

Comprendre l'Infiniment Grand - Introduction to Cosmology - Part II

Ch. Yèche, CEA-Saclay, IRFU/DPhP

July 1, 2026

Summary of Part I

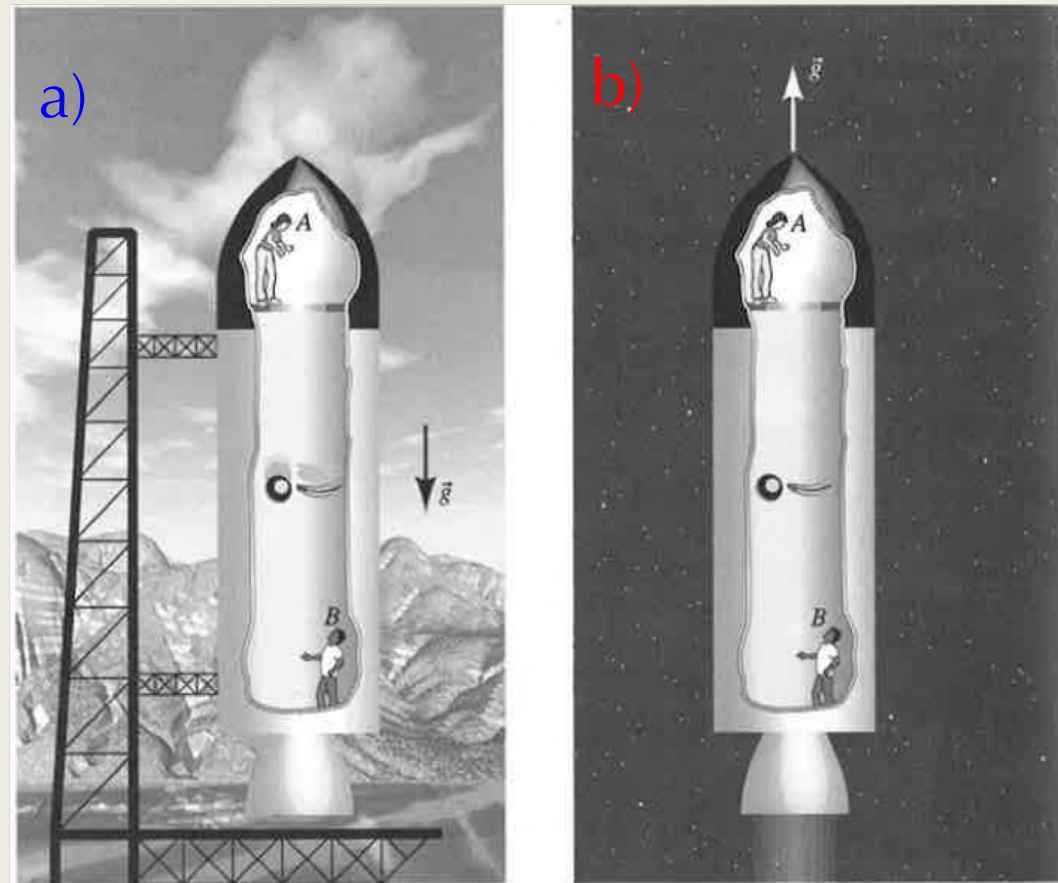
Equivalence principle - 1907

a) $m_i a = m_g g \Rightarrow$
the lead ball and the feather
experience the same
Acceleration

$\Rightarrow m_i = m_g$ and $a = g$

b) they have the same
constant speed but appear
with the same acceleration

- uniform gravitational field
= uniform acceleration

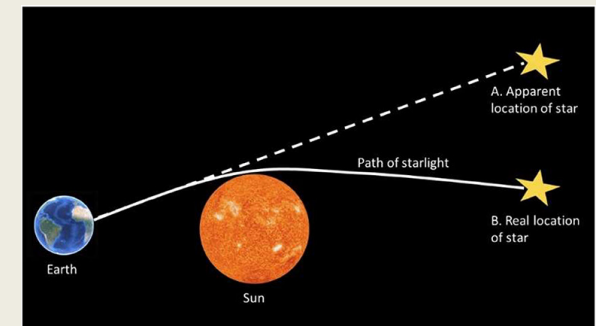
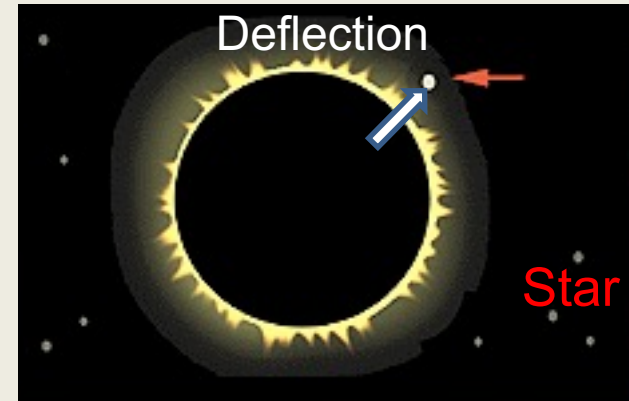


James B. Hartle

study effect of acceleration $\Rightarrow$ study gravitation

Light rays are bent - Clocks are delayed - 1911

- In 1919: [Arthur Eddington](#) observes light deviation by the sun during a solar eclipse:
 - 1.75 arc second = 8.5 μrad as predicted by Einstein
 - Twice the deflection predicted by first computation (Eq. principle alone)



- Times run slower in a gravitational field !

$$\Delta t_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right) \Delta t_A$$

At the surface of a star: $\phi_A = -GM/R$ and far away: $\phi_B = 0$

$$\Delta t_\infty = \left(1 + \frac{GM}{Rc^2}\right) \Delta t_*$$

Cosmology - Part II

1. Geometry of the Universe

- Curved spacetime – Metric
- Cosmological principles
- FLWR metric

2. Expansion of the Universe

- Cosmological redshift
- Friedman equation

3. History of the Universe

1) Geometry of the Universe

Reminder of differential geometry

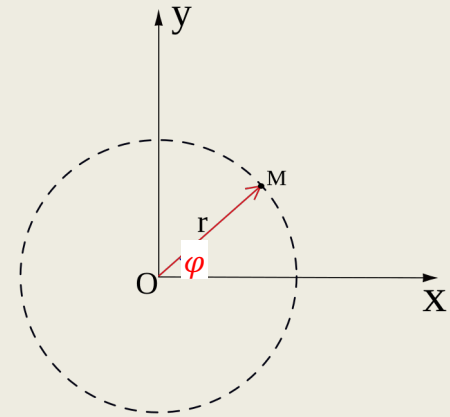
- 2D Euclidian space:

$$x = r \cos \varphi$$

$$y = r \sin \varphi$$

$$(ds)^2 = ds^2 = (dx)^2 + (dy)^2 = dx^2 + dy^2$$

$$ds^2 = dr^2 + r^2 d\varphi^2$$



- 3D Euclidian space:

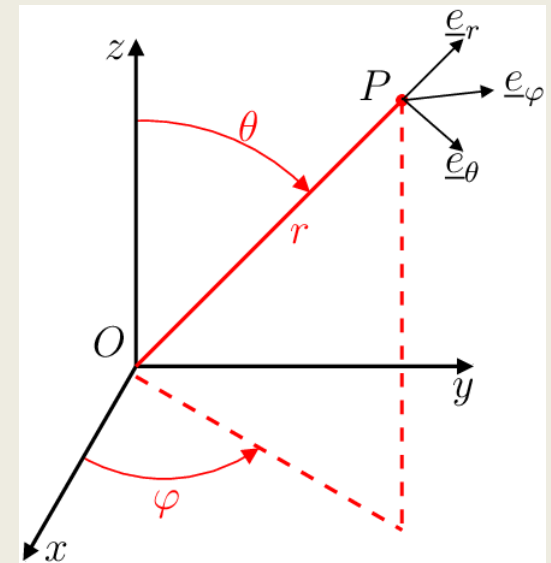
$$x = r \sin \theta \cos \varphi$$

$$y = r \sin \theta \sin \varphi$$

$$z = r \cos \theta$$

$$ds^2 = dx^2 + dy^2 + dz^2$$

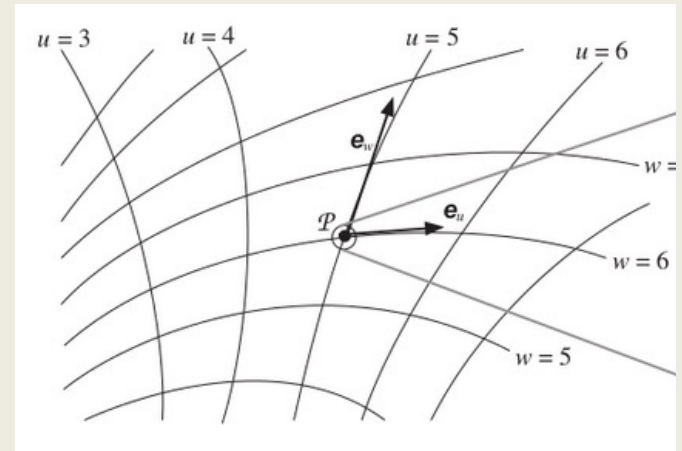
$$ds^2 = dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\varphi^2)$$



From 3D space to 4D spacetime

1) In the usual 3D Euclidian

- Define a coordinate system
 x^i = a labeling of space
ex plan (x,y) or (r,ϕ)



- We can measure distances with a ruler: $dS^2 = \mathbf{g}_{ij}(\mathbf{x}) dx^i dx^j$ (sum over 3x3 terms)

- The metric $g_{ij}(x)$ alone totally defines the geometry

- but $dS^2 = dr^2 + r^2 d\phi^2$ and $dS^2 = dx^2 + dy^2$: same geometry
we mean $(dx)^2$ and not $d(x^2)$! length² not surface

2) We generalize to a non-Euclidian 4D spacetime

Curved spacetime – Metric

- We generalize the 3D metrics to 4D in special relativity

$$ds^2 = c^2 dt^2 - dx^2 - dy^2 - dz^2 = \eta_{\alpha\beta} dx^\alpha dx^\beta$$

- We generalize in GR with non constant terms $g_{\mu\nu}$

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \text{ (16 terms, sum over } \mu \text{ and } \nu \text{)}$$

- The equivalence principle tells, $\Delta\tau_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right) \Delta\tau_A$

- GR : for a weak and static field, the metric is :

$$ds^2 = \underbrace{\left(1 + \frac{2\Phi(x)}{c^2}\right)}_{\text{equivalence principle}} c^2 dt^2 - \underbrace{\left(1 - \frac{2\Phi(x)}{c^2}\right)}_{\text{GR}} (dx^2 + dy^2 + dz^2)$$

fixed object $\Delta\tau^2 = \frac{ds^2}{c^2} = \left(1 + \frac{2\Phi}{c^2}\right) dt^2 \Rightarrow \Delta\tau_B = \left(1 - \frac{\Phi_A - \Phi_B}{c^2}\right) \Delta\tau_A$

Homogenous and isotropic

- **Cosmological principle**
 - Universe isotropic + homogeneous on large scales
 - Universe looks the same whoever and wherever you are
- **Isotropic** (on large scales)
 - CMB very isotropic
 - X ray background, radio galaxies
- **Homogeneous**
 - Test with 3D galaxy surveys
 - Only at large scales.... $> \text{Mpc}$

FLRW metric

- Homogeneous and isotropic $\Rightarrow$
Friedmann, Lemaitre, Robertson, Walker metric

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

- Isotropic: spherical coordinates $dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$
- Homogeneous: scale factor $R(t)$ due to expansion, it does not depend on (r, θ, ϕ)
- Dimensionless scale factor : $a(t) = R(t) / R(t_0)$
now $a(t_0) = 1$ **index 0, means today**
in the past $a(t) < 1$
Big Bang $a(t) = 0$

FLRW metric

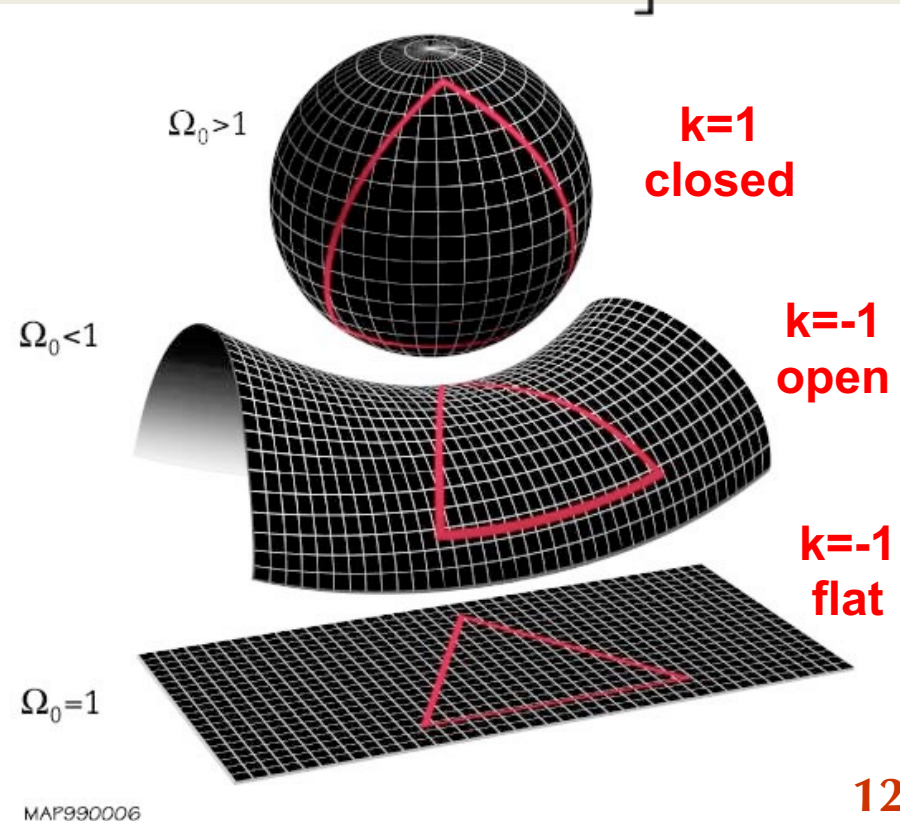
Friedmann, Lemaître, Robertson, Walker metric

$$ds^2 = dt^2 - R^2(t) \left[\frac{dr^2}{1 - kr^2} + r^2(d\theta^2 + \sin^2 \theta d\phi^2) \right]$$

$k = 1$: spherical geometry
or closed ($\sum \alpha > 180^\circ$)

$k = -1$: hyperbolic geometry
or open ($\sum \alpha < 180^\circ$)

$k = 0$: flat geometry ($\sum \alpha = 180^\circ$)



Comoving distance

- **Change of coordinates** $r = \sin \chi$ ($k=1$, closed)

$$r = \chi \quad (k=0, \text{ flat})$$

$$r = \sinh \chi \quad (k=-1, \text{ open})$$

$$ds^2 = dt^2 - R^2(t) \left[d\chi^2 + \begin{Bmatrix} \sin^2 \chi \\ \chi^2 \\ \sinh^2 \chi \end{Bmatrix} (d\theta^2 + \sin^2 \theta d\phi^2) \right] \begin{Bmatrix} \text{closed} \\ \text{flat} \\ \text{open} \end{Bmatrix}$$

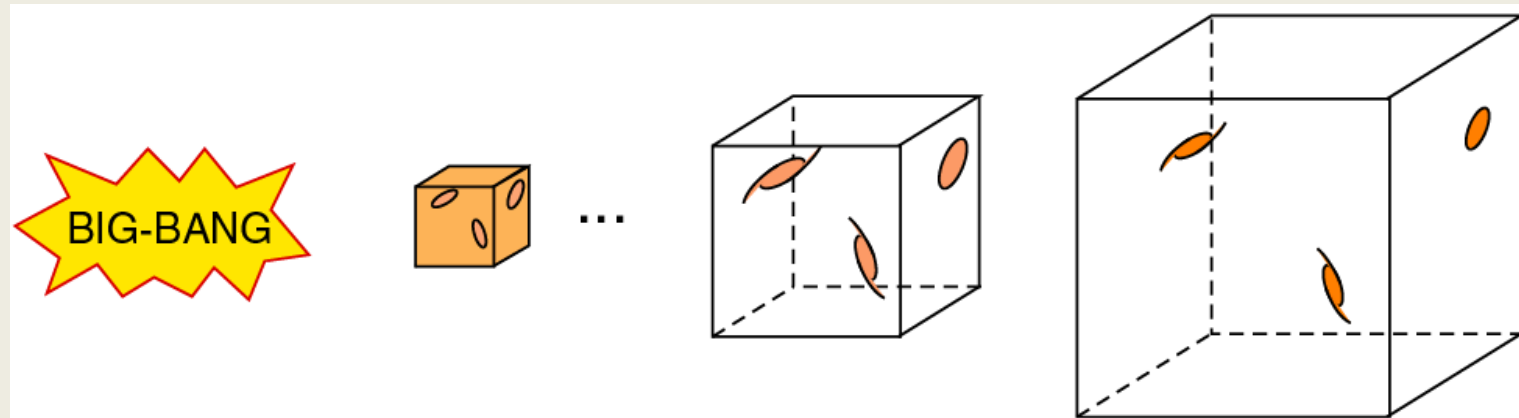
$\sin \rightarrow$ spherical $\sinh \rightarrow$ hyperbolical

- **Distance:**

- Galaxies remain at $\chi = \text{cst}$ (up to small local velocities)
- Physical distance between 2 galaxies : $R(t) \times \Delta\chi$ (Mpc)
increases with the expansion
- “comoving” distance : $R(\mathbf{t_0}) \times \Delta\chi$ is fixed (comoving Mpc)
= distance including the expansion up to $\mathbf{t=t_0}$
= independent from Universe expansion

The Big Bang

- Universe is expanding with H_0 (~ 70 (km/s)/Mpc)
- If we go back in time,
Universe more and more dense and hot: big bang



- Analogy with an inflated balloon
- Name invented by Fred Hoyle

2) Expansion of the Universe

Cosmological redshift

- Radial photon

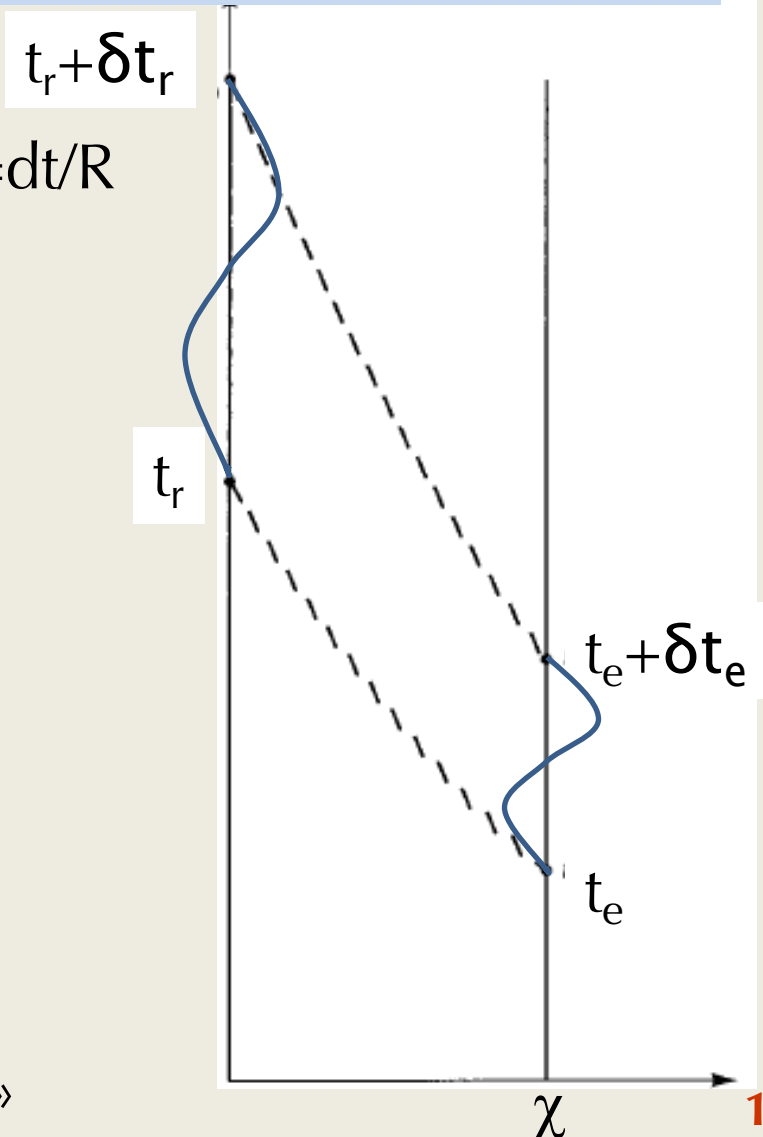
$$ds^2 = dt^2 - R^2(t) d\chi^2 = 0 \Rightarrow d\chi = dt/R$$

$$\begin{aligned} \chi &= \int_{t_e}^{t_r} \frac{dt}{R(t)} = \int_{t_e + \delta t_e}^{t_r + \delta t_r} \frac{dt}{R(t)} \\ \Rightarrow \int_{t_e}^{t_e + \delta t_e} \frac{dt}{R(t)} &= \int_{t_r}^{t_r + \delta t_r} \frac{dt}{R(t)} \\ &\Rightarrow \frac{\delta t_e}{R(t_e)} = \frac{\delta t_r}{R(t_r)} \end{aligned}$$

$$1 + z_e \equiv \frac{\lambda_r}{\lambda_e} = \frac{\delta t_r}{\delta t_e} = \frac{R(t_r)}{R(t_e)} \equiv \frac{1}{a(t_e)}$$

$$1 + z = \frac{1}{a}$$

« λ is dilating with the Universe »



Redshift: A fundamental concept in cosmology

- Measuring $z \rightarrow$ scale factor a when light emitted
- It is a cosmological redshift, $1+z = 1/a$ can be e.g. $z=1000$ (at CMB) cannot be interpreted as a simple Doppler effect
- In case of Hubble law ($v=H_0d$), it is locally interpreted as a Doppler effect
- z is also a measurement of time: e.g. CMB occurred at $z = 1100$ (i.e. when $a=0.0009$)

Hubble parameter

- Assume $t_e \sim t_0$ (locally) $\Rightarrow a \sim 1$, small z

$$1 + z = \frac{1}{a} \quad z = \frac{v}{c} = \frac{1 - a}{a} = \frac{\dot{a}\Delta t}{a} \quad \Rightarrow \quad v = \frac{\dot{a}}{a}(c\Delta t)$$

$$v = \frac{\dot{a}}{a}D \quad \text{Hubble law with} \quad H_0 = \frac{\dot{a}(t_0)}{a(t_0)} = H(t_0)$$

- Hubble parameter $H(t) \equiv \frac{\dot{a}(t)}{a(t)}$
- H_0 is not very precisely measured, we define

$$h \equiv \frac{H_0}{100 \text{ (km/s)/Mpc}} \approx 0.7$$

- cosmological results in units like $h^{-1}\text{Mpc}$
numerical result independent of h

Thermodynamic

- a volume V including a fixed number of particles
(i.e. galaxies !)

$$dE = -P dV$$

$$E = \rho V$$

- the physical volume is $V = a^3(t) V_{\text{com}}$ ($V_{\text{com}} = \text{comoving volume}$)

$$d_t (\rho a^3 V_{\text{com}}) = -P d_t (a^3 V_{\text{com}}) \quad \text{but } V_{\text{com}} = \text{cst} = V_0$$

$$d_t [\rho(t) a^3(t)] = -P(t) d_t [a^3(t)]$$

matter, radiation

- **Matter:** $d_t [\rho a^3] = -P d_t [a^3]$

Galaxies may be approximated as a pressure-less gas:
galaxies have no velocity relative to the overall expansion

$$\Rightarrow d_t [\rho_m a^3] = 0$$

$$\rho_m(t) = \rho_m(t_0) a^{-3}(t)$$

- **Pure radiation** (black body) Stefan's law: $\rho_r = g \frac{\pi^2 (k_B T)^4}{30 (\hbar c)^3}$
Thermodynamics: $P_r = (1/3) \rho_r$

$$d_t [\rho a^3] = -(1/3) \rho d_t [a^3] \Rightarrow 4pa^3 d(a) + a^4 d(\rho) = 0$$

$$\rho_r(t) = \rho_r(t_0) a^{-4}(t) \quad \begin{array}{l} a^{-3} \text{ for volume} \\ a^{-1} \text{ since } E \propto \lambda^{-1} \end{array}$$

$$T(t) = T(t_0) / a(t)$$

Vacuum

- “Vacuum is not empty”
virtual particle-antiparticle pairs
- Results in a vacuum energy density constant in space and time

$$d_t [\rho a^3] = -P d_t [a^3] \quad \Rightarrow \quad \rho d_t [a^3] = -P d_t [a^3]$$

$$P_v = - \rho_v = \text{cst} < 0$$

- Vacuum pressure is negative !
- Vacuum energy equivalent to cosmological constant or a form of dark energy: $\rho_v = \Lambda/(8\pi G)$ in Einstein equation

Friedman equation

- Einstein Eq => $\left(\frac{\dot{R}}{R}\right)^2 + \frac{k}{R^2} = \frac{8\pi\rho}{3}$ (Friedmann Eq.)

- Critical density today for which the Universe is flat ($k=0$)

$$t=t_0: \quad \frac{8\pi\rho_c}{3} = \left(\frac{\dot{R}}{R}\right)_0^2 = \left(\frac{\dot{a}}{a}\right)_0^2 = H_0^2$$

$$\boxed{\rho_c = \frac{3H_0^2}{8\pi}} = 1.88 \times 10^{-29} h^2 \text{ g/cm}^3 \sim 5 \text{ protons / m}^3$$

- We introduce $\Omega_m \equiv \frac{\rho_m(t_0)}{\rho_c}, \quad \Omega_r \equiv \frac{\rho_r(t_0)}{\rho_c}, \quad \Omega_v \equiv \frac{\rho_v(t_0)}{\rho_c}$

$$\Omega_T = \Omega_m + \Omega_r + \Omega_v = \rho_0 / \rho_c \quad (\Omega_x, \text{ at } t=t_0, \text{ should be } \Omega_x^0)$$

$$\begin{aligned} \rho(a) &= \rho_m(t_0)a^{-3} + \rho_r(t_0)a^{-4} + \rho_v(t_0) \\ &= \rho_c(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v), \end{aligned}$$

Friedman equation

$$\bullet \left(\frac{\dot{R}}{R} \right)^2 + \frac{k}{R^2} = \frac{8\pi\rho}{3} \xrightarrow{t=t_0} \frac{k}{R_0^2} = \frac{8\pi\rho_0}{3} - H_0^2 = H_0^2(\Omega_T - 1)$$

$\rho_0 = \rho_c \Omega_T$

$$\left(\frac{\dot{a}}{a} \right)^2 = \left(\frac{\dot{R}}{R} \right)^2 = \frac{8\pi\rho}{3} - H_0^2(\Omega_T - 1) \left(\frac{R_0}{R} \right)^2 = H_0^2 \left[\frac{\rho(a)}{\rho_c} + (1 - \Omega_T)a^{-2} \right]$$

$\frac{8\pi\rho_c}{3} = H_0^2$

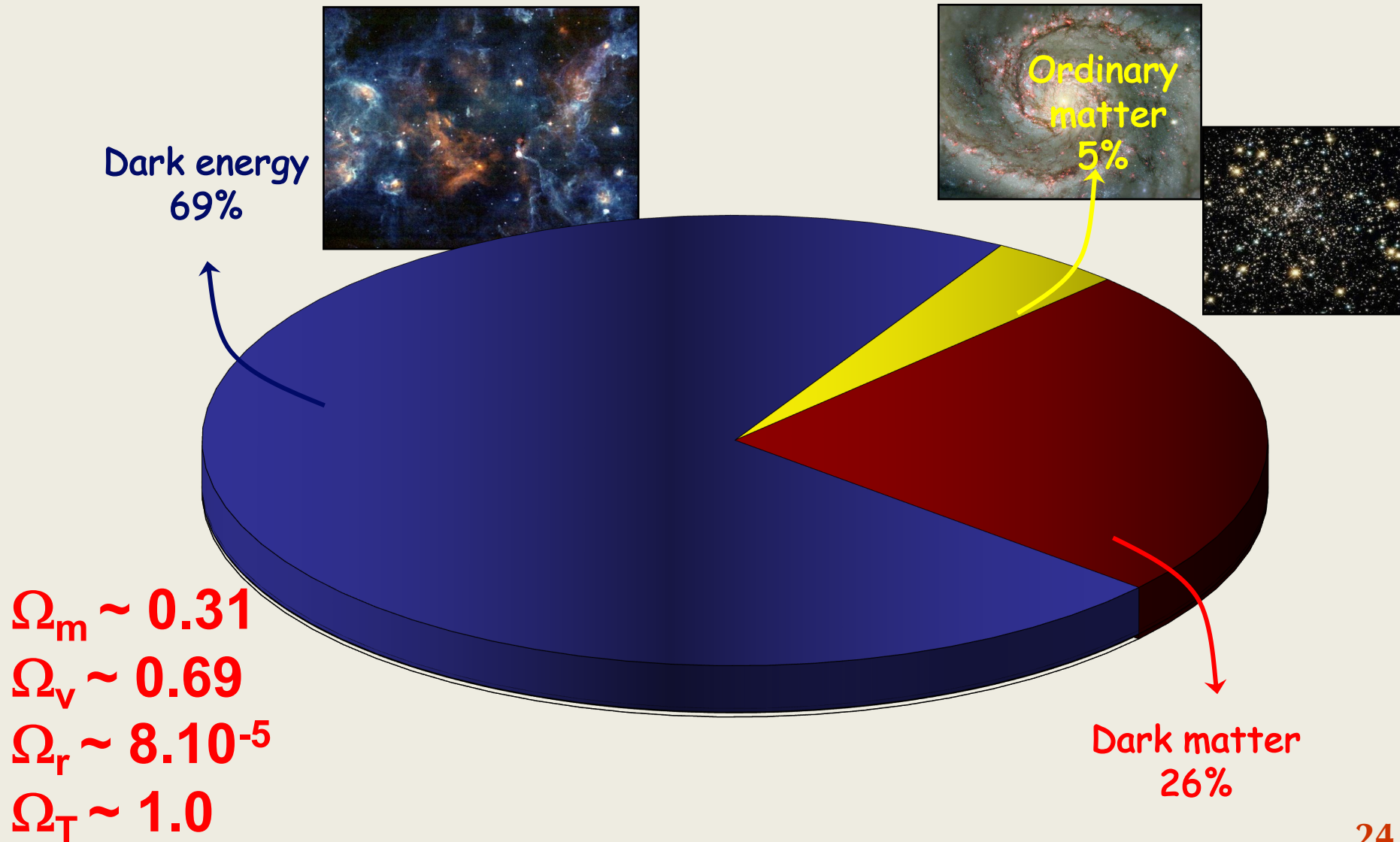
$$\bullet \rho(a) = \rho_m(t_0)a^{-3} + \rho_r(t_0)a^{-4} + \rho_v(t_0)$$

$$= \rho_c(\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v),$$

$$\left(\frac{\dot{a}}{a} \right)^2 = H_0^2 [\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + (1 - \Omega_T)a^{-2}]$$

Simplification: for a flat Universe ($k=0 \Rightarrow 1 - \Omega_T = 0$)

Content of the Universe



3) History of the Universe

Age of the Universe

$$\left(\frac{\dot{a}}{a}\right)^2 = H_0^2 [\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + (1 - \Omega_T) a^{-2}]$$

- many quantities may be computed from this equation by expressing in terms of 'a' and $\dot{a}/a$

- e.g. the age of the universe : $dt = \frac{dt}{da} da = \frac{da}{\dot{a}} = \frac{da}{a(\dot{a}/a)}$

$$t = H_0^{-1} \int_0^1 \frac{da}{a [\Omega_m a^{-3} + \Omega_r a^{-4} + \Omega_v + (1 - \Omega_T) a^{-2}]^{1/2}}$$

- $H_0 = 70(km/s)/Mpc = \frac{70km/s}{10^6 \times 3.262 \times 1an \times 300000km/s}$

$$H_0^{-1} = 14.10^9 \text{ years}$$

Age of the Universe

- Note: - our Universe is flat ($k=0 \Rightarrow \Omega_T = 1$)
- one may often neglect $\Omega_r = 9 \cdot 10^{-5}$ ($\Omega_m=0.3, \Omega_v=0.7$)
- Simplification of the equation:

$$t = H_0^{-1} \int_0^a \frac{da}{a (\Omega_r a^{-4} + \Omega_m a^{-3} + \Omega_v)^{1/2}}$$

- Universe with just matter $\Omega_m \approx 1$ and $a \rightarrow 1$

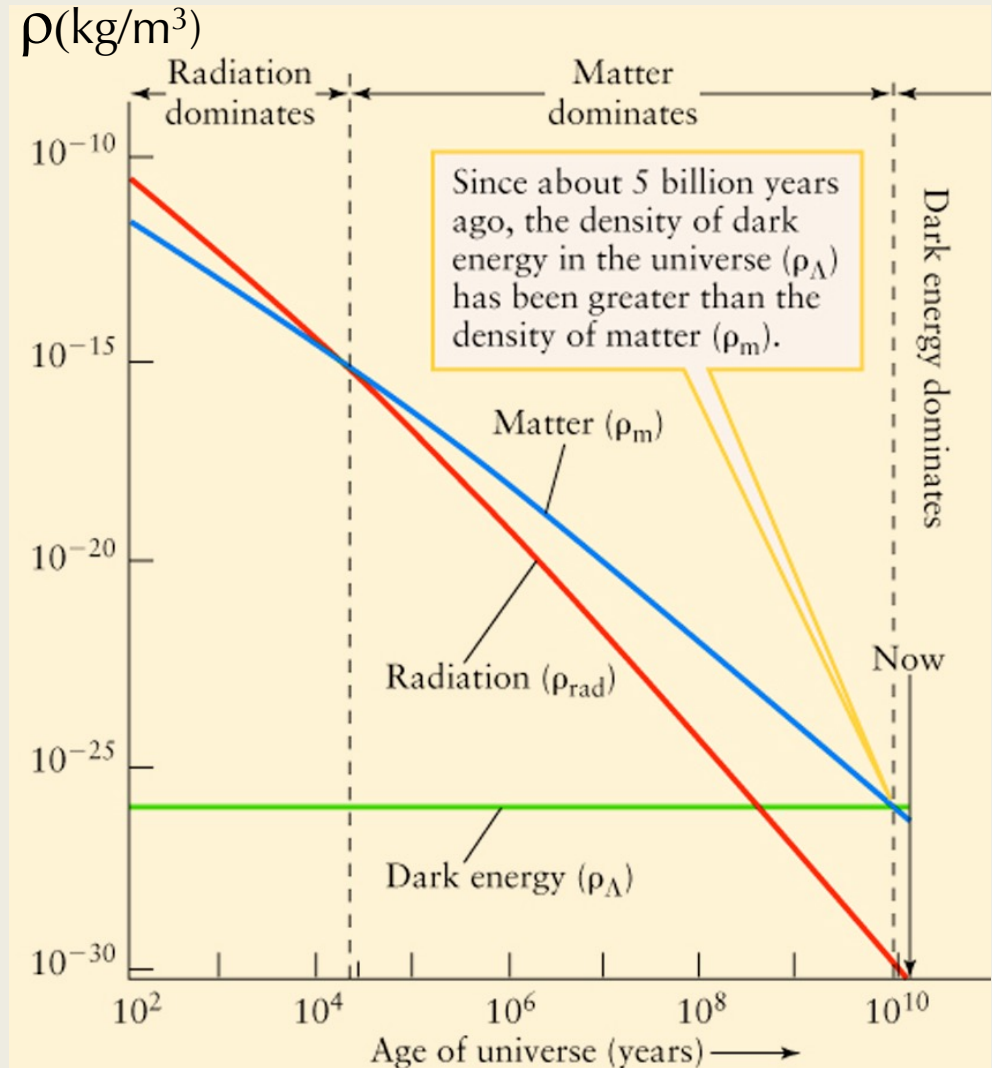
$$t(a) = H_0^{-1} \int_0^a \frac{da}{a^{-1/2}} = H_0^{-1} \int_0^a a^{1/2} da = \frac{2}{3} H_0^{-1} a^{3/2}$$

T $\sim 9 \cdot 10^9$ years, incompatible with the age of the first stars in MW (globular clusters)

Epochs of the universe

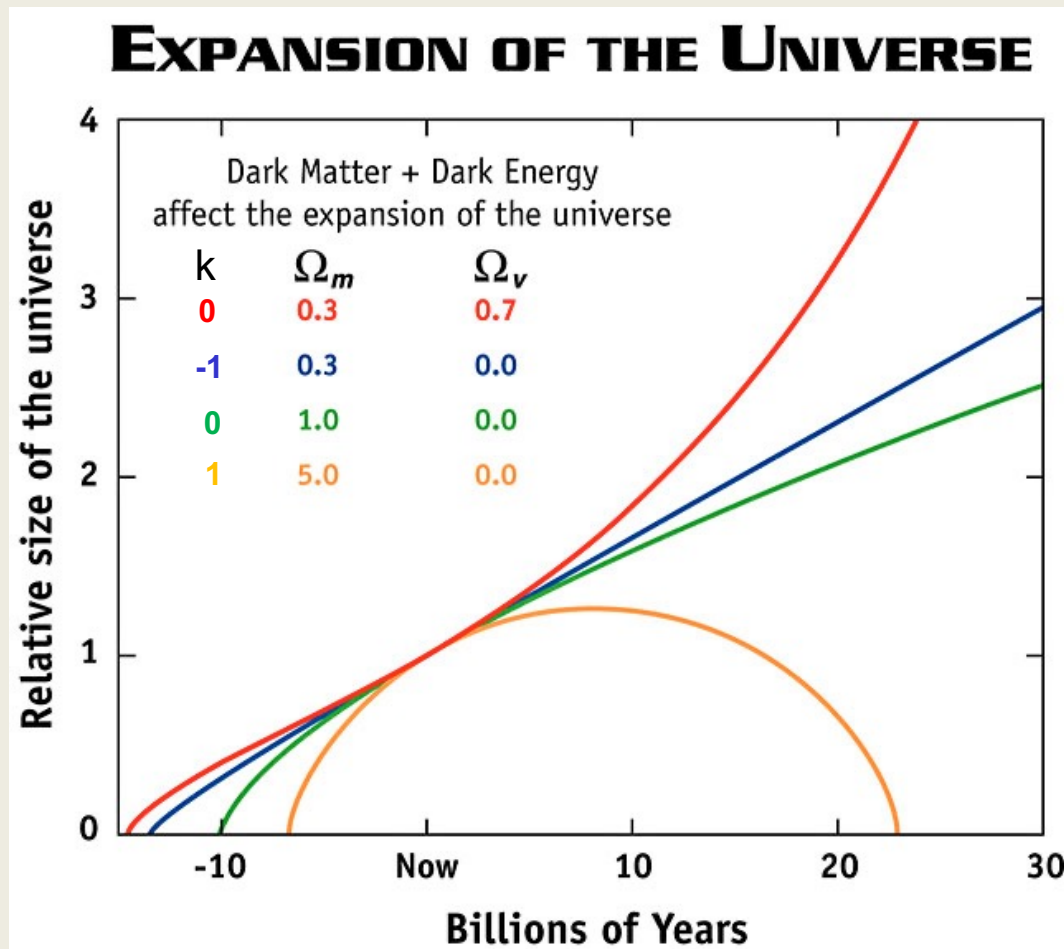
$$\rho(a) = \rho_{crit} \left(\Omega_v + \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} \right)$$

- beginning
('a' very small)
radiation dominates
- then matter dominates
- “recently” vacuum
(or dark energy)
dominates

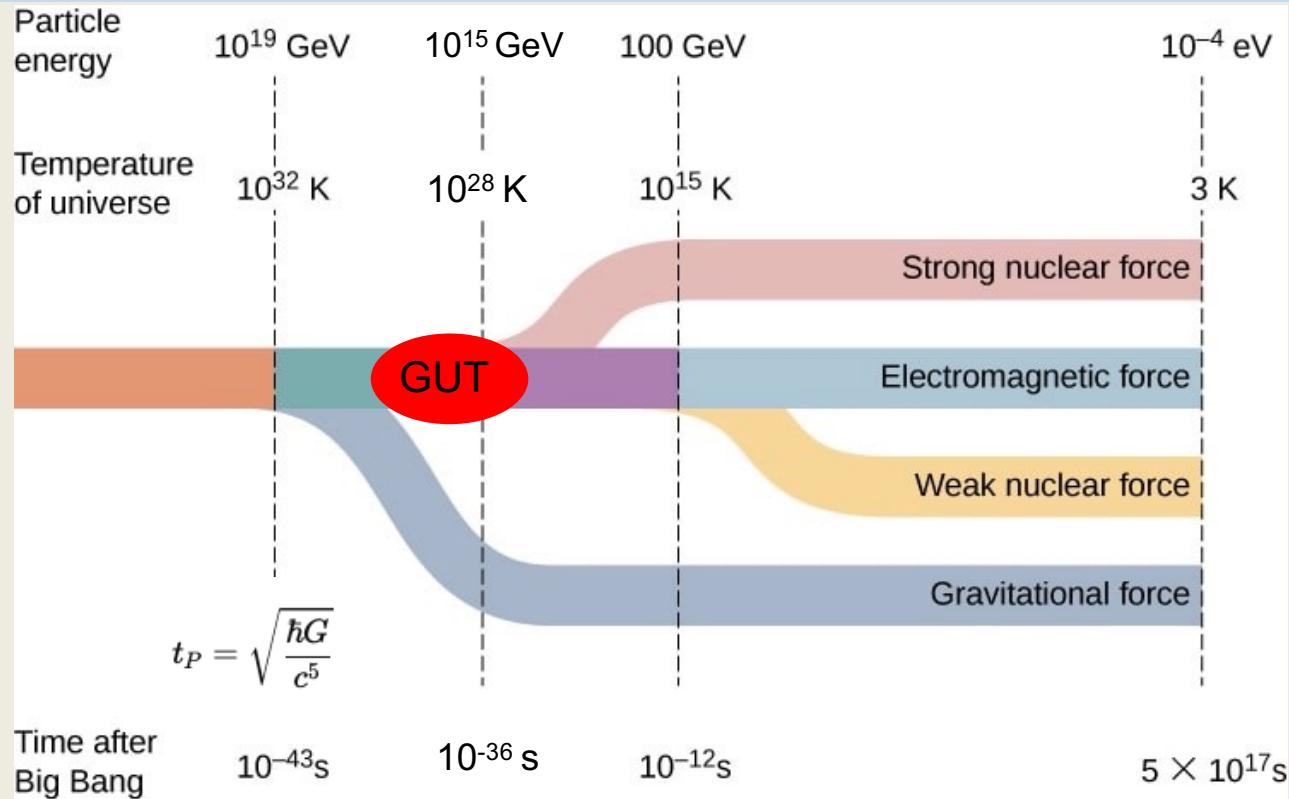


Fate of the Universe

- Fate of the Universe strongly related to the metric, i.e. k values (close, open or flat) and to the fraction of Dark Energy

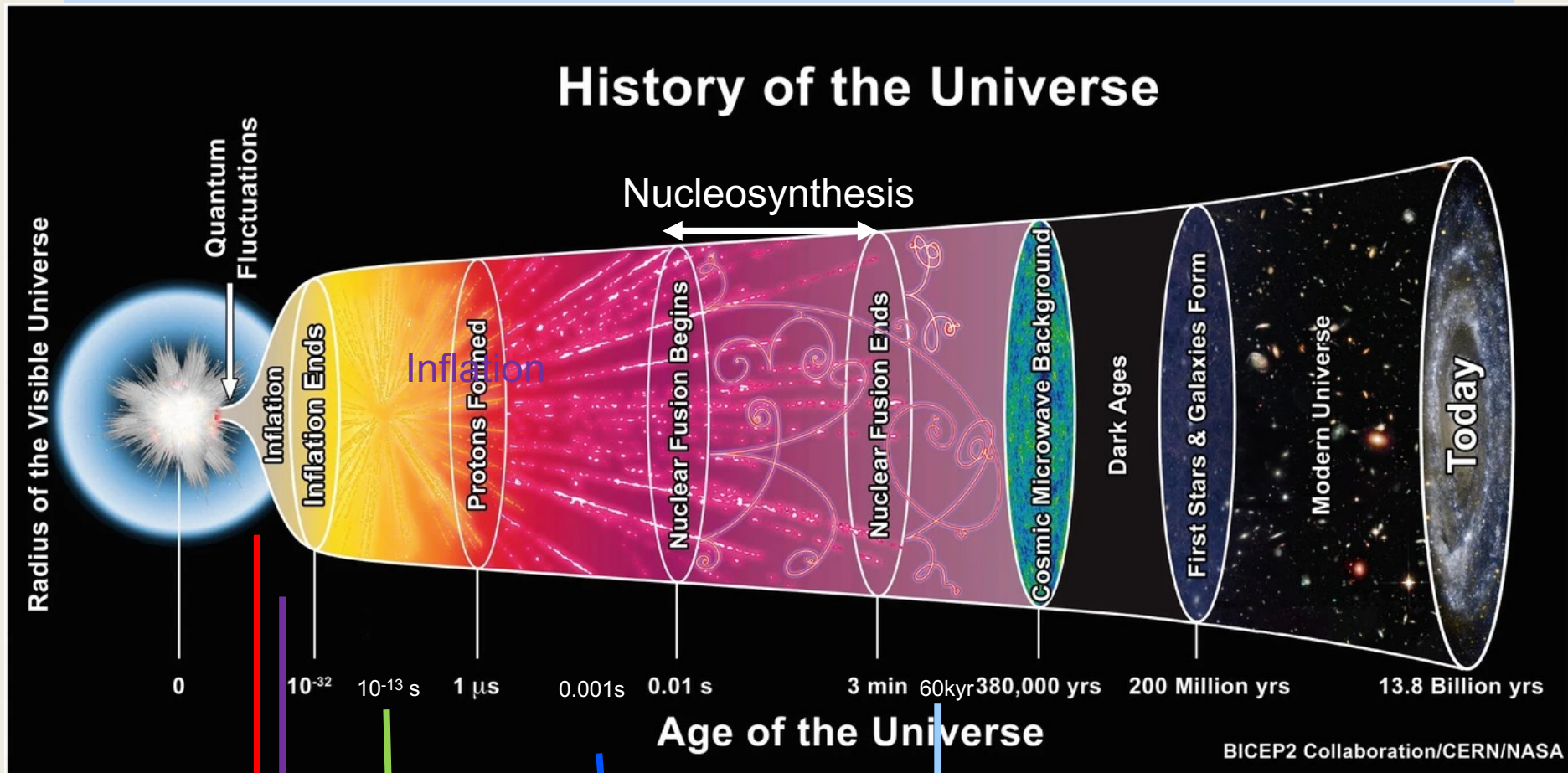


Unification of forces



- EW force $\sim 100 \text{ GeV}$
- Grand unification Theory (GUT) $\sim 10^{15} \text{ GeV}$ at 10^{-36} s after BB
- Theory of Everything $\sim 10^{19} \text{ GeV}$ (Planck scale)
- $t < t_P$ quantum effects of gravity become extremely strong, we need a theory of quantum gravity

Timeline of Universe history



GUT

EW

Inflation

Neutrinos
decoupling

Matter/radiation
equality