IPHC DRS, Theory group, University of Strasbourg

Axions and cosmic strings with extra space dimensions

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Aims of the internship

The **Standard Model** is a quantum field theory describing the known fundamental particles and their interactions:

- built from strong **symmetry principles**
- tested to high precision but with persistent **open questions**, among them:

Dark matter

Strong CP problem

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1. What is the **axion**?
2. What are the so-called **axion cosmic strings**?
3. What is their possible **cosmological role**?
4. How can they emerge from a **higher-dimensional framework**?

Axion and strong CP problem

Nature violates **CP symmetry** significantly in weak interactions but there is another contribution:

- Yukawas from the weak sector: $\theta_y = \arg(\det(M_q))$
- topological term in the strong sector: $\sim \theta \; G^a{}_{\mu\nu} \tilde{G}^{a\mu\nu}, \; a = 1...8$

$$\boxed{\text{physical parameter:} \quad \bar{\theta} = \theta + \theta_y}$$

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$$\underbrace{\boxed{\text{physical parameter:} \quad \bar{\theta} = \theta + \theta_y}}_{\text{experimentally } \bar{\theta} < 10^{-10}}$$

$\rightarrow$ **Strong CP problem**: $\bar{\theta}$ comes from both strong **and** weak interactions!
Why is $\bar{\theta}$ so small in the absence of any symmetry enforcing it?

$\rightarrow$ $\bar{\theta} = 0$ should come from symmetry considerations

→ introduce a new global ~~U(1)$_{\text{PQ}}$~~ symmetry: $\psi \to e^{i\alpha\gamma^5}\psi$, $\phi \to e^{i\alpha}\phi$

→ introduce a **complex scalar field** $\phi(x)$ charged under U(1)$_{\text{PQ}}$

$$\phi(x) = \frac{f_a + \rho(x)}{\sqrt{2}} e^{ia(x)/f_a}$$

$\rho(x)$: radial mode, $a(x)$: angular mode, f_a: symmetry breaking scale

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QCD dynamics generate an effective potential for $a(x)$:

$$V_{\text{eff}} \propto \cos\left(\bar{\theta} + \xi \frac{a}{f_a}\right), \qquad \text{with minimum for} \quad \boxed{\bar{\theta} = -\xi \frac{\langle a \rangle}{f_a}} \leftarrow \textbf{axion}$$

→ the axion **dynamically resolves the strong CP problem**!

As the early universe expanded and cooled, it underwent **phase transitions**

$$T \sim f_a \qquad \Longrightarrow \qquad \text{~~Peccei–Quinn symmetry~~}$$

→ gave birth to the axion field which began to settle into its vacuum state...

As the early universe expanded and cooled, it underwent **phase transitions**

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...which consists of degenerate vacua!

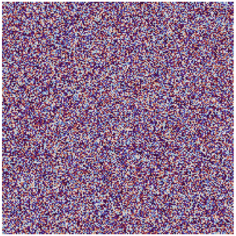

Figure: Simulation of the phase map before PQ phase transition...

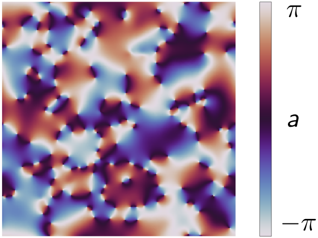

Figure: ...and simulation of the phase map after PQ phase transition

Topological defects appear in regions where mismatches in the phase cannot be smoothed out. *What is the appropriate probe?*

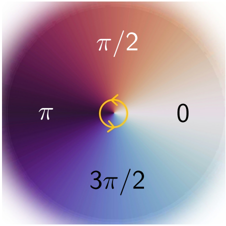

Figure: Phase values in the vacuum state

Cosmic strings appear when there are **non-contractible loops**

- formally, this is **homotopy** theory
- related to symmetry breaking

Axion cosmic strings

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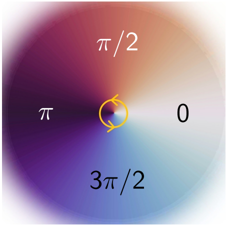

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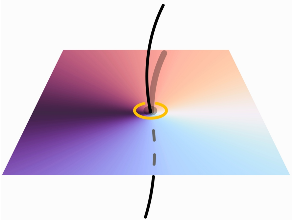

Figure: Cosmic string in 3D space

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These strings form a **complex network** with its own dynamics and interactions:

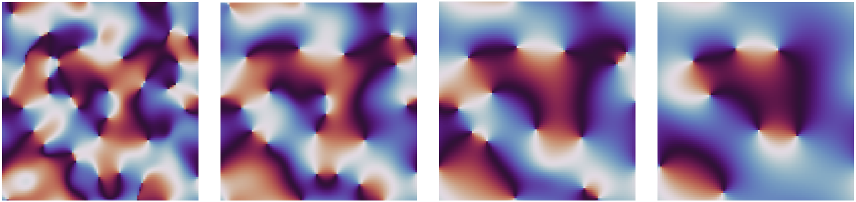

Figure: Simulation in a slice of universe of the evolution of a string network

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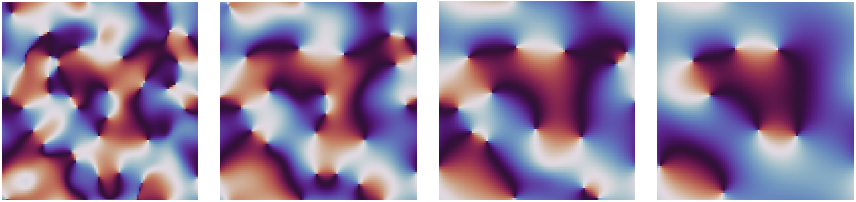

Figure: Simulation in a slice of universe of the evolution of a string network

→ string interactions
→ decay
→ **emission of axions**

We can interpret the relic abundance of axions as a **dark matter candidate**

$$f_a \sim 10^{9-10} \text{ GeV}$$

Extra dimensions

Investigate axion cosmic strings emerging from extra-dimensional defects

Existence of **extra space dimensions** beyond the familiar three?
To escape experimental detection, they could be:

- **Compact**: curled up into tiny circles
- **Large but hidden**: with Standard Model confined to a 4D surface
- **Warped**: certain regions are effectively hidden from low-energy physics

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Example: Kaluza-Klein model for compact 5^{th} dimension

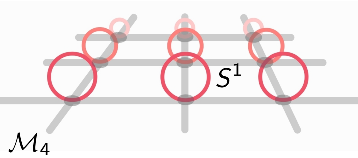

Figure: Four-dimensional spacetime with one compact extra dimension

Extra-dimensional axion

The model we consider is a spontaneously broken SU(2) gauge theory, including a gauge field ${A^a}_M$ coupled to a real scalar field ϕ^a valued in $\mathfrak{su}(2)$. This lives on a **5D spacetime** compactified as $\mathcal{M}_4 \times S^1$:

$$\mathrm{SO}(1,4) \qquad - \text{compactification} \rightarrow \qquad \underbrace{\mathrm{SO}(1,3)}_{\text{Lorentz in 4D}}$$

$M = [0,3] \cup \{5\}$, $a = 1,2,3$ (Lie algebra index), $A_M = {A^a}_M t^a$ (generators)

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In 5D, fields have additional dimensions in which they can live:

$$\begin{pmatrix} A_0 \\ \vec{A} \\ A_5 \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} A_0 \\ \vec{A} \end{matrix}}\right\} \quad \text{4D gauge field (4 components)} \\ \longleftarrow \quad \text{behaves like a 4D scalar} \end{matrix}$$

5^{th} component of a higher-dimensional gauge field = candidate for the **axion**

$M = [0,3] \cup \{5\}$, $a = 1,2,3$ (Lie algebra index), $A_M = {A^a}_M t^a$ (generators)

Extra-dimensional string?

Then what is the **cosmic string equivalent** in this framework?

- $V(\phi) = \frac{\lambda}{4}(1 - |\phi|^2)^2$ so that ϕ develops a vacuum expectation value
- SU(2) is broken to U(1)

$\rightarrow$ topological **magnetic monopoles**

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4D solution: *'t Hooft* and *Polyakov* (1974)

$$\mathcal{L} = \frac{1}{8}\mathrm{Tr}\{F_{\mu\nu}F^{\mu\nu}\} - \frac{1}{4}\mathrm{Tr}\{D_\mu\phi D^\mu\phi\} - \frac{\lambda}{4}\left(1 - |\phi|^2\right)^2$$

with $F_{\mu\nu} = \partial_{[\mu}A_{\nu]} + [A_\mu, A_\nu]$ and $D_\mu = \partial_\mu\phi + [A_\mu, \phi]$

Varying the action with respect to the fields:

$$D^i D_i\phi = -\lambda\left(1 - |\phi|^2\right)\phi \qquad D^i F_{ij} = -[D_j\phi, \phi]$$

Extra-dimensional string?

Setting $A_0 = 0$, the most general **ansatz** is: $(r = \sqrt{|x^i x_i|})$

$$\phi = H(r)\frac{x^a}{r}t^a \qquad A_{i>0} = -[1-K(r)]\epsilon_{ija}\frac{x^j}{2r^2}t^a \qquad H(0,\infty) = 0, 1 = 1-K(0,\infty)$$

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Solving the equations of motion:

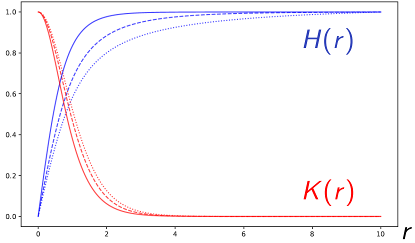

Figure: Profile functions in 4D

- topological magnetic charge
- analytic solution only for $\lambda \to 0$

...along with more exotic properties:

$B_i + D_i\phi = 0$ (Bogomol'nyi bound)

In 4D, $F_{\mu\nu} = \partial_{[\mu}A_{\nu]}$ has **6** components $\left(\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}F^{\rho\sigma}\right)$:

- **3** for the **electric field** (vector)

$$\partial^\mu F_{\mu\nu} = J_\nu^{elec}$$

- **3** for the **magnetic field** (vector)

$$\partial^\mu \tilde{F}_{\mu\nu} = J_\nu^{mag}$$

In 4D, the electromagnetic charge carriers must be **point-like**

Maxwell's equations

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In 4D, the electromagnetic charge carriers must be **point-like**

In 5D, $F_{MN} = \partial_{[M} A_{N]}$ has **10** components $\left(\tilde{F}_{PQR} = \frac{1}{2}\epsilon_{MNPQR} F^{MN}\right)$:

- **4** for the **electric field** (vector)

$$\partial^M F_{MN} = \boldsymbol{J^{elec}_N}$$

- **6** for the **magnetic field** (tensor)

$$\partial^P \tilde{F}_{PQR} = \boldsymbol{J^{mag}_{QR}}$$

In 5D, the magnetic charge carriers must be **string-like**

As we add extra dimensions, the monopole gains a **spatial extension**:

Figure: Relative extension of a monopole depending on spacetime dimension

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Figure: Relative extension of a monopole depending on spacetime dimension

5D magnetic monopole = candidate for the **cosmic string**

We can localize it along different spatial directions, we considered 2 cases:

- extended along x^5: monopole loop
- extended along x^3: monopole string

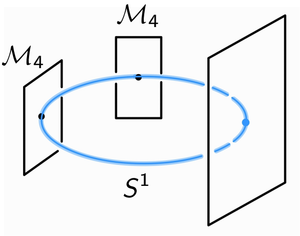
Monopole string along x^5

The first case we consider is the **monopole loop** in the extra dimension:

$$\partial_M \tilde{F}^{MN0} = q_m\ \delta^N_5 \delta^{(3)}(x_\perp)$$

$$\swarrow \qquad\qquad \searrow$$

$$\partial_M \tilde{F}^{M\mu 0} = 0 \qquad\qquad \partial_M \tilde{F}^{M50} = q_m\ \delta^{(3)}(x_\perp)$$



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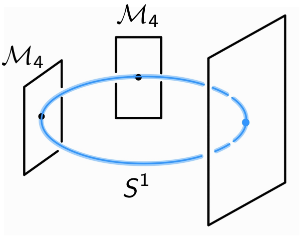

→ effective **point-like** monopole in 4D

Adding an extra dimension introduces new contributions to the equations of motion, embedded in:

$$F_{5i} = -\partial_i A_5 + [A_5, A_i] \qquad\qquad D_5 \phi = [A_5, \phi]$$

→ let us do some **developments**!

Monopole string along x^5

We recover the usual 't Hooft-Polyakov monopole's equations setting $A_{0,5} = 0$

- the mass is now interpreted as a **tension**
- the total mass is integrated over the extra dimension

Benchmark: the effective 4D theory describes the usual magnetic monopole!

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Benchmark: the effective 4D theory describes the usual magnetic monopole!

...but we can consider a non-zero A_5! We propose the following **ansatz**:

$$\underbrace{\phi = H(r)\frac{x^a}{r}t^a \qquad A_{i>0} = -[1-K(r)]\epsilon_{ija}\frac{x^j}{2r^2}t^a}_{\text{'t Hooft-Polyakov monopole}} \qquad \underbrace{A_5 = S(r)\frac{x^a}{r}t^a}_{\text{new scalar field}}$$

$S(0) = 0,\ S(\infty) = 1$

Solving the equations of motion, we obtain the following profiles:

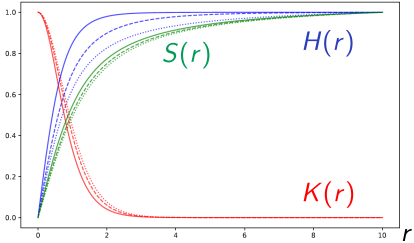

Figure: Profile function for a 5D magnetic monopole with additional scalar

- proof of **existence**
- monopole with an additional scalar

scalar-dressed monopole^{TM}

But magnetic monopoles have some pretty exotic properties!
$\rightarrow$ can we reproduce them with this new object?

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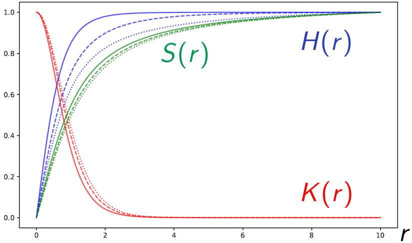

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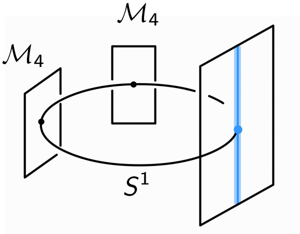
But magnetic monopoles have some pretty exotic properties!
$\rightarrow$ can we reproduce them with this new object?

$\rightarrow$ yes: define a complex adjoint triplet: $\Phi = \phi + iA_5$

$$B_i + e^{i\alpha} D_i \Phi = 0, \ \forall \alpha \qquad \text{(generalized Bogomol'nyi bound)}$$

Then we consider a **monopole string** extended along a non-compact direction:

$$\partial_M \tilde{F}^{MN0} = q_m \, \delta^N_3 \delta^{(3)}(x_\perp)$$



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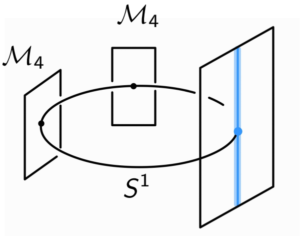

→ effective **string-like** monopole in 4D

Δ We must be extremely careful about **topological obstructions** when adding compact extra dimensions!

——[Beaudoin, Delafosse (2025)]
In a spacetime with compact extra dimensions admitting a factorizable scalar charge density, any charge localized in the compact space must be paired

First step: investigate the possible topological obstructions

- what is the behaviour of the higher-dimensional magnetic field?

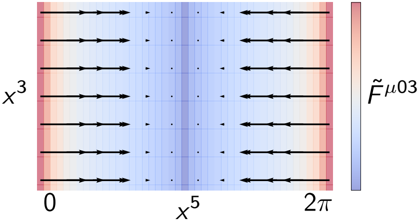

Figure: Magnetic field propagation along the extra dimension

$\rightarrow$ the field wraps around S^1

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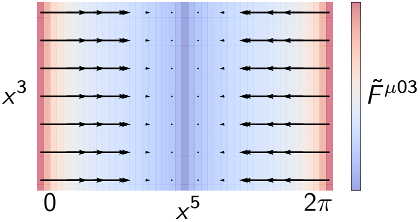

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Second step: unify scalars and strings

- what is the topological charge?
- how to formalize it?

$$Q \propto \oint_{S^1_\varphi} \mathrm{d}\ell^i \, \partial_i \left[\int_{S^1_{x^5}} \mathrm{d}x^5 \, \mathrm{Tr} \left\{ \hat{\phi}(x^i, x^5) A_5(x^i, x^5) \right\} \right] \qquad i = 1, 2$$

$\hat{\phi}$: orientation of ϕ far from the core, $S^1_{x^5} \times S^1_\varphi$: torus around the string

Effective axion cosmic string

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The model now has:

- **axion**: 5^{th} component of a higher-dimensional gauge field
- **cosmic string**: extended magnetic monopole string

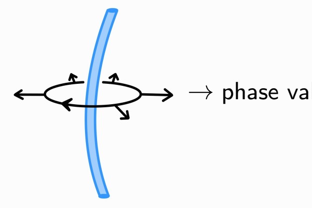
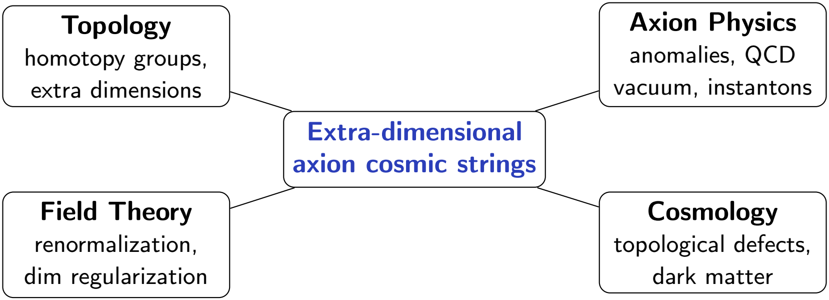


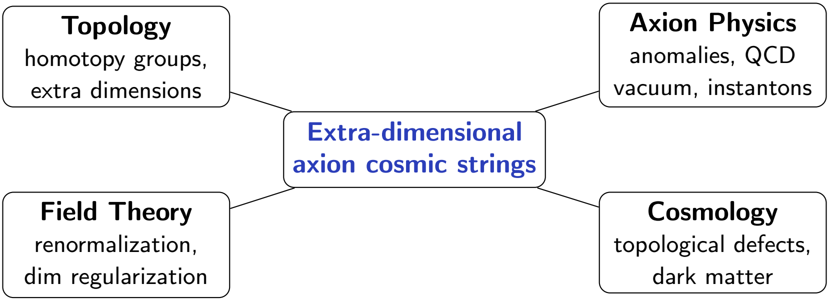
Figure: Winding around a cosmic string

→ acquires a **winding** around the core of the monopole string

→ mimics the behavior of an axion field circling a cosmic string

Summary and Open Questions





Open questions:

- What are the properties of this particular axion?
- What signals could we expect from such objects?
- How can we study the dynamics of topological defects? (Ongoing article with L. Delafosse: "*A sheaf-theoretic approach to path integrals with applications to topological solitons and anomalies*")

Thank you for your attention

- M. Nakahara, *Geometry, Topology and Physics* (2003)
- T. Kaluza, *On the Problem of Unity in Physics* (1953)
- L. Randall, R. Sundrum, *Large Mass Hierarchy from a Small Extra Dimension* (1999)
- M. Schwartz, *Quantum Field Theory and the Standard Model* (2013)
- R. Peccei, *The Strong CP Problem and Axions* (2006)
- A. Vilenkin, *Radiation of Goldstone Bosons from Infinitely Long Cosmic Strings* (1991)
- O. Wantz, *Axion Cosmology Revisited* (2011)
- G. 't Hooft, *Magnetic Monopoles in Unified Gauge Theories* (1974)

The neutron is a composite particle made of quarks and gluons, the CP-violating QCD term distorts the spatial distribution of charge in the neutron, creating a **permanent electric dipole moment**: $d_n \sim \bar{\theta} \cdot 10^{-16}$ e·cm.

$\rightarrow$ use ultracold neutrons in an electromagnetic field:

$$\underbrace{\omega \equiv 2\pi f}_{\text{precession frequency}} \sim \underbrace{2\mu_n B}_{\text{magnetic precession}} \pm \underbrace{2d_n E}_{\text{electric contribution}}$$

$\rightarrow$ reversing the direction of $\vec{E}$ while keeping $\vec{B}$ fixed allows to measure the change in precession frequency that is attributed to d_n alone!

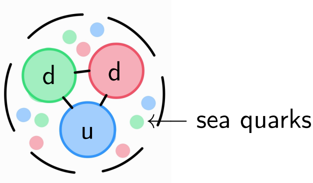


Figure: Valence quark content (and sea) of a neutron

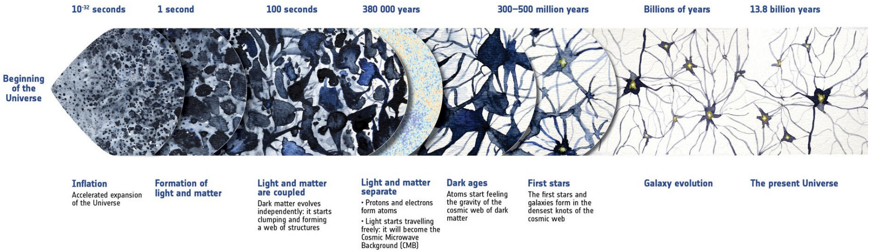


Figure: Cosmic history (Source: ESA)

- **Inflationary**: $10^{15} - 10^{16}$ GeV
- **Peccei-Quinn**: $10^{9} - 10^{12}$ GeV
- **Electroweak**: 100 GeV
- **QCD chiral**: 200 MeV
- **Neutrino decoupling**: 1 MeV
- **Recombination**: 0.3 eV

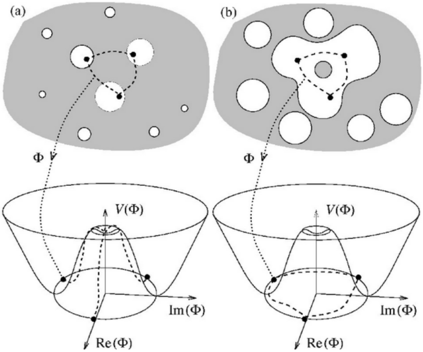


Figure: U(1) symmetry breaking of a complex scalar field produces cosmic strings (Source: Kibble, 1980)

- **(a)** Patches with true vacuum energies start growing as the symmetry is broken. Gray regions represent false vacua;
- **(b)** As the patches with true vacua merge, false vacuum regions are squeezed and form topological defects.

vacuum manifold $\cong S^1$

Axion potential

$$\mathcal{L} \supset \underbrace{\frac{1}{2}\partial_\mu a \partial^\mu a}_{\text{kinetic term}} + \underbrace{\mathcal{L}_{int}\left(\partial^\mu \frac{a}{f_a}, \psi\right)}_{\text{interaction term}} + \underbrace{\frac{a}{f_a}\xi \frac{g^2}{32\pi^2} G^{a\mu\nu}\tilde{G}^a{}_{\mu\nu}}_{\text{topological coupling}}$$

This **anomalous coupling** modifies the effective $\bar{\theta}$-parameter: $\bar{\theta}_{\text{eff}} = \bar{\theta} + \xi \frac{a}{f_a}$, and since QCD dynamics generate a potential that depends periodically on $\bar{\theta}_{\text{eff}}$, the resulting effective potential for the axion takes the form:

$$V_{\text{eff}} \propto \cos\left(\bar{\theta} + \xi \frac{a}{f_a}\right), \qquad \text{with minimum for } \langle a \rangle = -\frac{f_a}{\xi}\bar{\theta}$$

which dynamically cancels the CP-violating term and ensures $\bar{\theta}_{\text{eff}} = 0$ in the vacuum. Thus, defining **the physical axion field** as $a_{phys} = a - \langle a \rangle$, the Lagrangian becomes manifestly CP-conserving.

Kaluza-Klein

The 5D metric can be written as:

$$g^{(5)}_{MN} = \phi^{-1/3} \begin{pmatrix} g_{\mu\nu} - \phi A_\mu A_\nu & \phi A_\mu \\ \phi A_\nu & -\phi \end{pmatrix}$$

The 5D Einstein equations are obtained from:

$$S = \int \mathrm{d}^4 x \sqrt{|g|} \left[-\frac{R}{16\pi G} - \frac{1}{4}\phi F^{\mu\nu} F_{\mu\nu} + \frac{1}{6\kappa^2\phi^2} \partial_\mu \phi \partial^\mu \phi \right]$$

Compactifying the extra dimension on a circle S^1 of radius R, any quantity inherits a periodic dependence on x^5 and the fields can be Fourrier expanded:

$$\phi(x^\mu, x^5) = \sum_n \phi^{(n)}(x^\mu) e^{in\frac{x^5}{R}}$$

If $\Box^{(5)}\phi = 0$, the Fourier modes $\phi^{(n)}(x^\mu)$ satisfy $\Box\phi^{(n)} = (n/R)^2 \phi^{(n)}$, where the righthand side can be looked on as an effective mass term $m_n{}^2 = (n/R)^2$.

$$ds^2 \equiv g^{(5)}_{MN} dx^M dx^N = e^{-2\sigma(x^5)} \eta_{\mu\nu} dx^\mu dx^\nu + (dx^5)^2$$

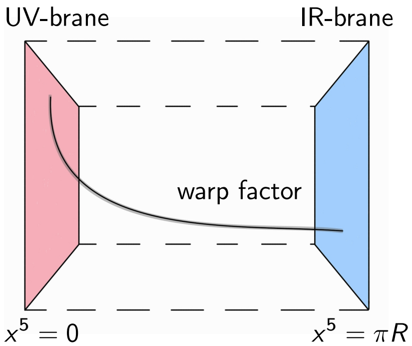


Figure: Representation of the AdS_5 setup. The fifth dimension is bounded by the UV- and IR-branes, where the warp factor is largest and smallest respectively

$$m_{IR} = m_{UV} \cdot e^{-k\pi R}$$

$\rightarrow$ geometric interpretation of couplings in terms of **wavefunction overlaps**

An **orbifold** is a topological space that is locally modeled on the quotient of Euclidean space by a finite group action:

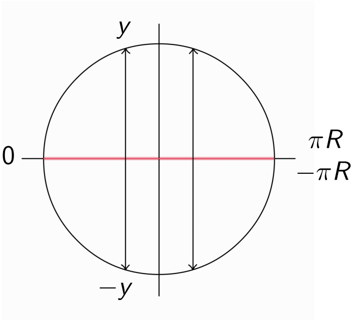


Figure: Line segment obtained by identification of opposite points on S^1

Hence, topologically, the orbifold $S^1/\mathbb{Z}_2$ is a **closed interval** $[0, \pi R]$, but with additional structure at the end-points inherited from the original circle and the group action.

$\rightarrow$ fields defined on $S^1/\mathbb{Z}_2$ must satisfy specific **boundary conditions** compatible with the orbifold symmetry (ex: parity)

Bogomol'nyi monopoles

The presence of magnetic fields raises the question of interactions between several monopoles and can be investigated from the mass in the $\lambda \to 0$ limit:

$$
\begin{aligned}
m &= -\frac{1}{4}\int \mathrm{d}^3x \left[\mathrm{Tr}\left\{B^i B_i\right\} + \mathrm{Tr}\left\{D^i\phi D_i\phi\right\}\right] \\
&= -\frac{1}{4}\int \mathrm{d}^3x \left[\mathrm{Tr}\left\{(B_i + D_i\phi)^2\right\}\right] + \frac{1}{2}\int \mathrm{d}^3x \left[\partial_i \,\mathrm{Tr}\left\{B^i\phi\right\}\right] \\
&= -\frac{1}{4}\int \mathrm{d}^3x \left[\mathrm{Tr}\left\{(B_i + D_i\phi)^2\right\}\right] + \underbrace{\frac{1}{2}\oint_{S^2_\infty} \left[\mathrm{Tr}\left\{B_i\phi\right\}\mathrm{d}S^i\right]}_{-\oint_{S^2_\infty} \mathcal{B}_i \mathrm{d}S^i \equiv -q_m}
\end{aligned}
$$

$\to B_i + D_i\phi = 0$, known as the **Bogomol'nyi equation**. The emerging idea is that if N well-separated monopoles each satisfy this equation, their total energy is just the sum of individual energies without interaction.

Property on scalar charges

Property:
In a spacetime of the form $\mathbb{R} \times N^p \times S^q$, where $\mathbb{R}$ denotes time, N^p is a p-dimensional Euclidean type space and S^q is a q-dimensional sphere, and with a factorizable scalar charge density, the presence of any charge on the compact space necessarily implies the presence of an associated anticharge of same magnitude.

Factorization hypothesis:
We restrict ourselves to a fixed-time slice of spacetime $\Sigma = N^p \times S^q$ and define $\rho(x, y)$ a scalar charge density function defined on Σ, where $x \in N^p$ and $y \in S^q$. The total charge in a region of Σ is obtained by integrating $\rho(x, y)$ over that region. Assume that the charge density comes from the divergence of some vector field, in analogy with Gauss's law. More precisely, we suppose that there exist a vector field $\boldsymbol{A}(x)$ defined on N^p, $\boldsymbol{B}(y)$ defined on S^q such that the charge density completely factorizes:

$$\rho(x, y) = \nabla_x \boldsymbol{A}(x) \nabla_y \boldsymbol{B}(y)$$