UNIVERSITY OF ALBERTA

QED in Ashtekar-Barbero Variables & Its Implications

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Content

- Study constraints in the presence of fermions coupled to gravity

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- Modified Ashtekar-Barbero variables

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Background

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  $M \cong \Sigma_t \times \mathbb{R}$ with
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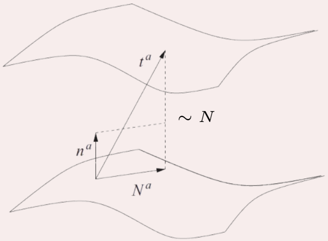

n^a: normal vector, N: lapse function, N^a: shift vector.

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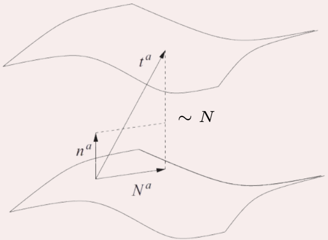
  - Time-evolution vector field $t^a = Nn^a + N^a$



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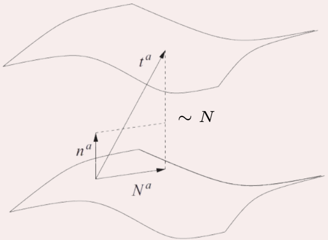

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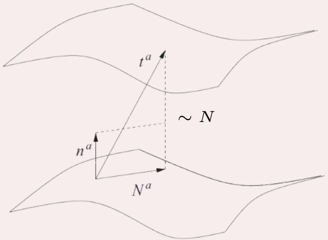

n^a: normal vector, N: lapse function, N^a: shift vector.

- Tetrads e^a_I s.t. $g_{ab} e^a_I e^b_J = \eta_{IJ}$
  - New covariant derivative

$$\mathcal{D}_a v^I = \nabla_a v^I + \underbrace{\omega_a{}^I{}_J}_{\text{connection 1-form}} v^J$$

Vacuum

Generalization of the Einstein-Hilbert action: Holst action

$$S = \frac{1}{2\kappa}\int d^4x\,|\det \mathrm{e}|\, e^a_I e^b_J P^{IJ}_{KL} F^{KL}_{ab}(\omega)$$

where

$$P^{IJ}{}_{KL} = \delta^{[I}_K \delta^{J]}_L - \frac{1}{2\beta}\varepsilon^{IJ}{}_{KL}\,, \qquad \kappa = 8\pi G\,,$$

$$F^{IJ}(\omega) = d\omega^{IJ} + \omega^{IK} \wedge \omega_K{}^J\,, \qquad \omega^{IJ}_a = e^{bI}\nabla_a e^J_b$$

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- From variation of the action w.r.t. ω_a^{IJ}, compatibility condition

$$P^{KL}{}_{IJ} \mathcal{D}_b \left(|\det \mathrm{e}| e_K^{[a} e_L^{b]} \right) = 0$$

- Using the parametrization

$$e_I^a = \mathcal{E}_I^a - n^a n_I$$

where $\mathcal{E}_I^a n_a = \mathcal{E}_I^a n^I = 0$.

- Imposing the time gauge:
$e_0^a = n^a \implies n^I = \delta_0^I$

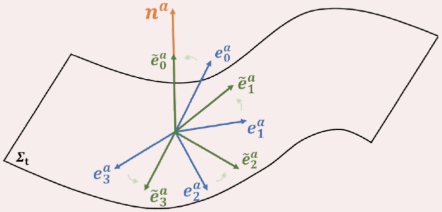

Figure: Time-gauge: $\tilde{e}_I^a$ (green) is the frame after the gauge fixing.

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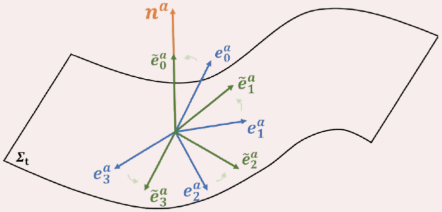

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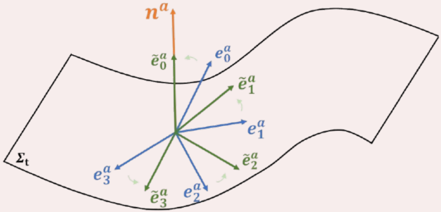

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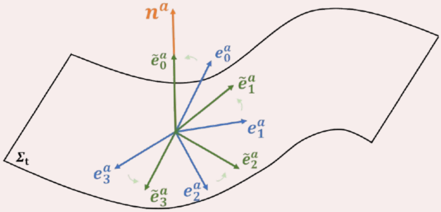

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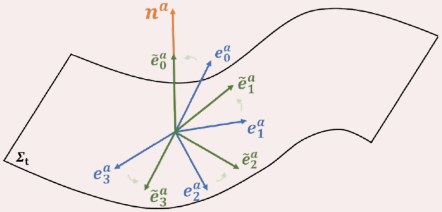

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$$A^i_a := \beta K^i_a + \Gamma^i_a \quad \text{Ashtekar-Barbero connection}$$

Vacuum Constraints

After imposing the compatibility condition

- $\mathcal{G}_i^G[A, E] = \frac{1}{\kappa\beta}\mathcal{D}_a^{(A)} E_i^a = \frac{1}{\kappa}\epsilon_{ij}{}^{k} K_a^j E_k^a$

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where $\mathcal{F}_{ab}^i(A) = 2\partial_{[a}A_{b]}^i + \varepsilon^i{}_{jk}A_a^j A_b^k$.

Hence

$$H = \int d^3x\left(-\Lambda^i\mathcal{G}_i^G + N\mathcal{H}^G + N^a\mathcal{H}_a^G\right)$$

with $\Lambda^i = \beta\omega_t^{0i} - \frac{1}{2}\varepsilon^i{}_{jk}\omega_t^{jk}$.

Fermionic Field

Dirac Fermions in Curved Spacetime

Covariant derivative in curved space-time (signature $(-+++)$)

$$\mathfrak{D}_a\Psi = \partial_a\Psi + \frac{1}{2}\omega_a^{IJ}\sigma_{IJ}\Psi$$

with $\sigma_{IJ} = \frac{1}{4}[\gamma_I, \gamma_J]$, Dirac spinors $\Psi = \begin{pmatrix}\psi_L \\ \psi_R\end{pmatrix}$ and gamma matrices defined as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}$$

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$\Longrightarrow$ *fermionic action* in curved space-time

$$S_F = \int d^4x|\det e|\left\{\frac{i}{2}\left(e^a_I\overline{\Psi}\gamma^I\mathfrak{D}_a\Psi - e^a_I\overline{\mathfrak{D}_a\Psi}\gamma^I\Psi\right) - m\overline{\Psi}\Psi\right\}$$

Fermionic Field Minimally Coupled to Gravity

- $$S_{G+F} = \int d^4x \, |\det e| \left\{ \frac{1}{2\kappa} e^a_I e^b_J P^{IJ}_{KL} F^{KL}_{ab}(\omega) + \left[\frac{i}{2} \left(e^a_I \overline{\Psi} \gamma^I \mathfrak{D}_a \Psi - e^a_I \overline{\mathfrak{D}_a \Psi} \gamma^I \Psi \right) - m \overline{\Psi} \Psi \right] \right\}$$

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- From $\frac{\delta S_{G+F}}{\delta \omega^{IJ}_a} = 0$, *new compatibility condition*:

$$\mathcal{D}_b(|\det \mathrm{e}| e^{[a}_K e^{b]}_L) = \frac{\kappa}{4} \frac{\beta^2}{1+\beta^2} |\det \mathrm{e}| \Big(\epsilon_{KL}{}^{MN} e^a_M J_N - \frac{1}{\beta} \left(e^a_K J_L - e^a_L J_K \right) \Big)$$

with $J^I = \overline{\Psi} \gamma_5 \gamma^I \Psi$ fermionic axial current

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$\Longrightarrow$ connection must be modified

New Ashtekar Connection

From $\frac{\delta S_{G+F}}{\delta \Gamma^i_a} = 0$ & modified compatibility condition

- **New spin connection**

$$\Gamma^i_a \to \Gamma^i_a + \underbrace{\frac{\kappa}{4}\frac{\beta^2}{1+\beta^2}\left(e^i_a J^0 - \frac{1}{\beta}\epsilon^i{}_{jk} e^j_a J^k\right)}_{C^i_a}$$

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- Modified Ashtekar connection

$$A^i_a \to A^i_a + \mathrm{C}^i_a$$

Fermionic & Gravitational Constraints

Decomposing the space-time as before and imposing the new compatibility condition

- $\mathcal{G}_i^{G+F}[A, E, \Psi, \bar{\Psi}] = \mathcal{G}_i^{G}[A, E] + \frac{1}{2}\frac{\beta^2}{1+\beta^2}\sqrt{\det h}J_i$

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- $\mathcal{H}^{G+F}[A, E, \Psi, \bar{\Psi}] = \mathcal{H}^{G}[A, E] - \frac{i}{2}E_i^a\left(\overline{\Psi}\gamma^i D_a^{(A)}\Psi - \overline{D_a^{(A)}\Psi}\right) - \frac{1}{2}\epsilon^i{}_{jk}E_i^a K_a^j J^k + \frac{\beta}{2}E_i^a K_a^i J^0 + \sqrt{\det h}\, m\,\overline{\Psi}\Psi$
- $\mathcal{H}_a^{G+F}[A, E, \Psi, \bar{\Psi}] = \mathcal{H}_a^{G}[A, E] + \frac{i}{2}\left(\overline{\Psi}\gamma^0 D_a^{(A)}\Psi - \overline{D_a^{(A)}\Psi}\gamma^0\Psi\right) - \frac{\beta}{2}\sqrt{\det h} K_a^i J_i$

with $D_a^{(A)}\Psi = \partial_a\Psi - iA_a^i\gamma_5\sigma_{0i}\Psi$

Issue

- Until now the canonical pairs are (A_a^i, E_j^b), (Ψ, Π) with $\Pi = i\sqrt{\det h}\Psi^\dagger$
- In this case, the fermionic symplectic term is

$$\Theta = \int d^4x \, \Pi \, \dot{\Psi} + \underbrace{\frac{i}{2}\kappa\beta \int d^4x \, \Psi^\dagger \, \Psi \, e_a^i \, \dot{E}_i^a}_{\Longrightarrow \, A_a^i \text{ acquires an imaginary correction}} - \int d^4x \, \mathcal{L}_t(\Pi \Psi)$$

Half-Density Fermions

- Problem solved by half-density fermions

$$\xi = \sqrt[4]{\det h}\,\Psi \implies \pi = i\xi^\dagger$$

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and anti-Poisson brackets

$$\{\xi_A(x), \pi_B(y)\}_+ = \delta_{AB}\delta(x-y)$$

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$$\{\xi_A(x), \pi_B(y)\}_+ = \delta_{AB}\delta(x-y)$$

- Components of the densitized fermionic axial current

$$\overline{J}^i = \sqrt{\det h}\,J^i = -i\pi\gamma^0\gamma_5\gamma^i\xi$$

$$\overline{J}^0 = \sqrt{\det h}\,J^0 = -i\pi\gamma^0\gamma_5\gamma^0\xi$$

Fermionic EOM in Curved Spacetime

Fermionic EOM in Curved Spacetime

- Via Euler-Lagrange (E-L) eq. and considering foliation of space-time
    - Dirac EOM in curved spacetime

$$
\begin{aligned}
(t^a - N^a)[i\gamma^0 D_a^{(A)}\xi - \frac{1}{2}\beta K_a^i \gamma_5 \gamma_i \xi] - N[ie_i^a \gamma^i D_a^{(A)}\xi \\
+ \frac{1}{2}\epsilon^i{}_{jk} e_i^a K_a^j \gamma_5 \gamma^k \xi - \frac{1}{2}\beta e_i^a K_a^i \gamma_5 \gamma_0 \xi - m\xi] = 0
\end{aligned}
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- Via Hamiltonian formulation
  - Substituting ξ in $\mathcal{H}^{G+F}$ and $\mathcal{H}_a^{G+F}$
  - Using the anti-Poisson brackets

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- Via Hamiltonian formulation
  - Substituting ξ in $\mathcal{H}^{G+F}$ and $\mathcal{H}_a^{G+F}$
  - Using the anti-Poisson brackets

$\Longrightarrow$ fermionic EOM from Hamiltonian consistent with those from E-L method

QED

Total Action

$$S_{\text{full}} = S_G + S_F + S_\gamma$$
$$= \int d^4x\, |\det \mathrm{e}| \left\{ \frac{1}{2\kappa} e^a_I e^b_J P^{IJ}_{KL} F^{KL}_{ab}(\omega) + \left[\frac{i}{2} \left(e^a_I \overline{\Psi} \gamma^I \tilde{\mathfrak{D}}_a \Psi \right.\right.\right.$$
$$\left.\left.\left. - e^a_I \overline{\mathfrak{D}_a \Psi} \gamma^I \Psi \right) - m \overline{\Psi} \Psi \right] \right\} + \int d^4x \sqrt{-\det g} \left(-\frac{1}{4} g^{ac} g^{bd} F_{cd} F_{ab} \right)$$

where

$$\tilde{\mathfrak{D}}_a \Psi = \partial_a \Psi + \frac{1}{2} \omega_a{}^{IJ} \sigma_{IJ} \Psi + \underbrace{i\, q\, A_a \Psi}_{\text{QED interaction term}}$$

$$F_{ab} = (\nabla_a A_b - \nabla_b A_a)$$
$$\pi^a = \sqrt{\det h}\, h^{ab} n^c F_{cb}$$ conjugate momentum to A_a

Constraints

$$\frac{\delta S_{\text{full}}}{\delta \omega_a{}^{IJ}} = \frac{\delta S_F}{\delta \omega_a{}^{IJ}} = -\frac{1}{4}|e| e^a_K \epsilon^K{}_{IJL} J^L \implies$$ photons do **not** modify compatibility condition

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Total constraints:

- $\Lambda \mathcal{G}^{\text{full}}[A, E, \Psi, \Pi, A_a, \pi^a] = \Lambda^i \mathcal{G}_i^{G+F} + A_t \left(\nabla_a \pi^a - \sqrt{\det h} q \Psi^\dagger \Psi \right)$

Constraints

$$\frac{\delta S_{\text{full}}}{\delta \omega_a{}^{IJ}} = \frac{\delta S_F}{\delta \omega_a{}^{IJ}} = -\frac{1}{4}|e| e^a_K \epsilon^K{}_{IJL} J^L \implies$$ photons do **not** modify compatibility condition

Total constraints:

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- $$\mathcal{H}^{\text{full}}[A, E, \Psi, \Pi, A_a, \pi^a] = \mathcal{H}^{G+F} + \frac{\sqrt{\det h}}{4} h^{ac} h^{bd} F_{cd} F_{ab} + \frac{1}{2\sqrt{\det h}} h_{ab} \pi^a \pi^b + q E^a_i A_a \mathcal{J}^i$$

- $\mathcal{H}_a^{\text{full}}[A, E, \Psi, \Pi, A_a, \pi^a] = \mathcal{H}_a^{G+F} - \sqrt{\det h} q A_a \mathcal{J}^0 + F_{ab} \pi^b$

with $\mathcal{J}^I = \overline{\Psi} \gamma^I \Psi$ fermionic 4-current

EOM

Fermonic field

- Via E-L eq., fermionic EOM in curved space-time

$$(t^a - N^a)\left[i\gamma^0 D_a^{(A)}\xi - \frac{\beta}{2}K_a^i\gamma_5\gamma_i\xi - \mathrm{qA_a}\gamma^0\xi\right] - N\left[ie_i^a\gamma^i D_a^{(A)}\xi + \frac{1}{2}\epsilon^i{}_{jk}e_i^a K_a^j\gamma_5\gamma^k\xi - \frac{\beta}{2}e_i^a K_a^i\gamma_5\gamma^0\xi - m\xi - \mathrm{qA_a e_i^a}\gamma^{\mathrm{i}}\xi\right] = 0$$

- Via Hamiltonian formulation
  - Substituting ξ in $\mathcal{H}^{\text{full}}$ and $\mathcal{H}_a^{\text{full}}$
  - Using the anti-Poisson brackets

$\Longrightarrow$ fermionic EOM from Hamiltonian consistent with those from E-L method

EOM

Photon field

- Via E-L eq.
  - Photonic EOM in curved space-time

$$\nabla_c \left(g^{ac} g^{bd} F_{ad}\right) = q e_I^b \mathcal{J}^I$$

  - Substituting $g^{ab} = h^{ab} - n^a n^b$

$$\nabla_c \left(h^{ac} h^{bd} F_{ad}\right) + 2\nabla_c \left(\frac{1}{\sqrt{\det h}} n^{[b} \pi^{c]}\right) - q e_I^b \mathcal{J}^I = 0$$

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- Via Hamiltonian formulation
  - Using photonic Poisson brackets

$$\{A_a(x), \pi^b(y)\} = \delta^b_a \delta(x - y)$$

from which

$$\dot{\pi}^a = -\frac{\delta H^{\text{full}}}{\delta A_a}$$

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Parity Transformation

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- Fermionic contribution to extrinsic curvature

$$K_a^i \to K_a^i \underbrace{- \frac{\kappa}{4} \frac{\beta^2}{1+\beta^2} \left(\frac{1}{\beta} e_a^i J^0 + \epsilon^i{}_{jk} e_a^j J^k \right)}_{k_a^i}$$

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  - All constraints are invariant

Conclusion

- Modification to GR constraints in QED system
- Consistent results with EOM in curved space-time
- Invariance under parity operator

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- Polymerization $\overset{?}{\Longrightarrow}$ modification to propagators
- Transition to loop representation

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Thank you!

Constraints

Consider

$$S = \int_{t_1}^{t_2} L(q, \dot{q}) dt$$

If $det \left(\frac{\partial^2 L}{\partial \dot{q}^{n'} \partial \dot{q}^n} \right) = 0 \rightarrow$ Singular system $\implies p_n = \frac{\partial L}{\partial \dot{q}^n}$ not all independent

Hence, there are some relations

$$\phi_m(q, p) = 0 \text{ with } m = 1, ..., M \qquad \textbf{primary constraints}$$

that follow from the definition of the momenta.
From the consistency condition

$$\dot{\phi} = [\phi_m, H] + u^{m'} [\phi_m, \phi_{m'}] = 0 \rightarrow \varphi_k = 0 \text{ with } k = M + 1, ..., M + K$$

secondary constraints

with u^m Lagrange multipliers.

Weak Equality

A function f is weakly equal to a function g

$$f \approx g$$

if f and g are equal on the subspace defined by the primary constraints $\phi_m = 0$.

First and Second Class Constraints

A constraint is called "*first class*" if its Poisson bracket with all the constraints Ω_A vanishes weakly,

$$\{\Omega^{(1)}_{A_1}, \Omega_B\} \approx 0 \;\; ; \;\; A_1 = 1, ..., N^{(1)} \;,\;\; B = 1, ..., N$$

First class constraints generate gauge transformations.
A constraint that is not first class is called "*second class*".

Holonomies and fluxes

- Classical Poisson algebra of field theories is not strictly an algebra:
$$\{h_{ab}(x), p^{cd}(y)\} = \delta^c_{(a}\delta^d_{b)}\delta(x,y)$$
- To arrive at a well-defined algebra free of infinite coefficients $\rightarrow$ smearing
- BUT the space of all metrics is hard to control or to equip with a good measure $\implies$ consider connection variables (A^i_a, E^b_j)
- Well-defined quantum analogs: holonomies
$$h_c(A) = \mathcal{P}\exp\left(\int_c ds^a A^i_a \tau_i\right)$$
along a curve c, with $\tau_i = -\frac{i}{2}\sigma_i$, and fluxes of the densitized triad
$$E_n(S) := \int_S d^2y\, E^a_i n_a f^i\,,$$
with n^a co-normal to the surface and f^i Lie algebra-valued smearing function.