Quantum Walks on Random Combs

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(Work in progress)

L'esprit des cartes, 15-16 mai 2025, IPhT

Emmanuel was my first student

12 shared publications

None of them on random maps !

In this talk: here is my only contribution to planar maps

Tribute to W. T. Tutte, the father of planar maps

My last collaboration with Emmanuel …

… my first with Thordur

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An IOP and SISSA journal

Mass distribution exponents for growing trees

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Abstract. We investigate the statistics of trees grown from some initial tree by attaching links to pre-existing vertices, with attachment probabilities depending only on the valence of these vertices. We consider the asymptotic mass distribution that measures the repartition of the mass of large trees between their different subtrees. This distribution is shown to be a broad distribution and we derive explicit expressions for scaling exponents that characterize its behaviour when one subtree is much smaller than the others. We show in particular the existence of various regimes with different values of these mass distribution exponents. Our results are corroborated by a number of exact solutions for particular solvable cases, as well as by numerical simulations.

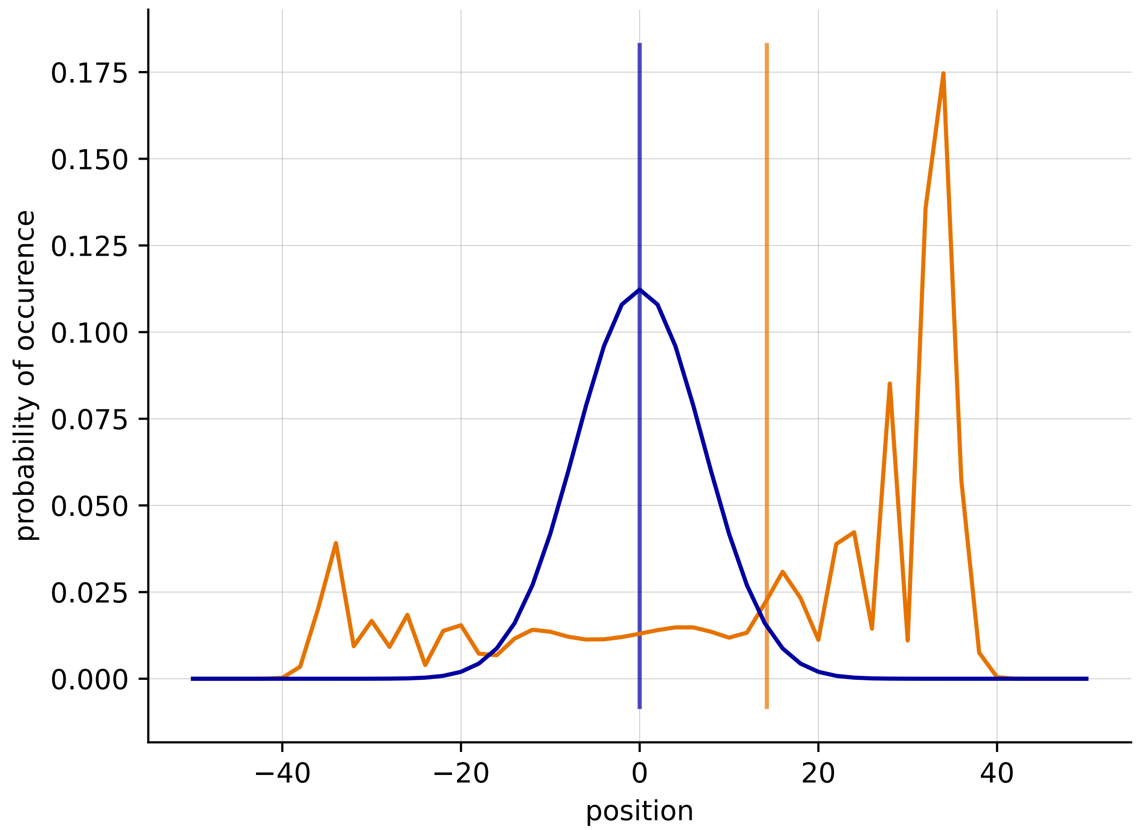
Keywords: exact results, growth processes, random graphs, networks

J. Stat. Mech. (2007) P02011

Quantum random walks versus classical random walk:

Classical: particle on $\mathbb{Z}$ + classical coin flip operator

Quantum: particle on $\mathbb{Z}$ + quantum coin flip operator



Simplest quantum random walk: just solve the continuous time Schrödinger equation on $\mathbb{Z}$.

It captures already some features of the full quantum random walk problem

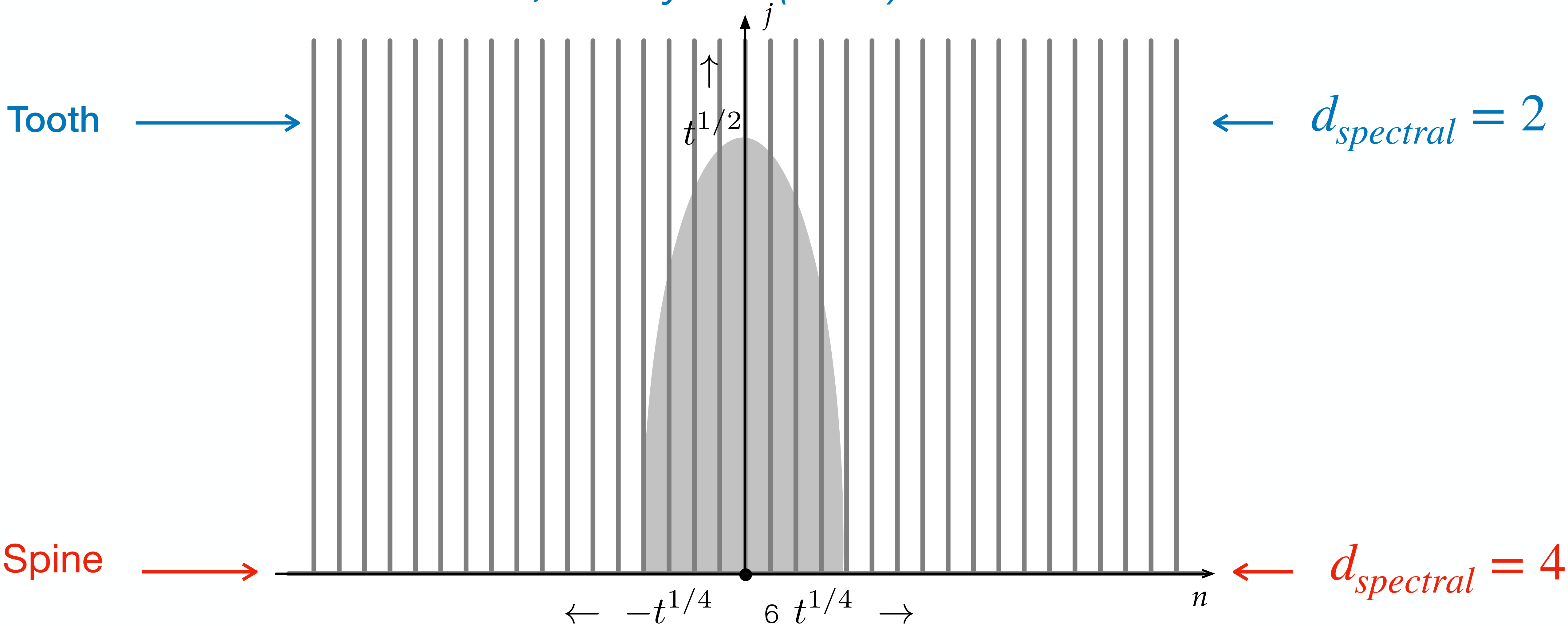
QRW in more complex geometries: the ∞ comb

This is already an interesting non-trivial example.

$\mathscr{C} = \mathbb{Z} \otimes \mathbb{N}$ (spine $\times$ teeth)

Question: How does a particle at a single point on the spine at time $t = 0$ diffuse as $t \to \infty$?

Classical diffusion (RW): *B. Durhuus, T. Jonsson and J. F. Wheater, Random walk on combs, J. Phys. A (2006)*

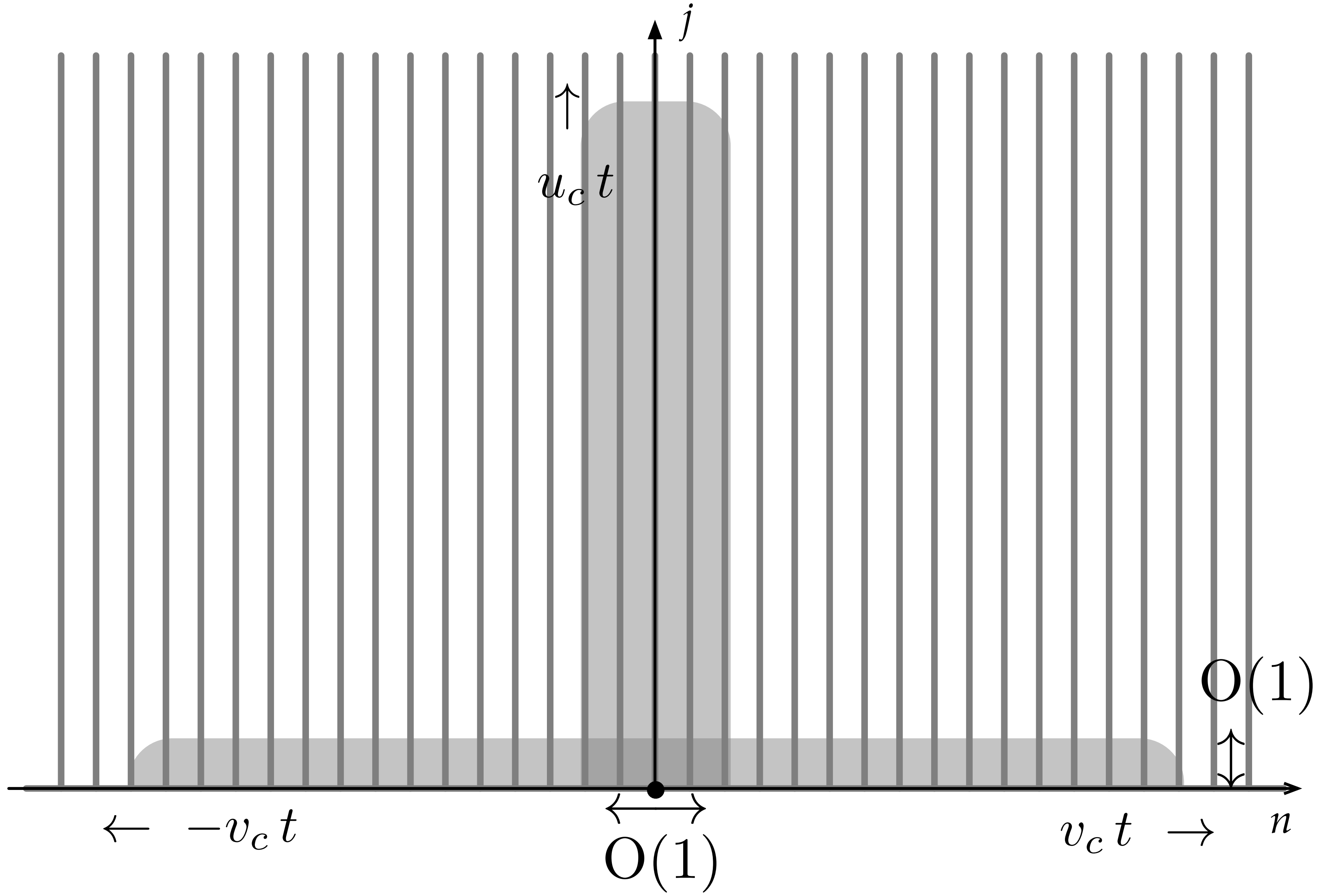


QRW on the ∞ comb

Same question: How does a particle at a single point on the spine at time $t = 0$ diffuse as $t \to \infty$?

F. D. & T. Jonsson, Quantum walk on a comb with infinite teeth (2022) J. Phys. A

Quantum ballistic along the teeth close to the origin, **and** along the spine, with non trivial asymptotics and scaling regimes (resurgence)



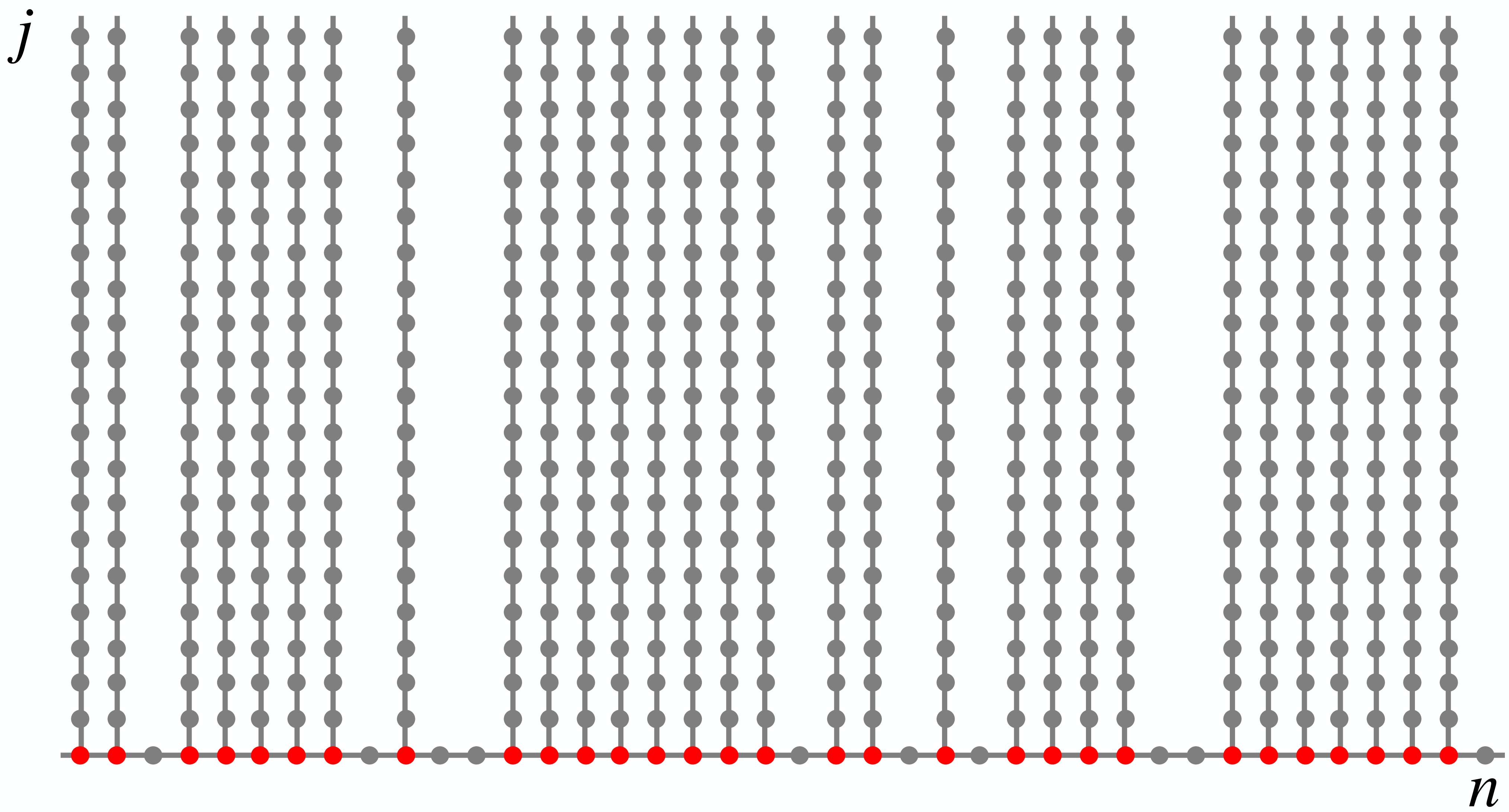
QRW on the infinite random comb:

Question: what is the effect of the geometric disorder.

Simplest geometrical model: the random comb

Teeth are i.i. distributed along the spine.

No tooth = « hole » with probability p , tooth with probability $1 - p$



The eigenstates equation

Energy eigenstates $H\phi = E\phi$

On the spine: $\phi(n,0) = C_n$

On the teeth:

if $0 \le E \le 4$, $\phi(n,j) = A_n e^{i\theta j} + B_n e^{-i\theta j}$, $A_n + B_n = C_n$, $E = 2 - 2\cos\theta$

if $4 \le E$, $\phi(n;j) = C_n (-1)^j e^{-\sigma j}$, $E = 2 + 2\cosh\sigma$

The $E > 4$ states are localized in the j direction (near the spine)

The $E \le 4$ states propagate along the teeth, $B_n = |\mathtt{IN}\rangle$, $A_n = |\mathtt{OUT}\rangle$

Eigenstate equation along the spine:

For $E \le 4$, if n is a tooth • $-C_{n-1} - C_{n+1} + A_n(1 + e^{i\theta}) + B_n(1 + e^{-i\theta}) = 0$

if n is a hole • $-C_{n-1} - C_{n+1} + C_n(2\cos\theta) = 0$

For $E > 4$, if n is a tooth • $-C_{n-1} - C_{n+1} + C_n(1 - e^{\sigma}) = 0$

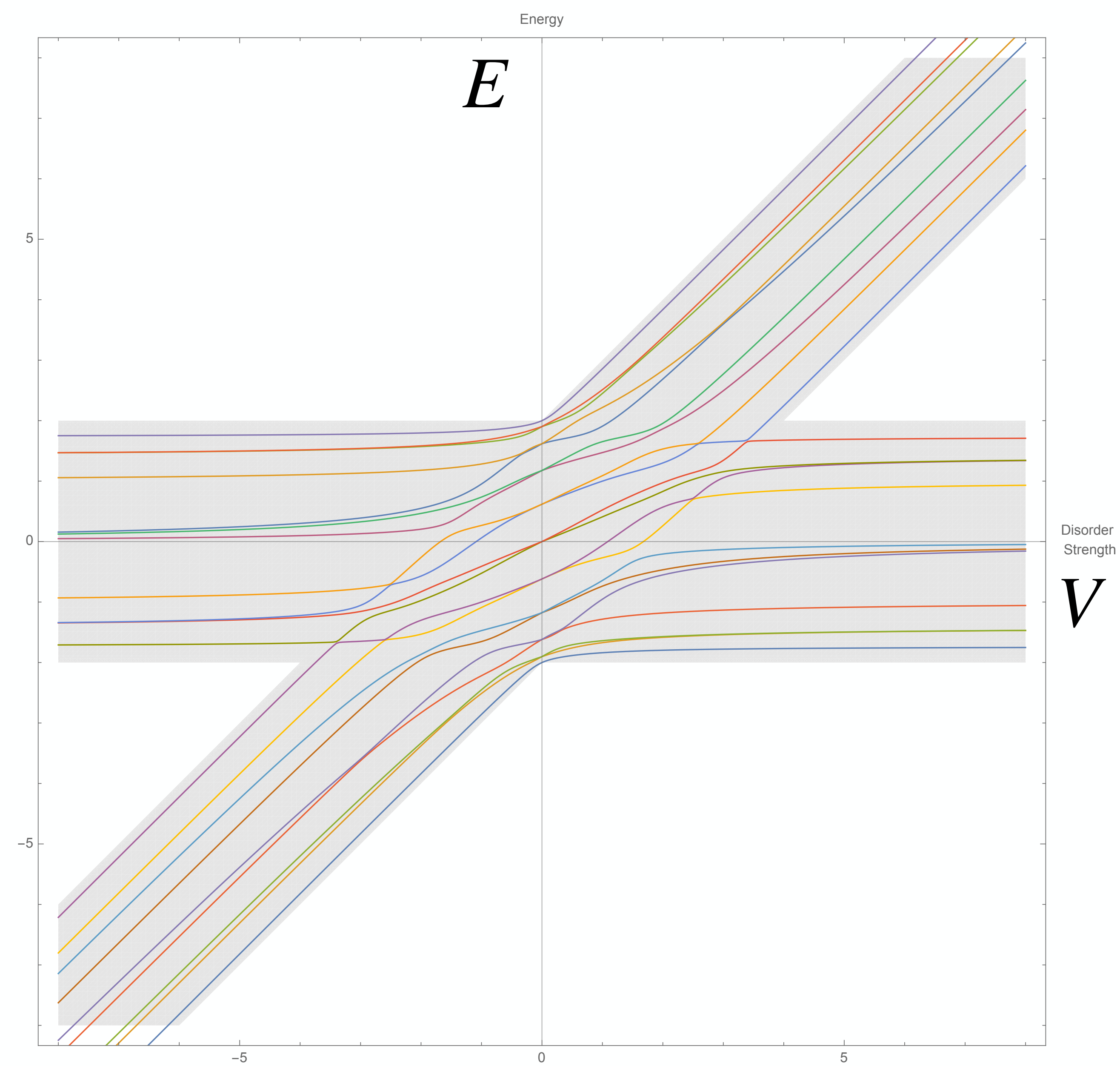
if n is a hole • $-C_{n-1} - C_{n+1} - C_n(2\cosh\sigma) = 0$

Mapping onto the random binary chain model

These equations can be mapped onto the prototypal model of a disordered conductor (Anderson), and of random harmonic chain (Dyson, Schmidt)

$$-\varphi(n-1) - \varphi(n+1) + V(n)\,\varphi(n) = E\,\varphi(n) \quad \text{with} \quad V(n) = 0, V \quad \text{iid random}$$

Spectral flow with disorder strength for chain of length $L = 20$ with equal probability $V(n) = 0$ or V.



Anderson localization:

$$|\varphi_E(n)| \leq C\,e^{-|n|/\xi(E)}$$

$\xi(E)$ localization length

$\xi = 1/\gamma$ Lyapunov exponent for a random dynamical process (Riccati equation, product of random matrices)

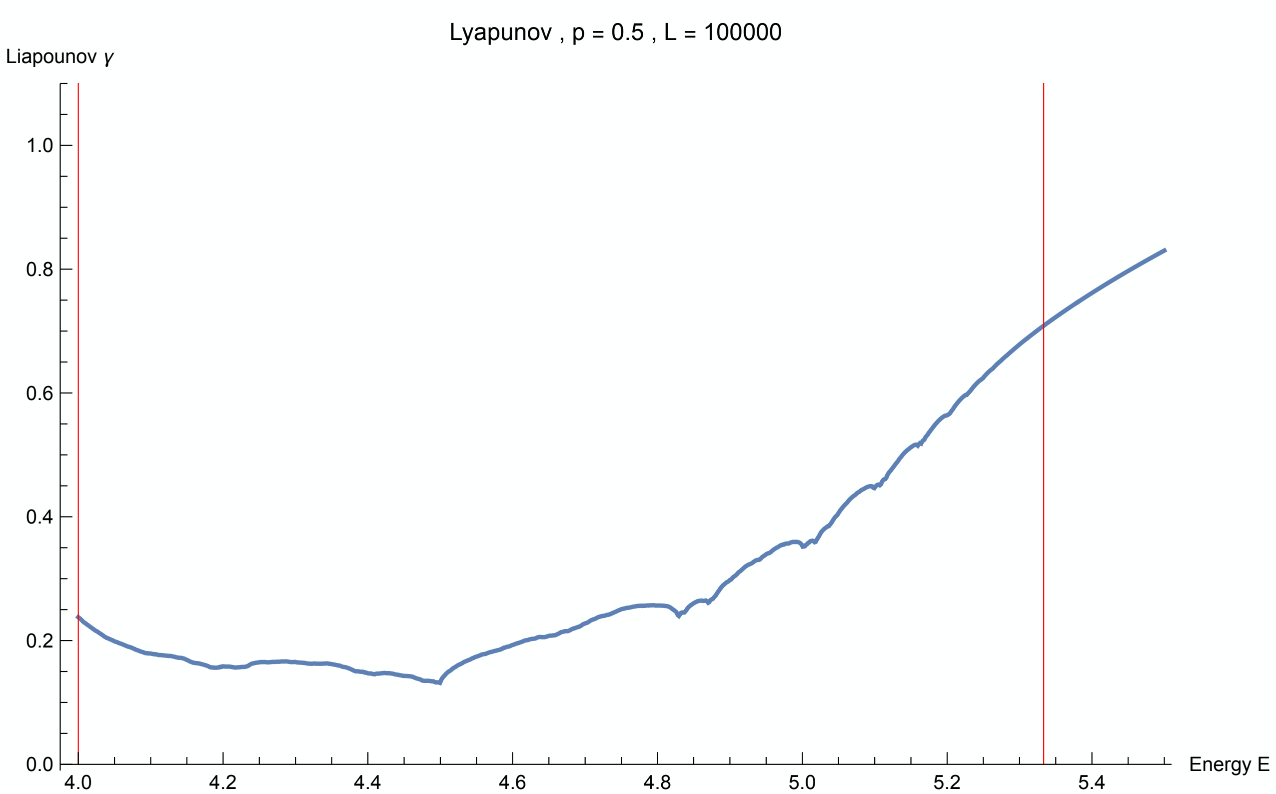
We can use the mathematical results (Furstenberg, Schmidt, Herbert-Jones, Thouless, …) and the numerical methods from localisation theory

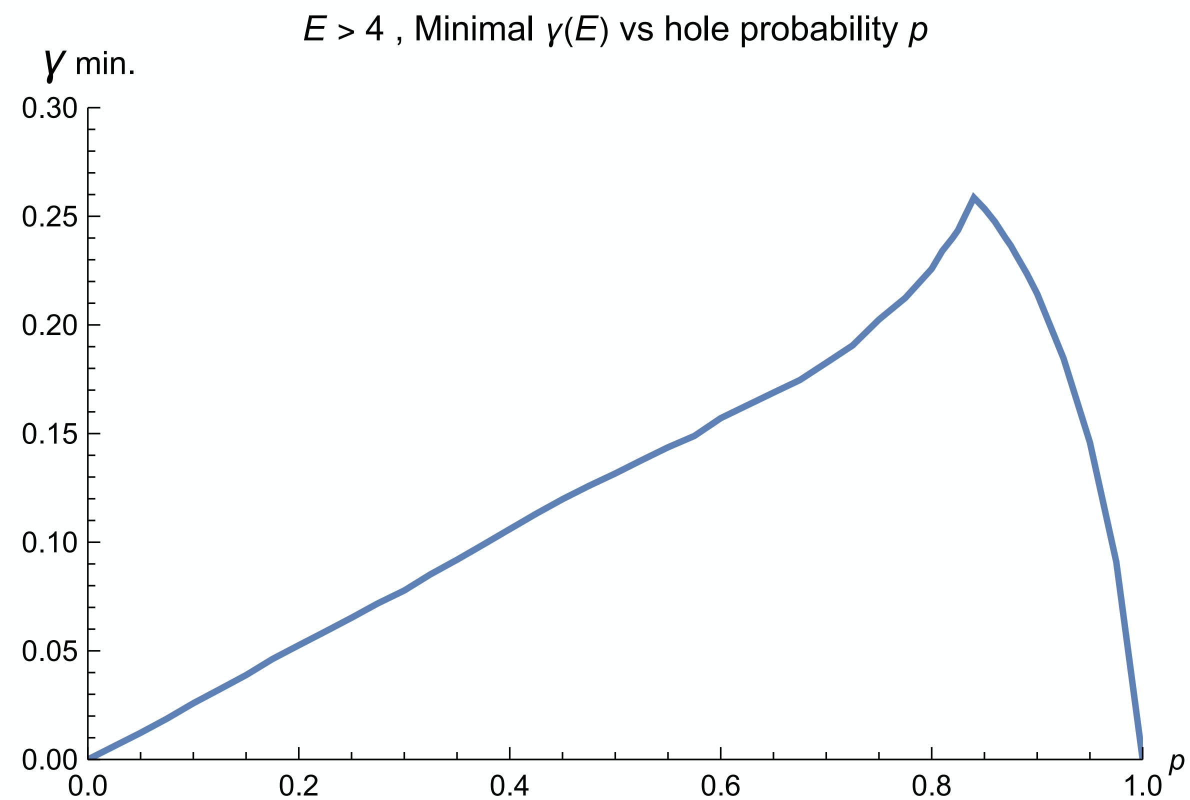
Consequences for the $E > 4$ (spine localised) states

The normalized number of $E > 4$ states converges as the length of the comb $L \to \infty$

$$\frac{\#\ E > 4 \text{ states}}{L} \xrightarrow[a.s]{} \frac{1-p}{2-p}$$

The $E > 4$ states are always localized if $p_{hole} > 0$





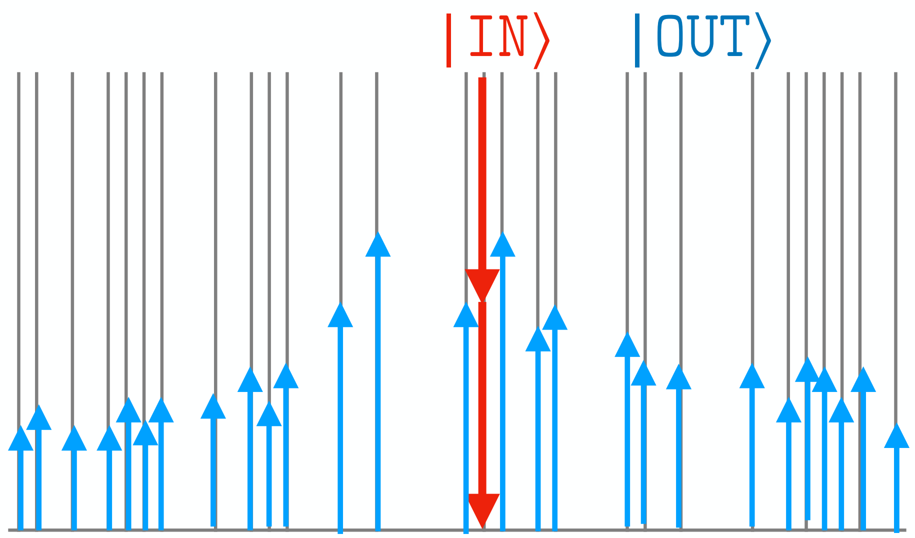
Consequences for the $E \leq 4$ states

These states correspond to a scattering/reflexion process: S-matrix

$$\phi(n,j) = A_n\, e^{i\theta j} + B_n\, e^{-i\theta j}$$

$$A = (A_n) = |\mathrm{OUT}\rangle \qquad B = (B_n) = |\mathrm{IN}\rangle$$

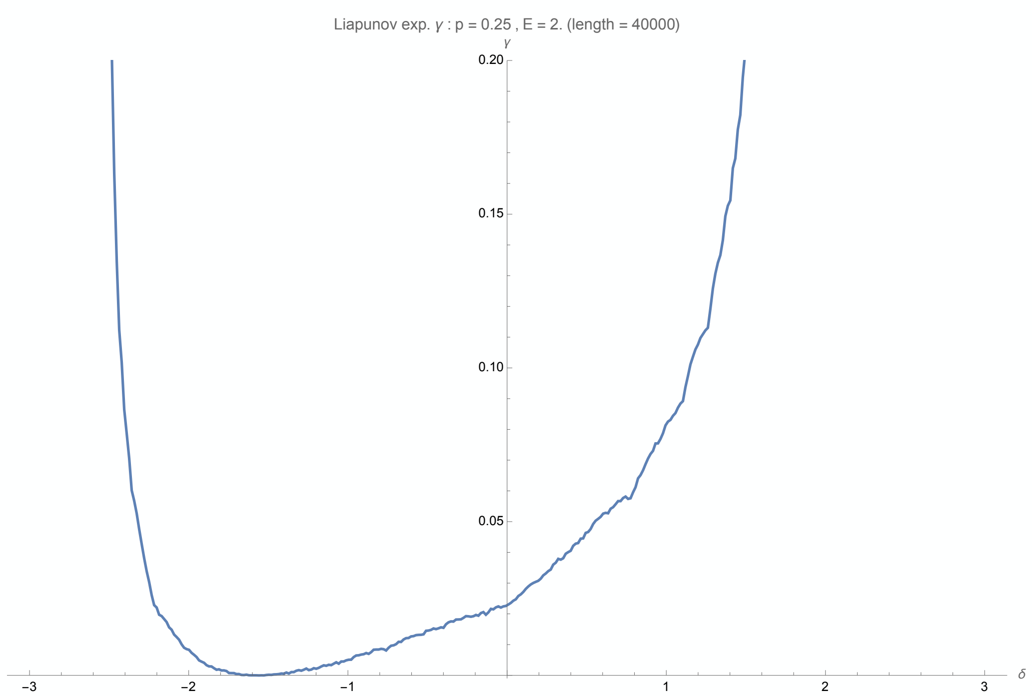
$$A \; = \; S(\theta)\, B \qquad T_{\mathrm{teeth}} \times N_{\mathrm{teeth}} \text{ matrix}$$

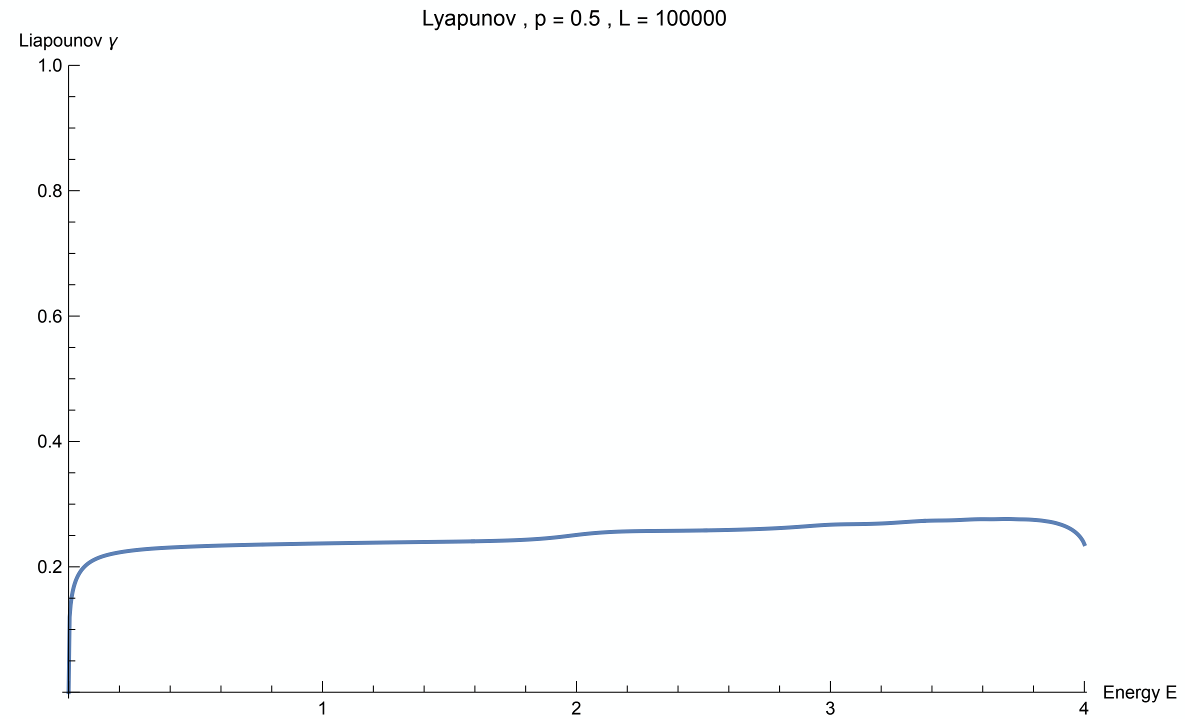


Study localization for:

1 - the matrix band structure of the S-matrix (row and column): $S_{nm}(\theta) = \langle n \,|\, \Upsilon_{\theta,m}\rangle$

2 - the eigenstates of the S-matrix: $S(\theta)\,|\,\Phi_{\theta,\delta}\rangle = e^{i\,\delta}\,|\,\Phi_{\theta,\delta}\rangle$, δ phase shift





Back to the initial problem: diffusion of a particle starting from the spine

Start from an initial state $|\phi_0\rangle$ localised on the spine at $(n_0, 0)$.

Probability to stay at finite distance from the spine at site n , $P_{\rm loc}(n_0, n)$.

Total probability to stay near the spine

$$P_{\rm loc}(n_0) = \sum_n P_{\rm loc}(n_0, n)$$

versus

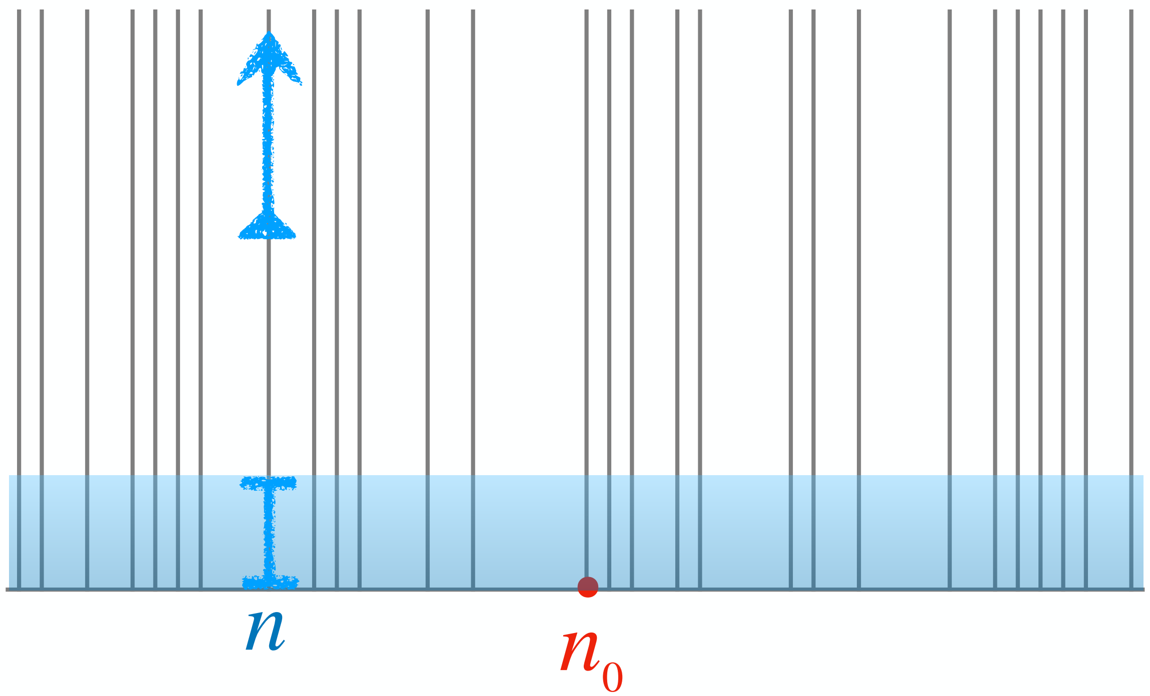
Probability to escape at ∞ along a tooth n

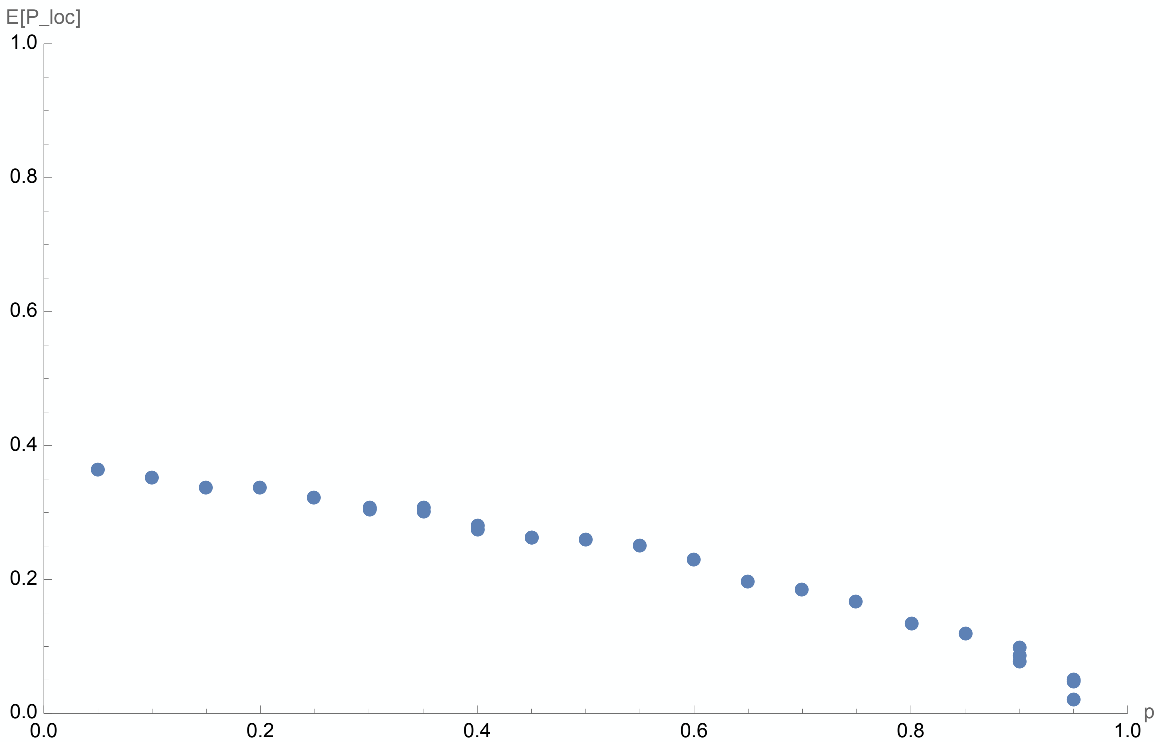
$$P_{\rm esc}(n_0, n)$$

These are random variables, but they are expected to converge in law when $L \to \infty$.

We have mostly numerical results.

e.g. $\mathbb{E}[P_{\rm loc}(n_0)]$ as function of $p_{\rm hole}$





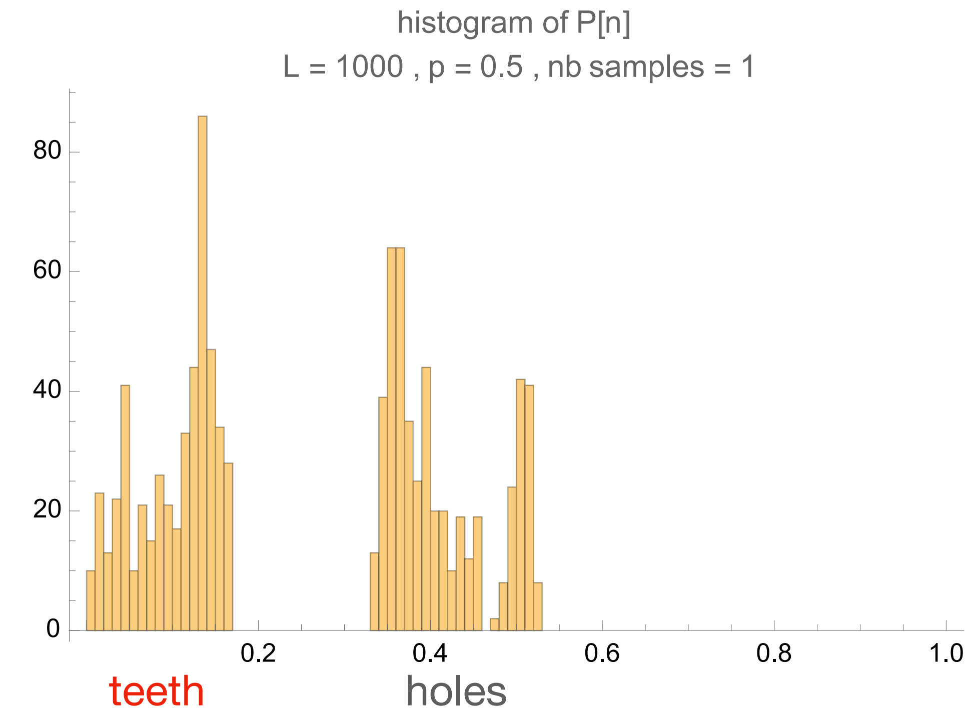
Some interesting questions

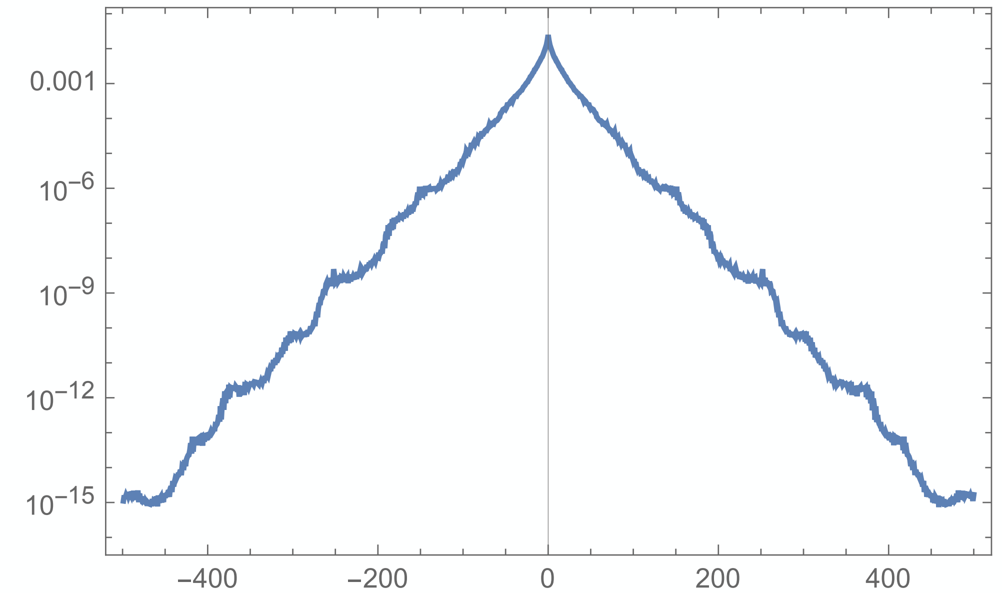
Numerical guesses:

The distribution of $P_{\rm loc}(n_0)$ has a gap ! Why ?

The (time average of the) probability to be localized near the spine at site n, $\overline{P_{\rm loc}(n_0, n)}$ decays exponentially with the distance $d = |n - n_0|$, hence exhibit localization

Disorder is relevant





Analytical guess:

For $p_{\rm hole} > 0$, the probability to escape at ∞ along a tooth n, $P_{\rm esc}(n_0, n)$, should decay with the distance $|n - n_0|$ as

$$\mathbb{E}[P_{\rm esc}(n_0, n)] \sim |n - n_0|^{-4}$$

as for the regular comb, $p_{\rm hole} = 0$.

Disorder is irrelevant

Thank you !

Joyeux anniversaire, et continue sur ta lancée, Emmanuel !