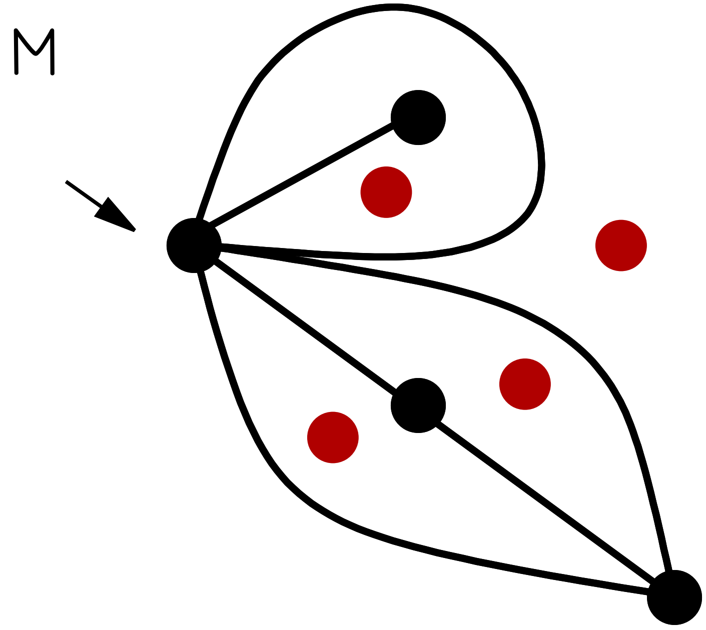

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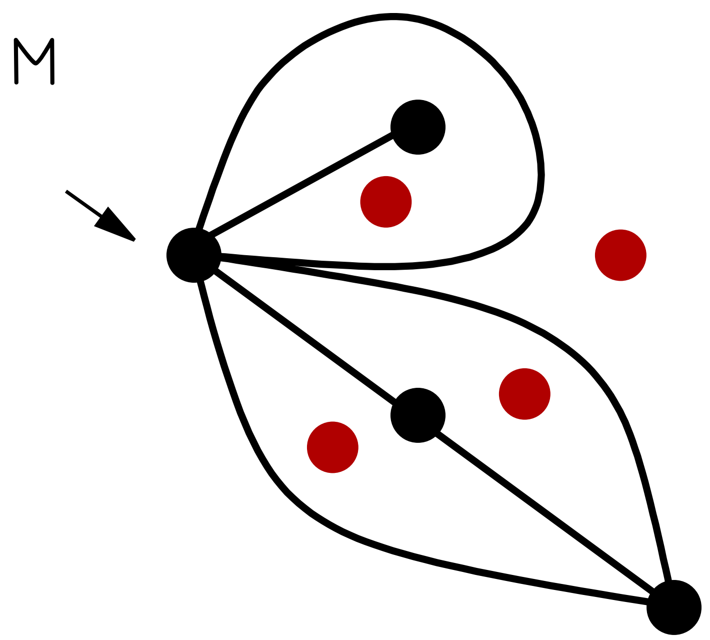

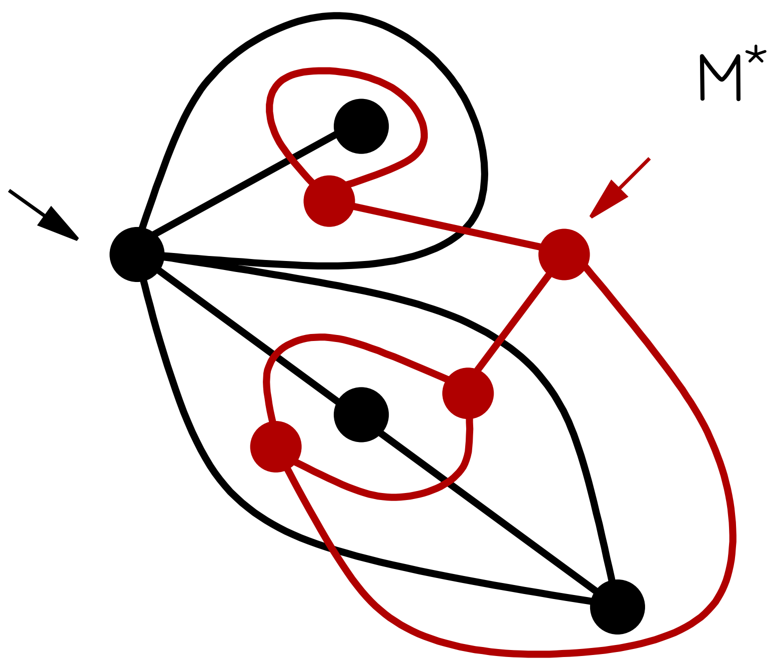

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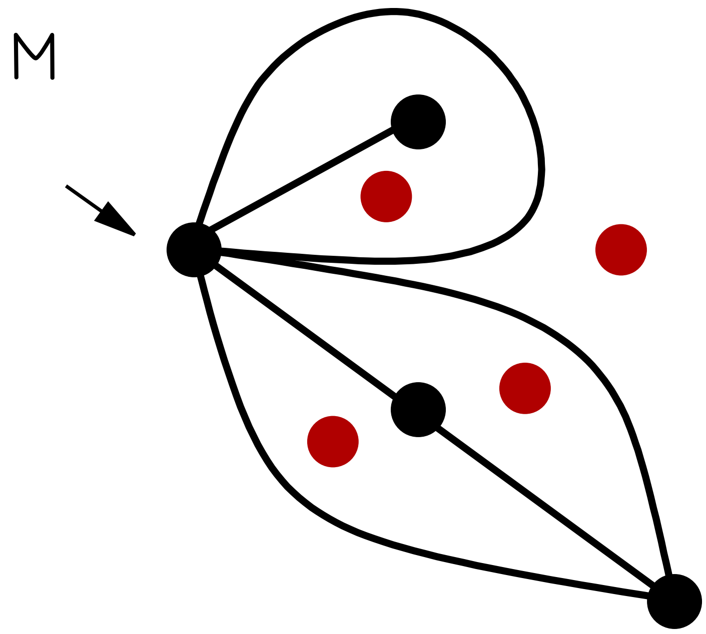
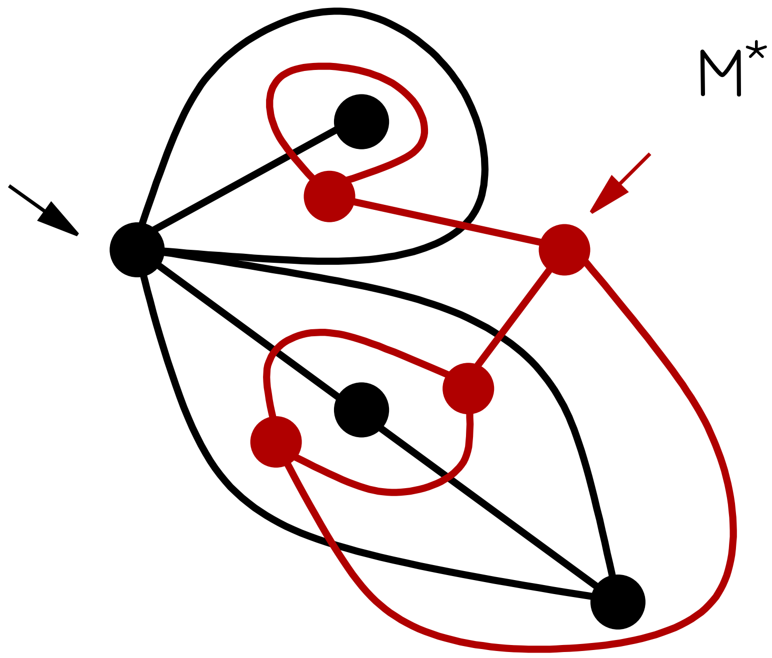


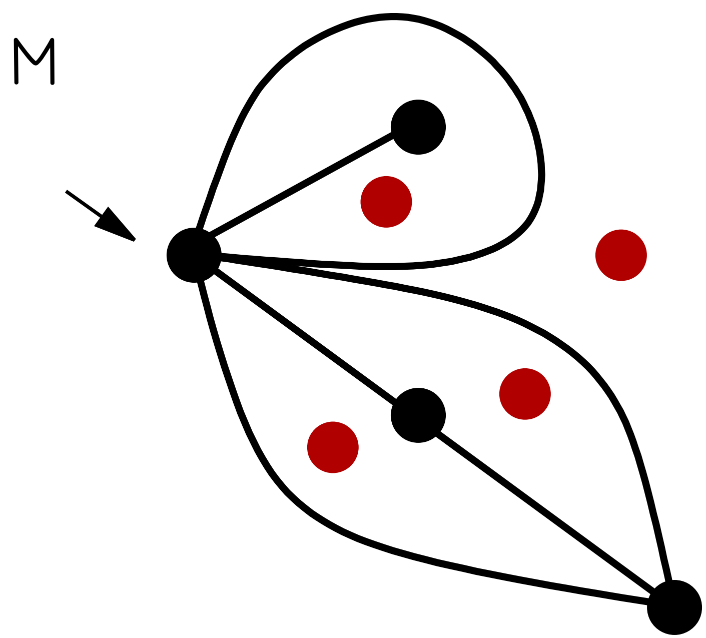
Triangulation

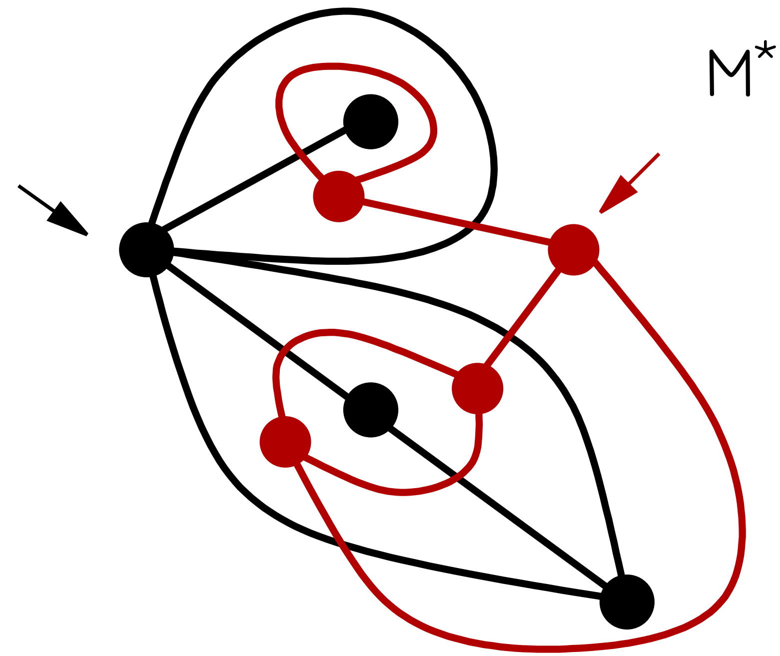


Cubic map

Duality

Exchange faces and vertices





Triangulation

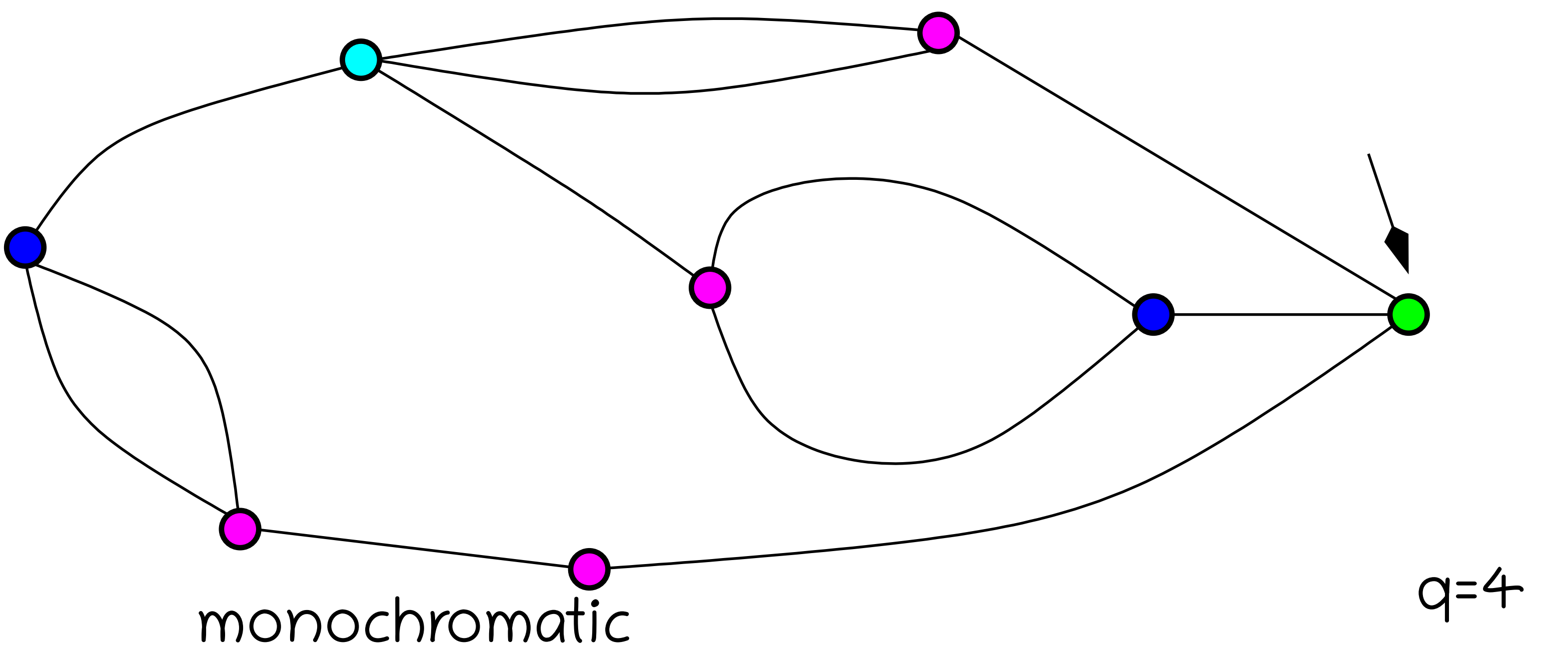
Near-triangulation

Cubic map

Near-cubic map

Vertex colourings of maps

Definition. Vertices are coloured in q colours



Proper colouring: neighbour vertices get different colours.
Potts model: a generalisation

Generating functions

- For a class of maps $\mathcal{C}$, equipped with some size (edge number...),

$$C := \sum_{M \in \mathcal{C}} t^{e(M)}.$$

- Multivariate versions, with more variables.

- The series C is **algebraic of degree k** if

$$P(C, t) = 0$$

for some irreducible polynomial P of degree k in its first variable.

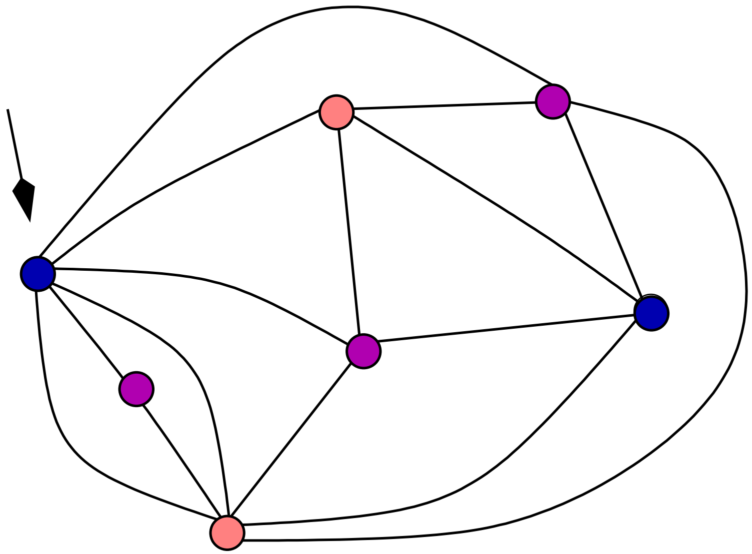
I. An old result, a conjecture, a new result

Properly 3-coloured triangulations

An old result [Tutte 63]

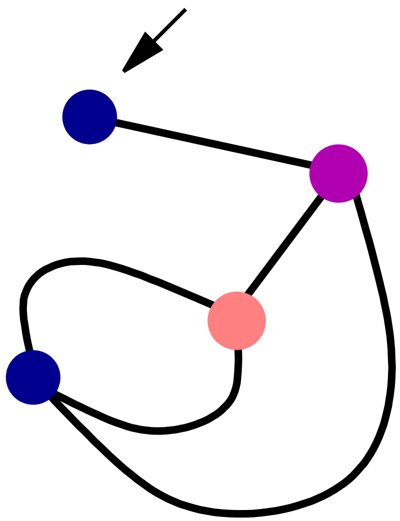
The generating function T_3 of properly 3-coloured triangulations (counted by vertices) is algebraic of degree 2:

$$18t^4 - 2t^3 + (24t^2 - 12t + 1)T_3 + 8T_3^2 = 0.$$

Properly 3-coloured cubic maps

A conjecture [Salvy~09, Bernardi-mbm 11]
The generating function C_1 of **properly 3-coloured near-cubic maps** of root degree 1 (counted by faces) is **algebraic of degree 11**.



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The generating function C_1 of **properly 3-coloured near-cubic maps** of root degree 1 (counted by faces) is **algebraic of degree 11**.

$$
\begin{aligned}
&9223372036854775808000000\, C_1^{11} + 9007199254740992\,(194560000\, t - 5971077)\, C_1^{10} \\
&+ 4294967296\,(280335535308800\, t^2 - 25398219177984\, t + 446991689475)\, C_1^{9} \\
&-1024\,(3799912185593856000000\, t^4 - 188284129271105978368\, t^3 + 7442656312099340288 0\, t^2 \\
&\qquad -3460024309515976704\, t + 60644726921050599)\, C_1^{8} \\
&-1024\,(8552566501857474641 92\, t^5 + 198557240861845880832\, t^4 + 7030700057733103616\, t^3 \\
&\quad -2005025500677518336\, t^2 + 6571937954614 7724\, t - 1261082394855783)\, C_1^{7} \\
&-64\,(13794761675403801133056\, t^6 + 1749420037224685109248\, t^5 - 2787711609861276958 72\, t^4 \\
&\quad +34432203597308620 80\, t^3 + 294527021649617744\, t^2 - 12400864344288084\, t + 586081179814293)\, C_1^{6} \\
&-16\,(3282933868861021224 9600\, t^7 - 5417040139462922731 52\, t^6 - 549137038895633924096\, t^5 \\
&\qquad +41876669882140680192\, t^4 - 936289577498747840\, t^3 \\
&\quad +12987916499676352\, t^2 + 208517314053540\, t - 54447680943015)\, C_1^{5} \\
&-32\,(124515522497539473408\, t^9 + 6242274275823592669184\, t^8 - 8988081837910576332 80\, t^7 \\
&\quad -52753292846413250 56\, t^6 + 653978506614911897 6\, t^5 - 361493662811609868\, t^4 \\
&\quad +9979948894517522\, t^3 - 432679480767965\, t^2 + 6248694091833\, t + 378858660750)\, C_1^{4} \\
&-8\,(747093134985236840448\, t^{10} + 5932367633073989222400\, t^9 - 1529736206124490686464\, t^8 \\
&\quad +1325858390725660508 16\, t^7 - 3048630269218258944\, t^6 - 1350875701987661 76\, t^5 \\
&\quad +5706147748413032\, t^4 - 229584590608200\, t^3 + 23755821897083\, t^2 - 152875558308\, t - 27738626328)\, C_1^{3} \\
&+(-336191910743356578 2016\, t^{11} - 6012198464670331305984\, t^{10} + 2332964327872863928320\, t^9 \\
&\quad -341248528343609901056\, t^8 + 24933054438553903104\, t^7 - 994662704339242816\, t^6 \\
&\qquad +33270083406272816\, t^5 - 1608971168541300\, t^4 + 7467003627448\, t^3 \\
&\qquad\quad +5037279798640\, t^2 - 194388001728\, t + 808501760)\, C_1^{2} \\
&+t\,(-840479776858391445504\, t^{11} - 1576185196591070576 64\, t^{10} + 157170928122096254976\, t^9 \\
&\quad -34691457904249143296\, t^8 + 3785139252232855552\, t^7 - 224694559056638912\, t^6 \\
&\qquad +6999136302319904\, t^5 - 197576502742812\, t^4 + 19551640345287\, t^3 \\
&\qquad\quad -1347626230088\, t^2 + 40099744688\, t - 404250880)\, C_1 \\
&-4\,t^4\,(196987447701185495 04\, t^9 - 802528937445320294 4\, t^8 + 1366977099830657024\, t^7 \\
&\quad -120213529404735488\, t^6 + 5234026490678784\, t^5 - 86995002866345\, t^4 \\
&\quad +4680668094111\, t^3 - 691486996440\, t^2 + 31610476208\, t - 404250880) = 0.
\end{aligned}
$$

Properly 3-coloured cubic maps

A new result [mbm-Notarantonio 25]

The generating function C_1 of **properly 3-coloured cubic maps** of root degree 1 (counted by faces) is **algebraic of degree 11**.

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Its derivative satisfies:

$$324t^2 = 655360\dot{C}_1^{11}+1245184\dot{C}_1^{10}+866304\dot{C}_1^{9}-80\left(8192t-1995\right)\dot{C}_1^{8}-2880\left(512t+49\right)\dot{C}_1^{7}$$
$$-504\left(2944t+219\right)\dot{C}_1^{6}-24\left(36640t+1383\right)\dot{C}_1^{5}-\left(16384t^2+334416t+3033\right)\dot{C}_1^{4}$$
$$-6\left(4096t^2+13584t-153\right)\dot{C}_1^{3}-9\left(1536t^2+1300t-33\right)\dot{C}_1^{2}-27\left(4t+1\right)\left(32t-1\right)\dot{C}_1.$$

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$$-6\left(4096t^2 + 13584t - 153\right)\dot{C}_1^{3} - 9\left(1536t^2 + 1300t - 33\right)\dot{C}_1^{2} - 27\,(4t + 1)\,(32t - 1)\,\dot{C}_1.$$

Moreover: the same holds for the GF that counts **all** 3-coloured near-cubic maps with a weight ν per monochromatic edge:

the 3-state Potts model on cubic maps

The q-state Potts model on planar maps

Definition. Let q be positive integer, M a map. The partition function of the (q-state) Potts model on M (or: **Potts polynomial** of M) is

$$P_M(q,\nu) := \sum_{c:V(M)\to\{1,\dots,q\}} \nu^{m(c)},$$

where m(c) is the number of monochromatic edges in the colouring c.

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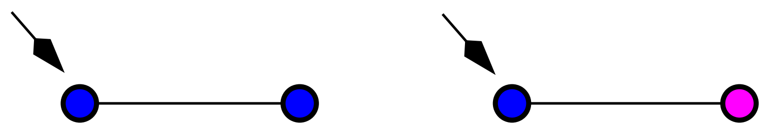
Example.

$$P_M(q,\nu) := q\nu + q(q-1).$$



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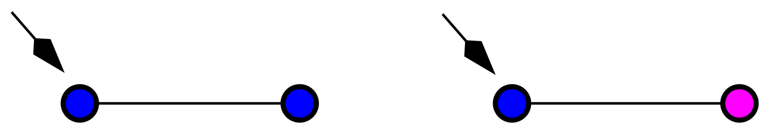

Properties

- polynomial in q and ν
- duality: for $q = (\nu-1)(\nu^*-1)$,

$$(\nu^*-1)^{f(M)-1} P_M(q,\nu) = (\nu-1)^{f(M^*)-1} P_{M^*}(q,\nu^*).$$

The Potts GF of near-triangulations

The Potts GF of (planar) near-triangulations is

$$T(y) \equiv T(q, \nu, t; y) = \sum_{M} P_M(q, \nu) t^{v(M)} y^{\mathrm{drf}(M)},$$

where the sum runs over all near-triangulations M and drf(M) is the degree of the root face.

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$$T(y) = \sum_{M,c} \nu^{m(c)} t^{v(M)} y^{\mathrm{drf}(M)},$$

where c is a q-colouring of the vertices of M.

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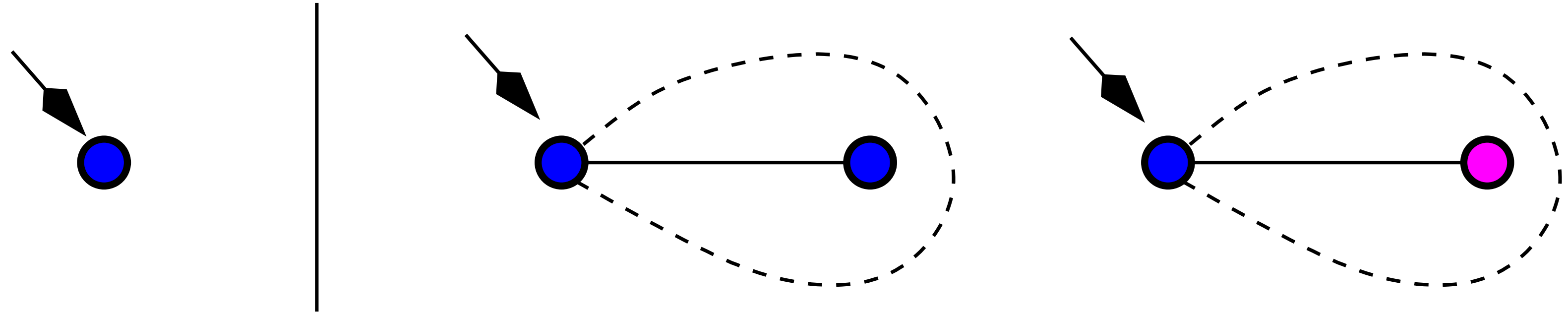
First coefficients:

$$T(y) = qt + yq\,(\nu + q - 1)\,(\nu + y)\,t^2 + \mathcal{O}(t^3).$$



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$$C(y) \equiv C(q, \nu, t; y) = \sum_{M} P_M(q, \nu) t^{f(M)} y^{\mathrm{drv}(M)},$$

where the sum runs over all near-cubic maps M and drv(M) is the degree of the root vertex.

Duality

Potts model

Near-triangulations — equiv. — Near-cubic maps

$T(q, \nu, t; y)$ $\qquad q = (\nu - 1)(\nu^* - 1) \qquad$ $C(q, \nu^*, t; y)$

Duality

Potts model

Near-triangulations — equiv. — Near-cubic maps

$T(q, \nu, t; y)$

$q = (\nu - 1)(\nu^* - 1)$

$C(q, \nu^*, t; y)$

$\nu = 0$

Near-triangulations

Proper colourings

Duality

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Near-triangulations

$T(q, \nu, t; y)$

equiv.

$q = (\nu - 1)(\nu^* - 1)$

Near-cubic maps

$C(q, \nu^*, t; y)$

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$\nu^* = 0$

Near-triangulations

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Potts model

Near-triangulations ⟷ equiv. ⟷ Near-cubic maps

$T(q, \nu, t; y)$

$$q = (\nu - 1)(\nu^* - 1)$$

$C(q, \nu^*, t; y)$

$\nu = 0$

$\nu = 1\text{-}q$

$\nu^* = 0$

Near-triangulations

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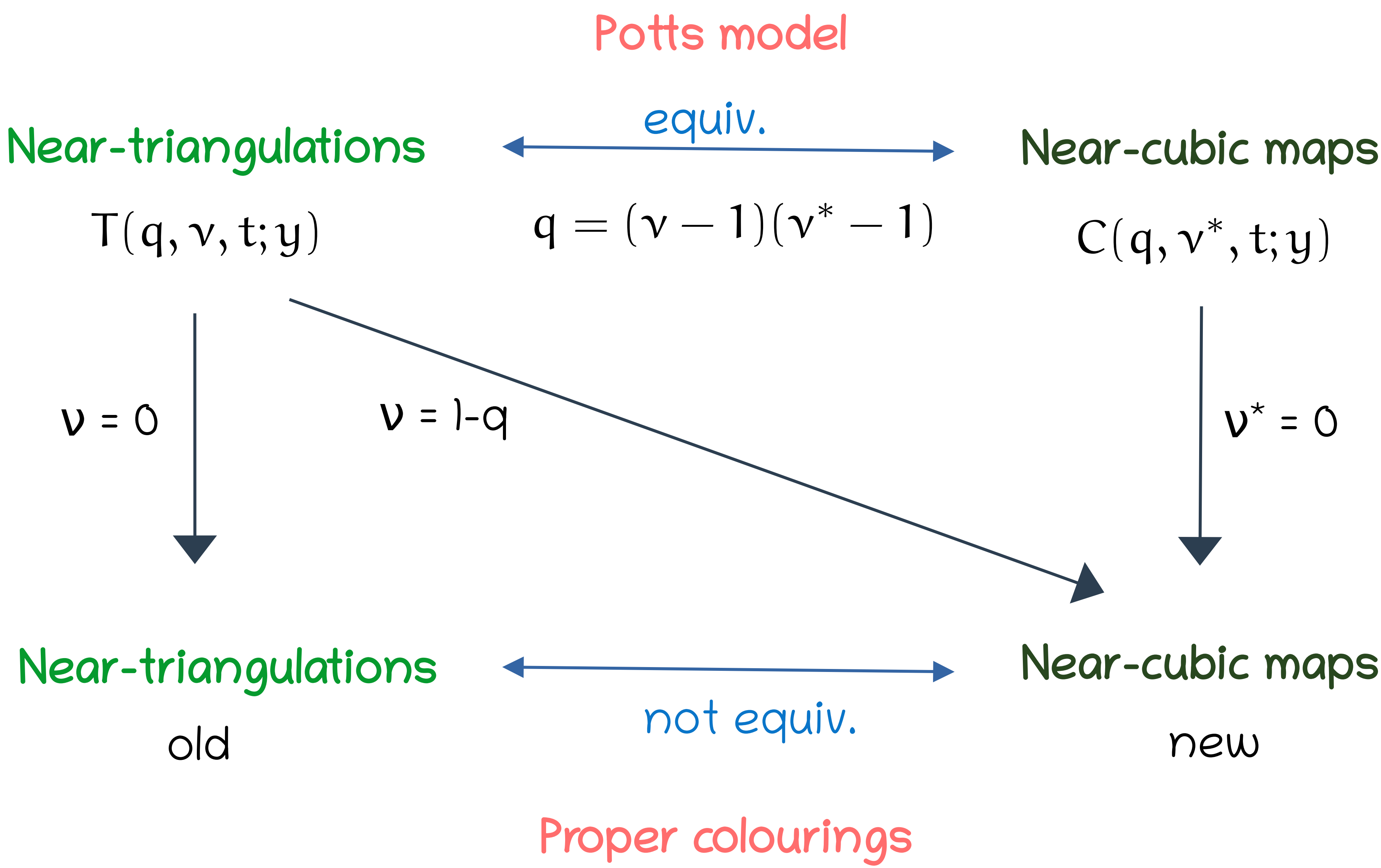

Potts model
Near-triangulations
$T(q, \nu, t; y)$
equiv.
$q = (\nu - 1)(\nu^* - 1)$
Near-cubic maps
$C(q, \nu^*, t; y)$
$\nu = 0$
$\nu = 1-q$
$\nu^* = 0$
Near-triangulations
old
not equiv.
Near-cubic maps
new
Proper colourings

A new result (3 colours) [mbm-Notarantonio 25]

Proposition. For any $i \geq 1$, the 3-Potts generating function T_i of near-triangulations of outer degree i is algebraic of degree 11.

A new result (3 colours) [mbm-Notarantonio 25]

Proposition. For any $i \geq 1$, the 3-Potts generating function T_i of near-triangulations of outer degree i is algebraic of degree 11.

Minimal polynomial of the derivative of T_1 (degree 2 in t):

$$\begin{aligned}
&276480\dot{T}_1^{11}\nu^7 - 27648\nu^6(31\nu+24)\dot{T}_1^{10} + 1152\nu^5\left(1021\nu^2+1678\nu+541\right)\dot{T}_1^9 \\
&\quad - 18\nu^4\left(46080\nu^3 t + 51935\nu^3 + 138243\nu^2 + 92253\nu + 17089\right)\dot{T}_1^8 \\
&+72\nu^3\left(1920\nu^3(17\nu+7)t + 6545\nu^4 + 25755\nu^3 + 26863\nu^2 + 10253\nu + 1144\right)\dot{T}_1^7 \\
&-4\nu^2\left(1008\nu^3\left(727\nu^2+586\nu+127\right)t + 38596\nu^5 + 219355\nu^4 + 322318\nu^3 + 190022\nu^2 + 43274\nu + 2915\right)\dot{T}_1^6 \\
&+4\nu\left(216\nu^3\left(2433\nu^3+2879\nu^2+1255\nu+153\right)t + 8027\nu^6 + 67626\nu^5 + 134820\nu^4 + 109109\nu^3 + 38007\nu^2\right. \\
&\left.+5103\nu + 188\right)\dot{T}_1^5 + \left(41472\nu^6(\nu-1)t^2 - 12\nu^3\left(78871\nu^4 + 122456\nu^3 + 80010\nu^2 + 19688\nu + 1375\right)t\right. \\
&\left.-3876\nu^7 - 53138\nu^6 - 145202\nu^5 - 151460\nu^4 - 71656\nu^3 - 14332\nu^2 - 958\nu - 18\right)\dot{T}_1^4 + \left(-13824\nu^5(5\nu+1)(\nu-1)t^2\right. \\
&+8\nu^2(5\nu+1)\left(6823\nu^4 + 11843\nu^3 + 9045\nu^2 + 2429\nu + 100\right)t + 208\nu^7 + 6088\nu^6 + 24600\nu^5 + 31836\nu^4 + 19256\nu^3 \\
&\left.+5040\nu^2 + 440\nu + 12\right)\dot{T}_1^3 + \left(1728\nu^4(\nu-1)(5\nu+1)^2 t^2 - 12\nu(3\nu+1)\left(1358\nu^5 + 2771\nu^4 + 2504\nu^3 + 868\nu^2\right.\right. \\
&\left.\left.+58\nu + 1\right)t - 312\nu^6 - 2401\nu^5 - 3747\nu^4 - 2821\nu^3 - 899\nu^2 - 78\nu - 2\right)\dot{T}_1^2 + \nu\left(-96\nu^2(\nu-1)(5\nu+1)^3 t^2\right. \\
&\left.+4(\nu+1)\left(1229\nu^5 + 2390\nu^4 + 2114\nu^3 + 697\nu^2 + 49\nu + 1\right)t + \left(104\nu^4 + 189\nu^3 + 177\nu^2 + 67\nu + 3\right)\right)\dot{T}_1 \\
&+2\nu^2(\nu-1)(5\nu+1)^4 t^2 - 2\nu^2(\nu+2)\left(104\nu^4 + 189\nu^3 + 177\nu^2 + 67\nu + 3\right)t = 0.
\end{aligned}$$

A new result (3 colours) [mbm-Notarantonio 25]

Proposition. For any $i \geq 1$, the 3-Potts generating function T_i of near-triangulations of outer degree i is algebraic of degree 11.

Minimal polynomial of the derivative of T_1 (degree 2 in t):

$$
\begin{aligned}
&276480\dot{T}_1^{11}\nu^7 - 27648\nu^6(31\nu+24)\dot{T}_1^{10} + 1152\nu^5\left(1021\nu^2+1678\nu+541\right)\dot{T}_1^9 \\
&\quad - 18\nu^4\left(46080\nu^3 t + 51935\nu^3 + 138243\nu^2 + 92253\nu + 17089\right)\dot{T}_1^8 \\
&+72\nu^3\left(1920\nu^3(17\nu+7)t + 6545\nu^4 + 25755\nu^3 + 26863\nu^2 + 10253\nu + 1144\right)\dot{T}_1^7 \\
&-4\nu^2\left(1008\nu^3\left(727\nu^2+586\nu+127\right)t + 38596\nu^5 + 219355\nu^4 + 322318\nu^3 + 190022\nu^2 + 43274\nu + 2915\right)\dot{T}_1^6 \\
&+4\nu\left(216\nu^3\left(2433\nu^3+2879\nu^2+1255\nu+153\right)t + 8027\nu^6 + 67626\nu^5 + 134820\nu^4 + 109109\nu^3 + 38007\nu^2\right. \\
&\left.+5103\nu+188\right)\dot{T}_1^5 + \left(41472\nu^6(\nu-1)t^2 - 12\nu^3\left(78871\nu^4 + 122456\nu^3 + 80010\nu^2 + 19688\nu + 1375\right)t\right. \\
&\left.-3876\nu^7 - 53138\nu^6 - 145202\nu^5 - 151460\nu^4 - 71656\nu^3 - 14332\nu^2 - 958\nu - 18\right)\dot{T}_1^4 + \left(-13824\nu^5(5\nu+1)(\nu-1)t^2\right. \\
&+8\nu^2(5\nu+1)\left(6823\nu^4 + 11843\nu^3 + 9045\nu^2 + 2429\nu + 100\right)t + 208\nu^7 + 6088\nu^6 + 24600\nu^5 + 31836\nu^4 + 19256\nu^3 \\
&\left.+5040\nu^2 + 440\nu + 12\right)\dot{T}_1^3 + \left(1728\nu^4(\nu-1)(5\nu+1)^2 t^2 - 12\nu(3\nu+1)\left(1358\nu^5 + 2771\nu^4 + 2504\nu^3 + 868\nu^2\right.\right. \\
&\left.\left.+58\nu+1\right)t - 312\nu^6 - 2401\nu^5 - 3747\nu^4 - 2821\nu^3 - 899\nu^2 - 78\nu - 2\right)\dot{T}_1^2 + \nu\left(-96\nu^2(\nu-1)(5\nu+1)^3 t^2\right. \\
&\left.+4(\nu+1)\left(1229\nu^5 + 2390\nu^4 + 2114\nu^3 + 697\nu^2 + 49\nu + 1\right)t + \left(104\nu^4 + 189\nu^3 + 177\nu^2 + 67\nu + 3\right)\right)\dot{T}_1 \\
&+2\nu^2(\nu-1)(5\nu+1)^4 t^2 - 2\nu^2(\nu+2)\left(104\nu^4 + 189\nu^3 + 177\nu^2 + 67\nu + 3\right)t = 0.
\end{aligned}
$$

Full series $T(y)$: degree 55

II. Proper 3-colourings: triangulations vs. cubic maps

$$18t^2 + (12t - 3)\dot{T}_3 + 2\dot{T}_3^2 = 0$$

$$324t^2 = 655360\dot{C}_1^{11} + 1245184\dot{C}_1^{10} + 866304\dot{C}_1^{9} - 80\,(8192t - 1995)\,\dot{C}_1^{8} - 2880\,(512t + 49)\,\dot{C}_1^{7}$$
$$-504\,(2944t + 219)\,\dot{C}_1^{6} - 24\,(36640t + 1383)\,\dot{C}_1^{5} - \left(16384t^2 + 334416t + 3033\right)\dot{C}_1^{4}$$
$$-6\left(4096t^2 + 13584t - 153\right)\dot{C}_1^{3} - 9\left(1536t^2 + 1300t - 33\right)\dot{C}_1^{2} - 27\,(4t + 1)\,(32t - 1)\,\dot{C}_1$$

What do these two results have in common?

The GF of properly 3-coloured triangulations is algebraic of degree 2

The GF of properly 3-coloured cubic maps is algebraic of degree 11

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- Algebraicity
- Universality class: number of maps $\sim \kappa\mu^n n^{-5/2}$

What do these two results have in common?

The GF of properly 3-coloured triangulations is algebraic of degree 2

$$\mu = 8$$

The GF of properly 3-coloured cubic maps is algebraic of degree 11

$$\begin{aligned} 0 = {} & 36622445679\mu^9 + 138511711692\mu^8 \\ & - 110121066732132\mu^7 + 2091641987340288\mu^6 \\ & - 12387476762689536\mu^5 - 25586598278489702 4\mu^4 \\ & + 435833605194566860 8\mu^3 - 2306758957375212748 8\mu^2 \\ & + 82199700398766292992\mu - 288230376151711744000 \end{aligned}$$

- Algebraicity
- Universality class: number of maps $\sim \kappa \mu^n n^{-5/2}$

How do these two results differ?

The GF of properly 3-coloured triangulations is algebraic of degree 2

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The GF of properly 3-coloured triangulations is algebraic of degree 2

$P(t, T_3)=0$ Genus 0

The GF of properly 3-coloured cubic maps is algebraic of degree 11

$Q(t, C_1)=0$ Genus 1

How do these two results differ?

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Well-understood combinatorics

- Bijection with bipartite maps

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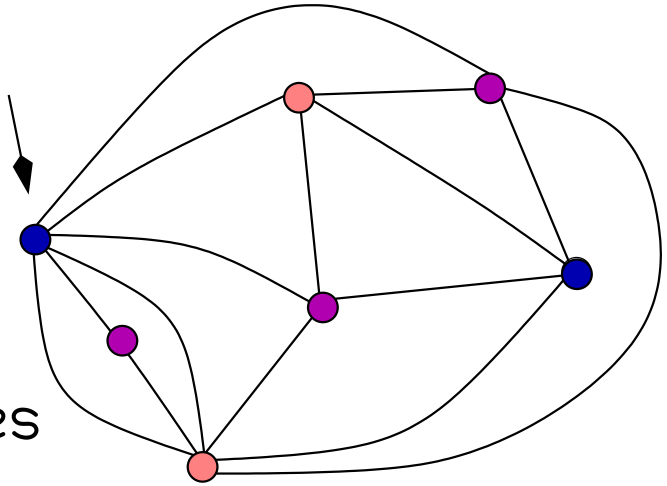

n vertices

How do these two results differ?

The GF of properly 3-coloured triangulations is algebraic of degree 2

$P(t, T_3) = 0$ Genus 0

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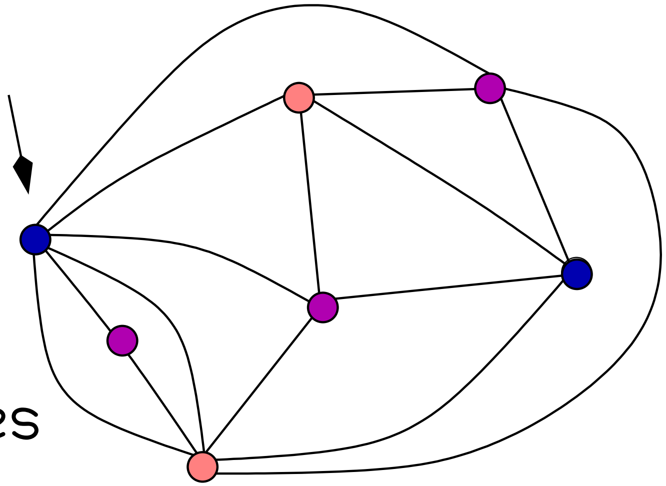
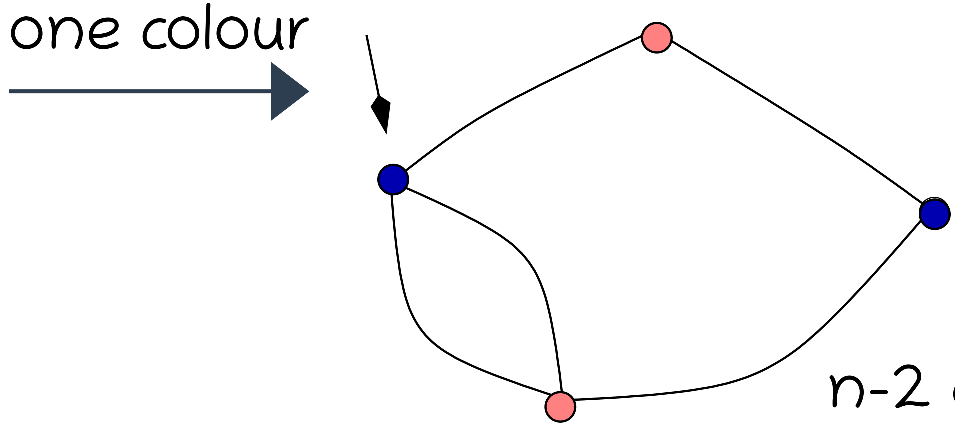


n vertices

delete one colour



n-2 edges

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The GF of properly 3-coloured cubic maps is algebraic of degree 11

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Well-understood combinatorics

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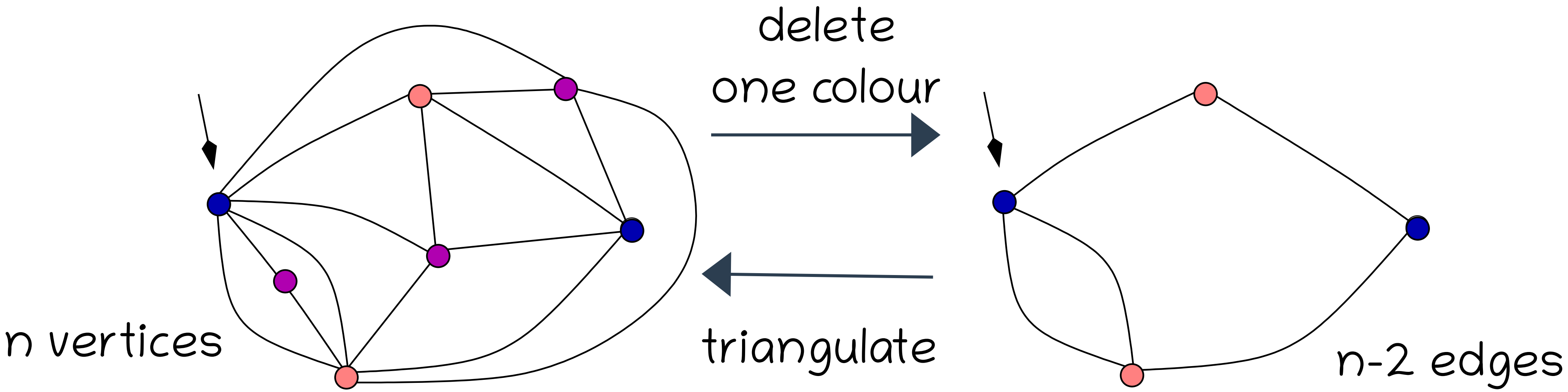

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- Bijections with **trees**

⇒ combinatorial explanation of algebraicity

[Schaeffer 98, Bouttier-Di Francesco-Guitter 02]

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Combinatorial explanation?

Every map has O(1) colourings

A random map has α^n colourings

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Every map has $O(1)$ colourings

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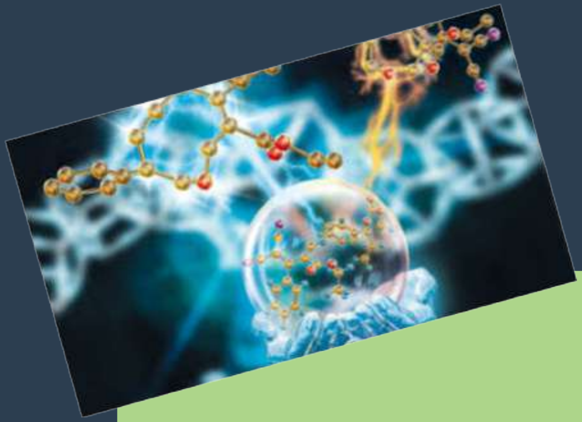
2-catalytic

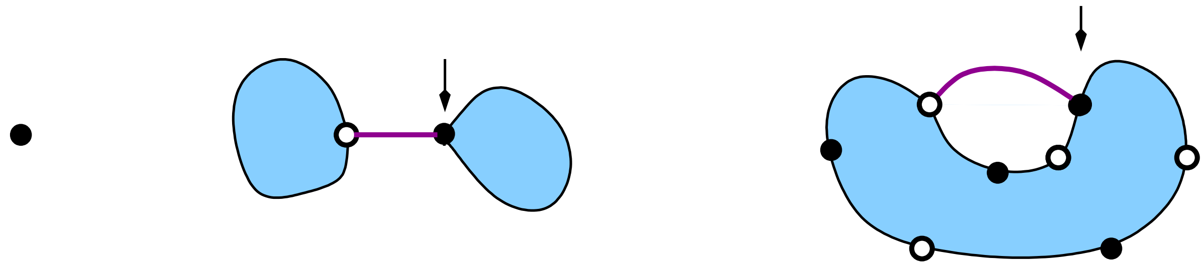


III. Equations, equations, equations

Properly 3-coloured triangulations are 1-catalytic

⇔ Bipartite maps counted by edges (t): series B

Tutte's approach: delete the root edge



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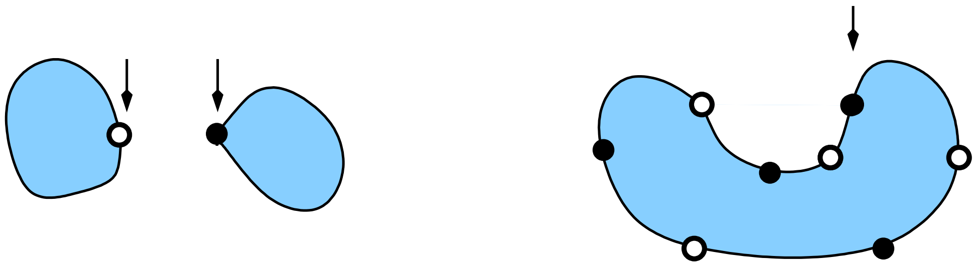

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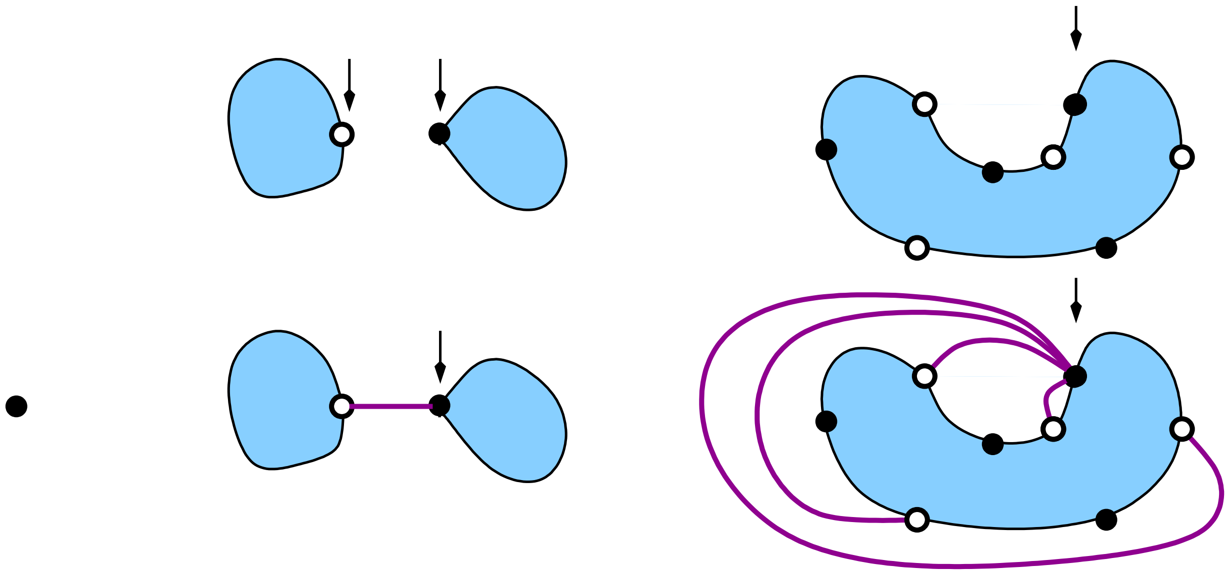

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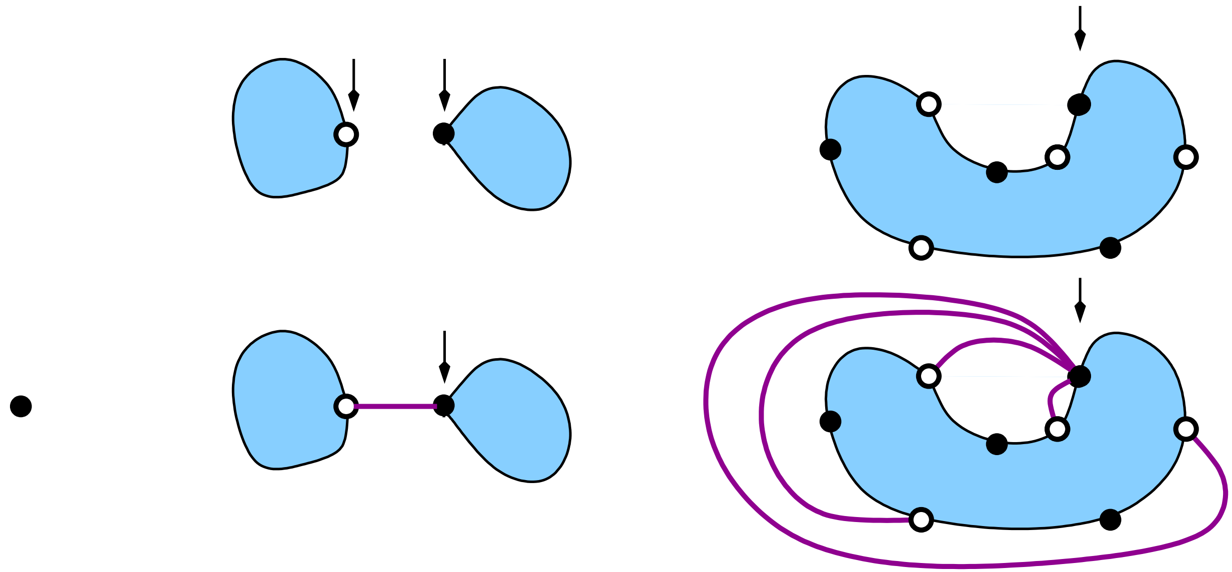

↪ **Record a new, "catalytic" statistic**: half the degree of the root face (new variable y)

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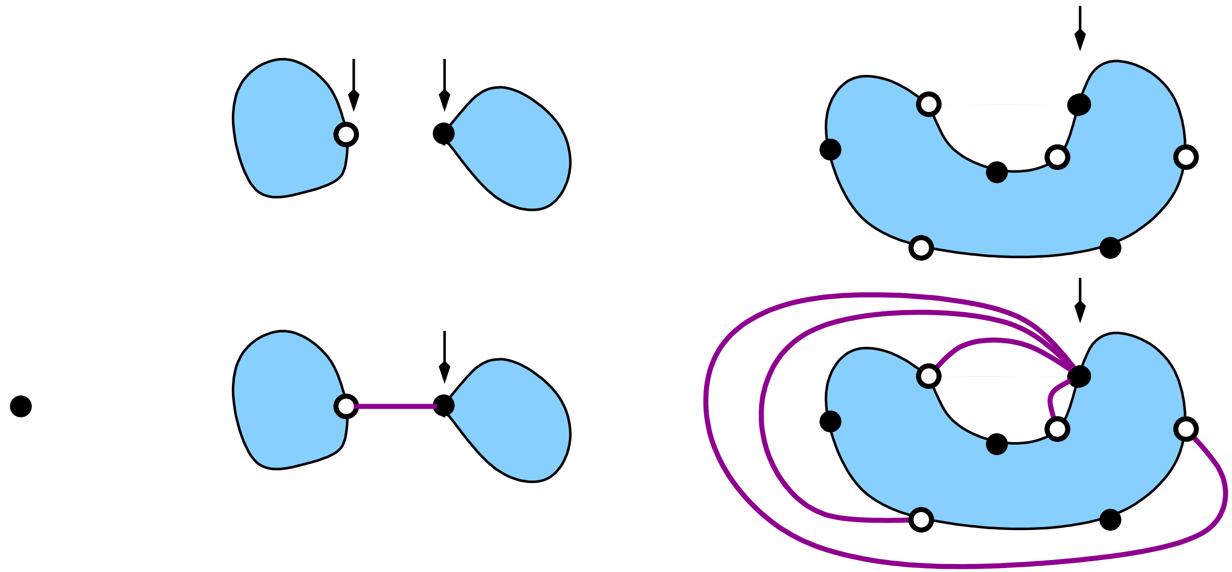

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$$B(y) = 1 + tyB(y)^2 + ty\frac{B(y) - B(1)}{y - 1}.$$

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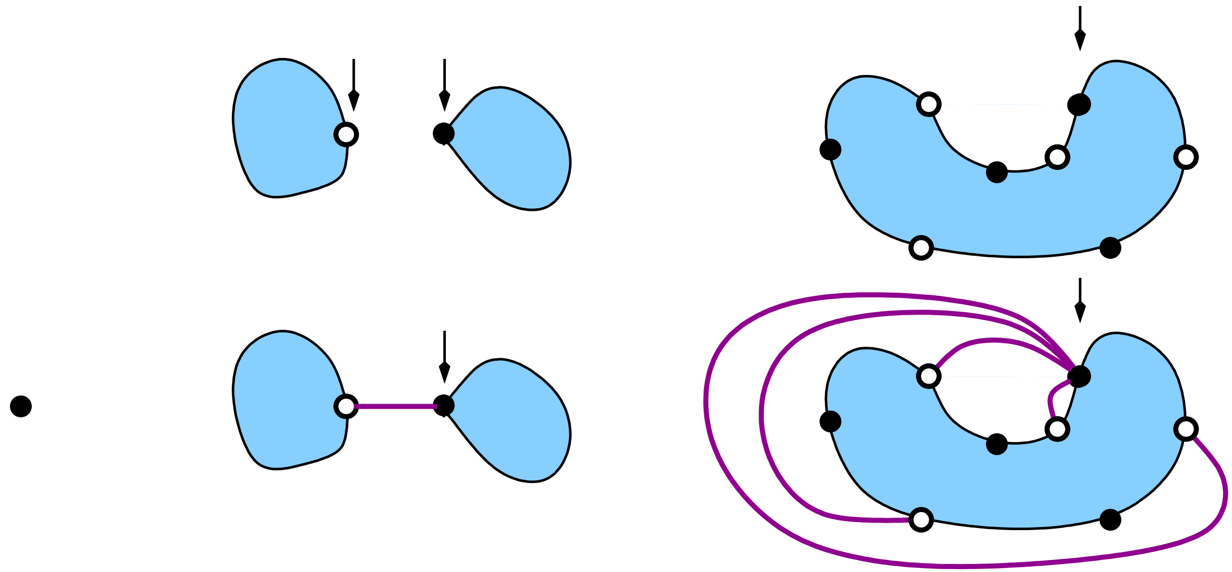

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An equation in one catalytic variable, y

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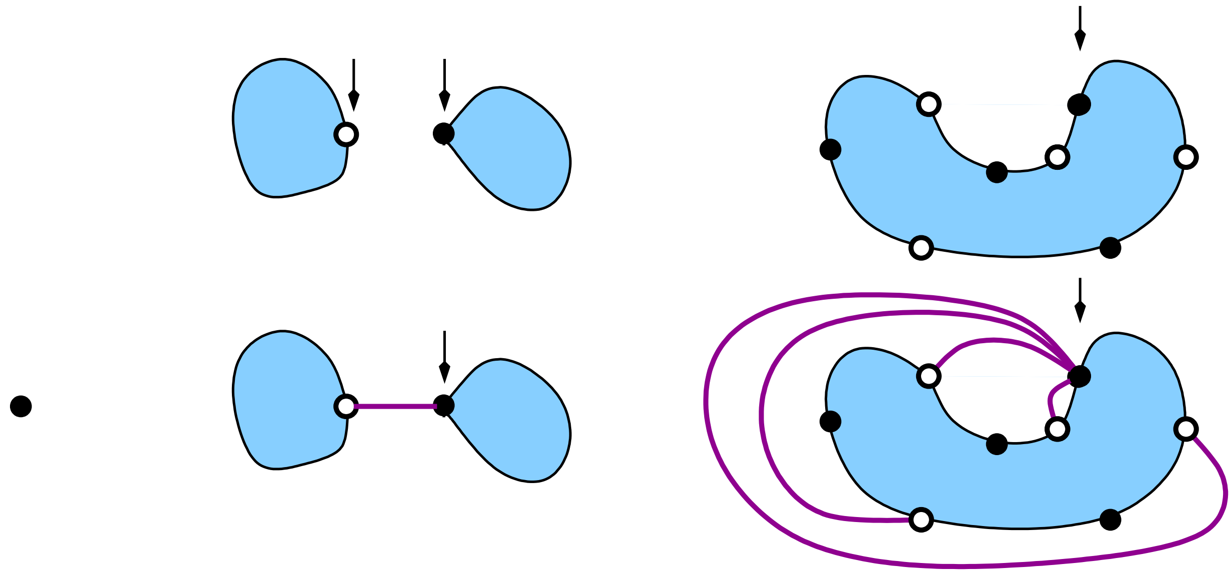

↪ **Record a new, "catalytic" statistic**: half the degree of the root face (new variable y)

$$\mathrm{Pol}(B(y), B(1), t, y) = 0$$

An equation in one catalytic variable, y

Properly 3-coloured cubic maps are 2-catalytic

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Why? Computing the Potts polynomial requires **deletion and contraction** of the root-edge e:

$$P_M(q, \nu) = P_{M\setminus e}(q, \nu) + (\nu - 1)P_{M/e}(q, \nu)$$

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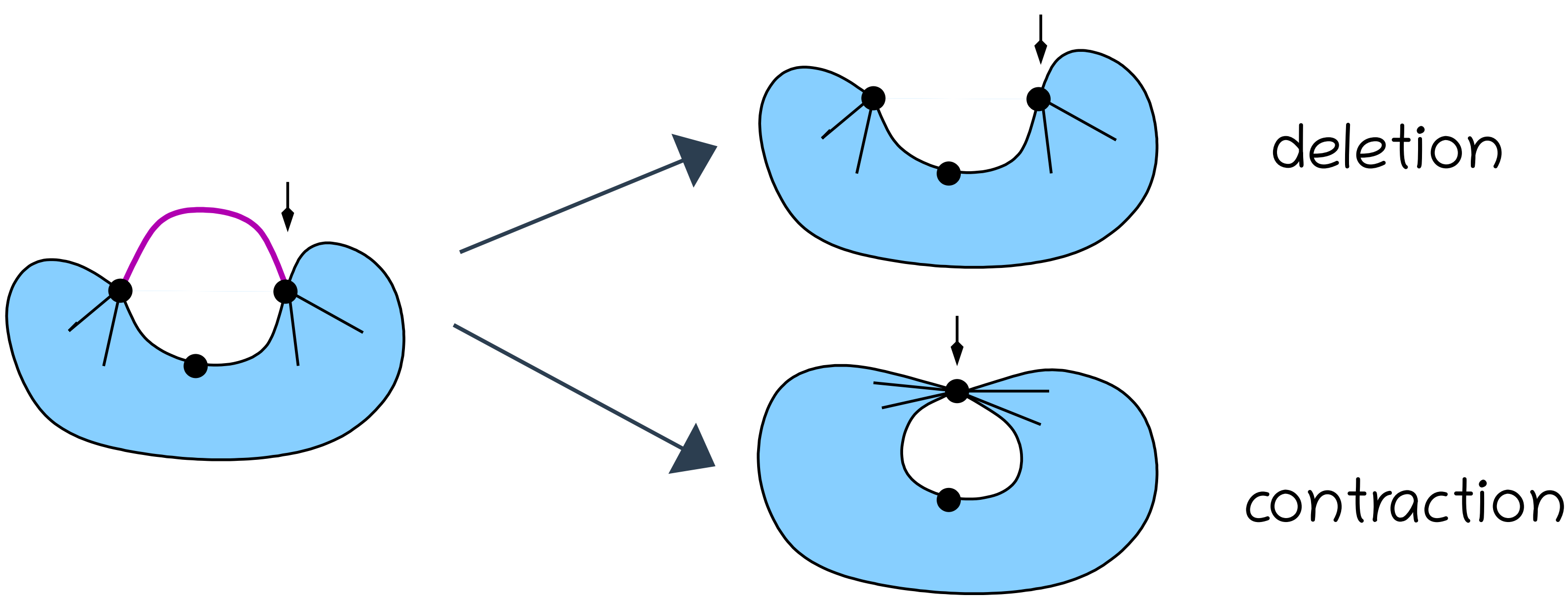

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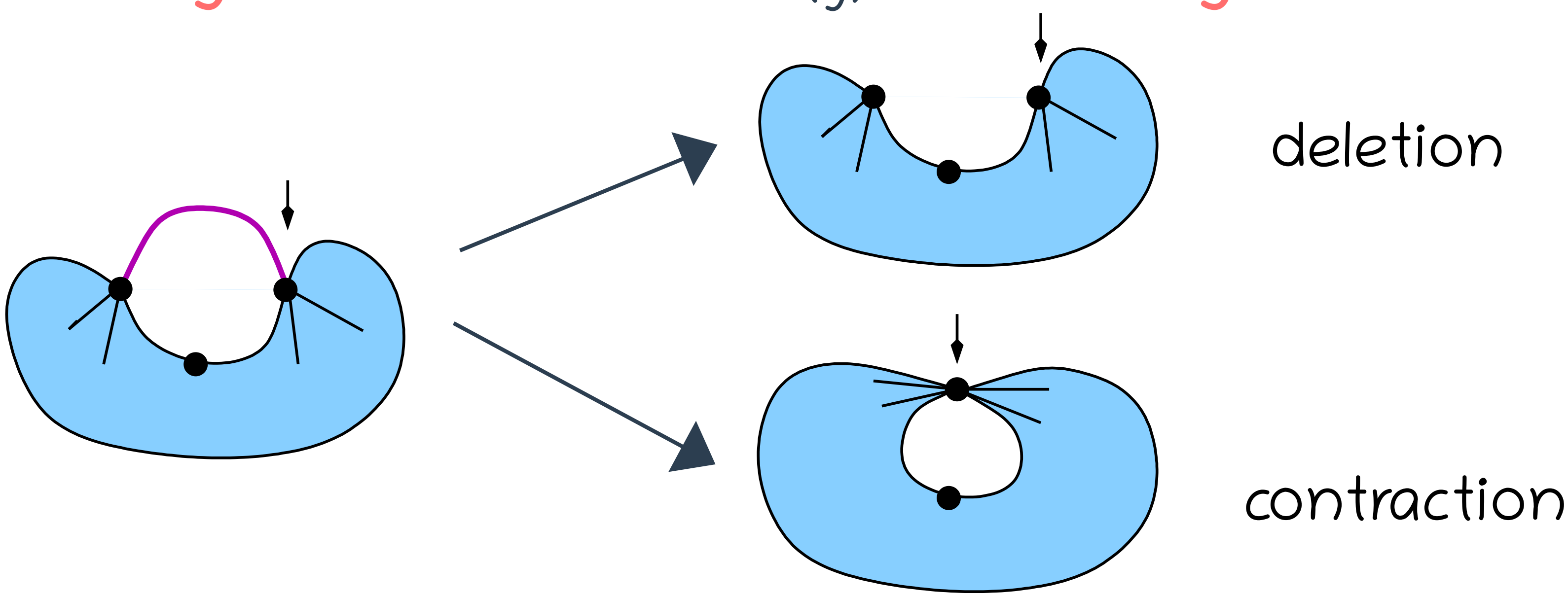
↪ Record the **degree of the root face** (y) and the **degree of the root vertex** (x)



The q-state Potts model on triangulations

Proposition. Let Q(x,y)=Q(q, ν, t; x,y) be the only formal series in t satisfying

$$Q(x,y) = 1 + t\,\frac{Q(x,y) - 1 - yQ_1(x)}{y} + xt(Q(x,y) - 1) + xytQ_1(x)Q(x,y)$$
$$+ yt(\nu - 1)Q(x,y)(2xQ_1(x) + Q_2(x)) + y^2t\left(q + \frac{\nu - 1}{1 - xt\nu}\right)Q(0,y)Q(x,y)$$
$$+ \frac{yt(\nu-1)}{1 - xt\nu}\,\frac{Q(x,y) - Q(0,y)}{x}$$

with

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[Bernardi-mbm 11]

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Then Q(0,y)=T(y) is the q-Potts GF of near-triangulations.

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Then Q(0,y) :=T(y) is the q-Potts GF of near-triangulations.

Proposition. When $q \neq 0,4$ is of the form $4\cos(k\pi/m)^2$, the series Q(0,y)=T(y) also satisfies an equation in **one** catalytic variable (y).

Includes q=2 (Ising), q=3.

[Bernardi-mbm 11]

3-Potts on near-triangulations is (also) 1-catalytic

Proposition. Take q=3. There exists an explicit polynomial such that

$$\mathrm{Pol}(T(y), T_1, T_3, T_5, T_7, \nu, t, y) = 0$$

where

$$T_i = [y^i]T(y)$$

is the contribution of near-triangulations with root-face of degree i.

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$$\begin{aligned}
0 = {} & 78732\,(\nu-1)^2\,\nu^5 y^{12} T(y)^5 \\
& + 729\,(\nu-1)^2\,\nu^3 y^9 \left(37\nu^2y^2 - 108\nu^2 y - 20\nu\, y^2 + 144\nu^2 - 36y\nu - 17y^2\right) T(y)^4 \\
& - 54\,(\nu-1)^2\,\nu\, y^6 \left(486\, T_1\, \nu^4 y^4 - 405\nu^4 t\, y^4 + 486\nu^4 t\, y^3 - 56\nu^4 y^4 - 81\nu^3 t\, y^4 \right. \\
& + 342\nu^4 y^3 + 53\nu^3 y^4 - 1044\nu^4 y^2 + 18\nu^3 y^3 + 60\nu^2 y^4 + 1458\nu^4 y - 99\nu^3 y^2 \\
& \left. - 333\nu^2 y^3 - 55\nu\, y^4 - 972\nu^4 + 486\nu^3 y + 171\nu^2 y^2 - 27\nu\, y^3 - 2y^4\right) T(y)^3 + \cdots
\end{aligned}$$

No combinatorial explanation

[Bernardi-mbm 11]

One-catalytic implies algebraic !

Theorem [Popescu 86]

If a system of polynomial equations of the form

$$\mathrm{Pol}_i(S_1(t;y),\dots,S_j(t;y),A_1(t),\dots,A_k(t),t,y)=0$$

with coefficients in some field $\mathbb{F}$ has a **unique solution** $S_1(t;y)$, ..., $S_j(t;y)$, $A_1(t)$, ..., $A_k(t)$ in **formal power series**, then all these series are **algebraic** over $\mathbb{F}(t,y)$.

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Proposition [mbm-Jehanne 06]

Same result for a single equation of a "proper" type...

+ effective procedure.

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- Extension to systems [Notarantonio-Yurkevich 23(a)]

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Proposition. Let q=3. There exists an explicit polynomial such that

$$\mathrm{Pol}(T(y), T_1, T_3, T_5, T_7, \nu, t, y) = 0 \qquad (1)$$

where

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Corollary. The 3-Potts GF of near-triangulations T(y) is algebraic.

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Now a solution...
What happened?

IV. Some tools

General approach to 1-catalytic equations [mbm-AJ]

Consider the 1-catalytic equation

$$\mathrm{Pol}(S(y), A_1, A_2, A_3, A_4, t, y) = 0.$$

Theorem: Let $\Delta(a_1, a_2, a_3, a_4, t, y)$ be the discriminant of $\mathrm{Pol}(s, a_1, a_2, a_3, a_4, t, y)$ in its first variable.

Then, as a polynomial in y, $\Delta(A_1, A_2, A_3, A_4, t, y)$ has 4 double roots Y_1, Y_2, Y_3, Y_4.

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⇒ system of 8 polynomial equations for $A_1, A_2, A_3, A_4, Y_1, Y_2, Y_3, Y_4$

$$\Delta(A_1, A_2, A_3, A_4, t, Y_i) = \partial_y \Delta(A_1, A_2, A_3, A_4, t, Y_i) = 0, \quad i = 1 \ldots 4.$$

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- **3-Potts on near-triangulations:** $\Delta(T_1, T_3, T_5, T_7, t, y)$ has degree 26 in y, total degree 10 in the T_i's.

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Theorem [Bernardi-mbm 15] There exist two polynomials $D_+(T_1, T_3, T_5, T_7, t, u)$ and $D_-(T_1, T_3, T_5, T_7, t, u)$, of degree 5 and 6 in u respectively, degree 2 in the T_i's, that have each 2 double roots in u (U_1, U_2 and U_3, U_4).

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Elimination via resultants ⇒ each T_i has degree 11

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Solution for 3-Potts on near-triangulations

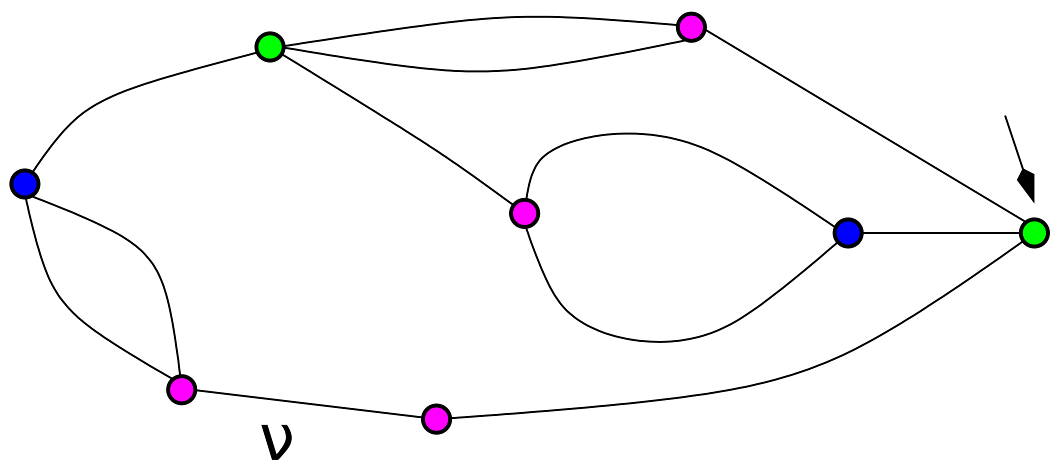
The case of general planar maps

Proposition. The 3-Potts generating function $M_1(\nu,t)$ of **general planar maps** is algebraic of **degree 22**, with an explicit minimal polynomial.

[mbm-Notarantonio 25]

Genus 4...

- Same starting point with D_+, D_-
- Alternative solution technique



V. Asymptotics

Asymptotics for 3-Potts on near-triangulations

Proposition. Fix $\nu > 0$. The 3-Potts GF T_1 of near-triangulations of outer degree 1 has radius of convergence ρ_ν where

$$\Delta_1(\nu, \rho_\nu) = 0 \quad \text{for} \quad 0 < \nu \leq \nu_c := 1 + 3/\sqrt{47},$$
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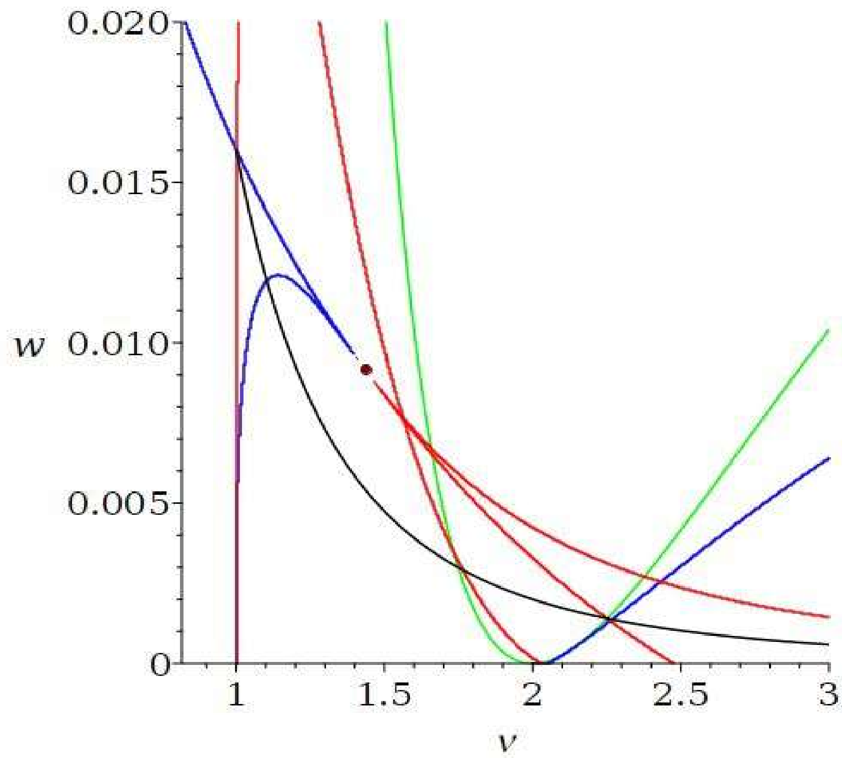

Roots of the discriminant

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As t approaches ρ,

$$T_1 = \alpha_\nu + \beta_\nu(1 - t/\rho_\nu) + \gamma_\nu(1 - t/\rho_\nu)^\alpha (1 + o(1)),$$

with

$$\alpha = 3/2 \quad \text{if} \quad \nu \neq \nu_c, \qquad \alpha = 6/5 \quad \text{if} \quad \nu = \nu_c.$$

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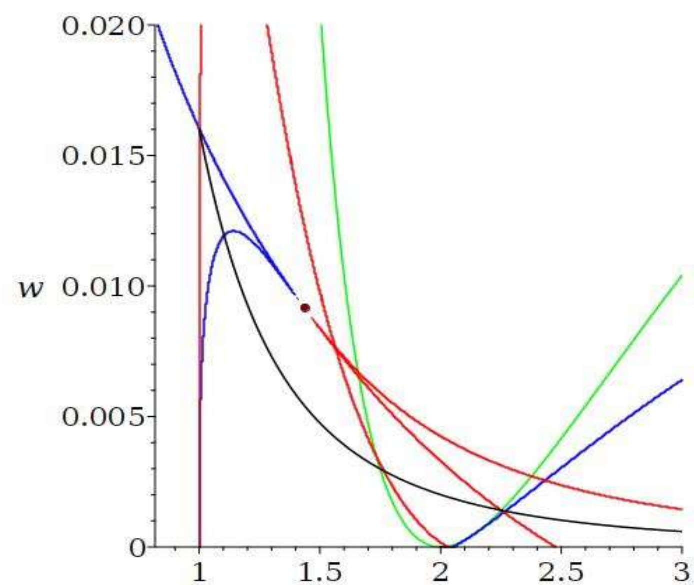

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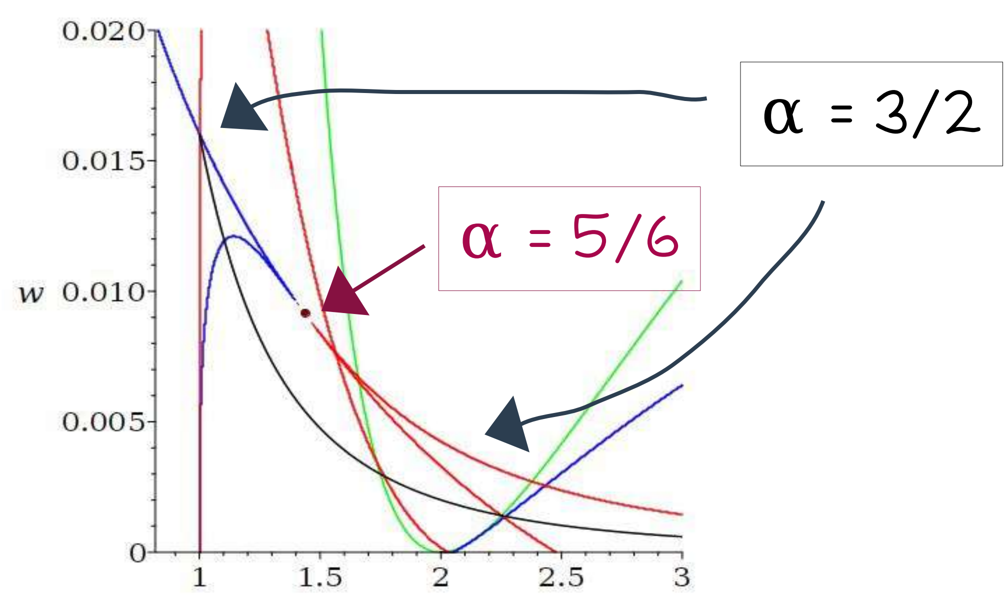

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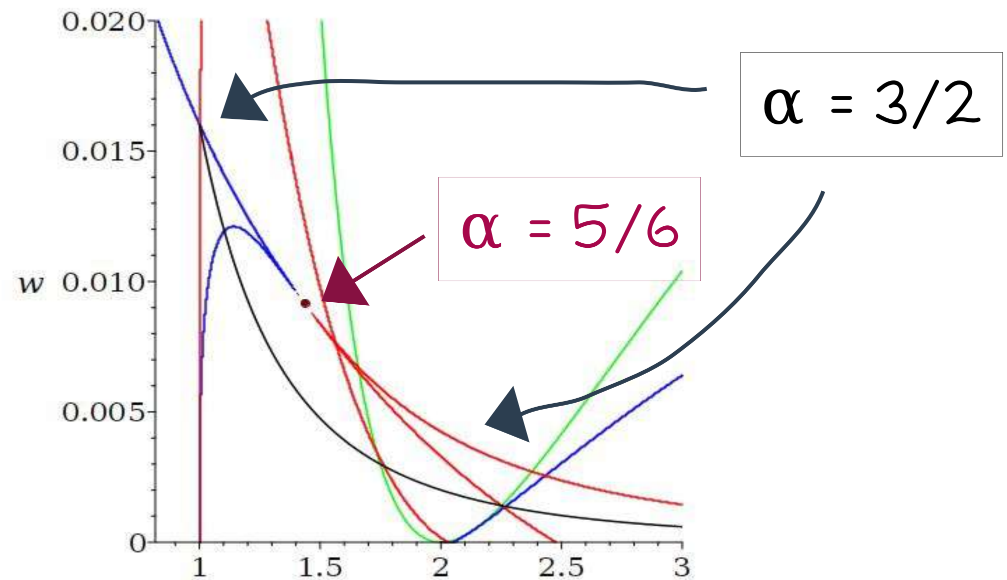

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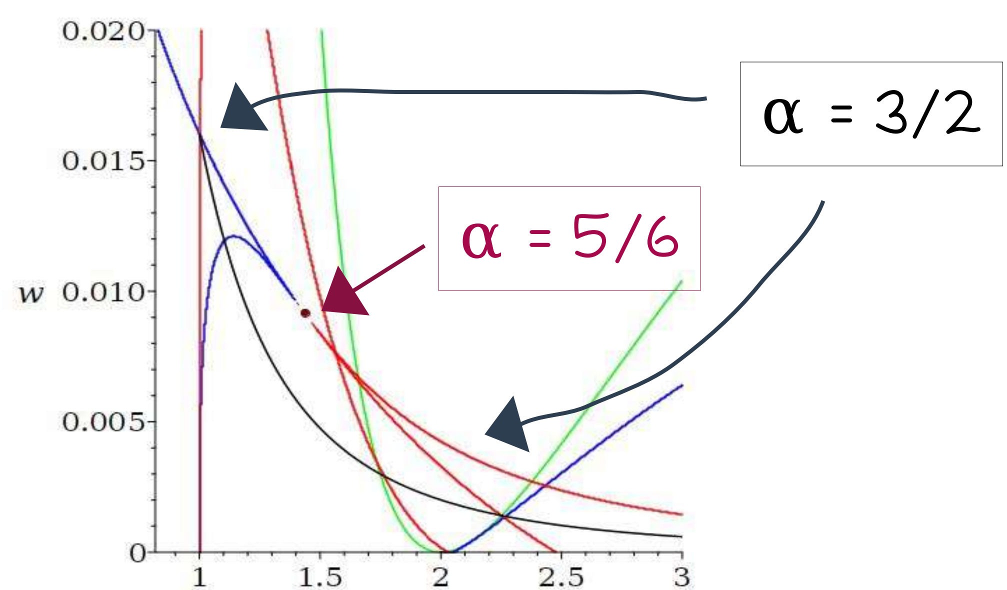

Merci !