The quantum group structure of QRF transformations

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1. Introduction

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Quantum Gravity and Quantum Information both call for a generalization of the concept of reference frame, and with it, of reference frame transformations.

In Quantum Gravity: Quantum-spacetime reference frames.

The quantum properties of spacetime affect the description of spacetime events.

Quantum groups can define quantum-spacetime reference frame transformations.

In Quantum Information: Quantum-mechanical reference frames.

The quantum properties of the reference frame affect the description of quantum systems.

The (centrally extended) Galilei group is the Lie group of CRF transformations.

QRF transformations also close a Lie group, but different from the Galilei group.

What if a quantum group structure underlies the Lie group of QRF transformations?

2. Quantum Reference Frames

- The Lie algebra $\mathfrak{g}$:

$$[P_0, P_1] = 0\,, \quad [K, P_0] = P_1\,, \quad [K, P_1] = M\,, \quad [M, \cdot] = 0\,.$$

- The Lie group $G = \exp \mathfrak{g}$:

$$G = e^{\theta M}\, e^{bP_0}\, e^{aP_1}\, e^{vK}\,.$$

- The Lie group product $G = G' \cdot G''$:

$$e^{\theta M}\, e^{bP_0}\, e^{aP_1}\, e^{vK} = e^{\theta' M}\, e^{b' P_0}\, e^{a' P_1}\, e^{v' K} \cdot e^{\theta'' M}\, e^{b'' P_0}\, e^{a'' P_1}\, e^{v'' K}\,,$$

$$(\theta, b, a, v) = (\theta' + \theta'' + v'a'' + \tfrac{1}{2}v'^2 b'', b' + b'', a' + a'' + v'b'', v' + v'')\,.$$

- The Lie algebra representation $\rho : \mathfrak{g} \to \mathrm{End}(\mathcal{H})$:

$$\rho(M) = m\hat{1}\,, \quad \rho(P_0) = \frac{\hat{p}^2}{2m}\,, \quad \rho(P_1) = \hat{p}\,, \quad \rho(K) = -m\hat{q} + t\hat{p}\,,$$

with $[\hat{q}, \hat{p}] = i\hbar\hat{1}$.

Let O, O' be classical reference frames. Let $|\Psi\rangle_B \in \mathcal{H}_B$ be the quantum state of a system B with respect to O.

A CRF transformation from O to O' is

$$
\begin{aligned}
\hat{U} : \mathcal{H}_B &\to \mathcal{H}_B\,, \\
|\Psi\rangle_B &\mapsto |\Psi'\rangle_B = \hat{U}\,|\Psi\rangle_B\,,
\end{aligned}
$$

with $\hat{U} \in \mathcal{U}(\mathcal{H}_B)$.

Examples:

- $\hat{U}_{P_1} = e^{\frac{i}{\hbar} a \hat{P}_B}$, with a the relative position between O and O'.
- $\hat{U}_K = e^{\frac{i}{\hbar} v \hat{K}_B}$, with v the relative velocity between O and O'.

Let A, B, C be quantum systems. We now attach the reference frame O to system C, so that $|\Phi\rangle_A \otimes |\Psi\rangle_B \in \mathcal{H}_A \otimes \mathcal{H}_B$ is the quantum state of the system $A \otimes B$ with respect to $O(C)$.

A QRF transformation from $O(C)$ to $O'(A)$ is

$$
\begin{aligned}
\hat{U} : \mathcal{H}_A \otimes \mathcal{H}_B &\to \mathcal{H}_C \otimes \mathcal{H}_B \,, \\
|\Phi\rangle_A \otimes |\Psi\rangle_B &\mapsto |\Phi'\rangle_C \otimes |\Psi'\rangle_B = \hat{U}(|\Phi\rangle_A \otimes |\Psi\rangle_B) \,,
\end{aligned}
$$

with $\hat{U} \in \mathcal{U}(\mathcal{H}_A \otimes \mathcal{H}_B)$.

Extended Galilei transformations:

$$
\hat{U}_{P_1} = e^{\frac{i}{\hbar} \hat{q}_A \otimes \hat{P}_B} \,, \quad \hat{U}_K = e^{\frac{i}{\hbar} \frac{1}{m_A} \hat{p}_A \otimes \hat{K}_B} \,,
$$

with $[\hat{q}_A, \hat{p}_A] = i\kappa \hat{1}_A$ and $[\hat{q}_B, \hat{p}_B] = i\hbar \hat{1}_B$.

*[F. Giacomini, E. Castro-Ruiz, C. Brukner, Nat. Commun. **10**(1), 494 (2019)]*

The Galilei Lie group is lost.

What about the group structure?

We consider the algebra structure on $\mathcal{H}_A \otimes \mathcal{H}_B$ and define

$$\hat{P}_{AB} = \hat{q}_A \otimes \hat{P}_B\,, \quad \hat{K}_{AB} = \frac{1}{m_A}\hat{p}_A \otimes \hat{K}_B\,.$$

Can we close a Lie algebra with them?

The dynamical Lie algebra $\mathcal{D}(7) \subset \mathfrak{sp}(4,\mathbb{R})$, with 5 more elements:

$$\hat{D}_A = \frac{1}{2}(\hat{q}_A\hat{p}_A + \hat{p}_A\hat{q}_A) \otimes \hat{1}_B\,, \quad \hat{D}_B = \hat{1}_A \otimes \frac{1}{2}(\hat{q}_B\hat{p}_B + \hat{p}_B\hat{q}_B)\,,$$

$$\hat{Q}_A = \frac{\hat{p}_A^2}{2m_A} \otimes \hat{1}_B\,, \quad \hat{Q}_B = \hat{1}_A \otimes \frac{\hat{p}_B^2}{2m_B}\,,$$

$$\hat{T} = \hat{p}_A \otimes \hat{p}_B\,.$$

The Galilei Lie algebra is recovered as a subalgebra in the limit $\kappa \to 0$.

*[Á. Ballesteros, F. Giacomini, G. Gubitosi, Quantum **5**, 470 (2021)]*

3. Quantum group structure

Apparently, the dynamical Lie group $D(7)$ and the Galilei Lie group are unrelated.

What if $D(7)$ emerges from (a particular limit of) a quantum Galilei group?

- A quantum group G_α:

$$G_\alpha = e^{\theta\otimes M}\, e^{b\otimes P_0}\, e^{a\otimes P_1}\, e^{v\otimes K}\ .$$

- The group structure $G_\alpha = G'_\alpha \cdot G''_\alpha$:

$$e^{\theta\otimes M}\, e^{b\otimes P_0}\, e^{a\otimes P_1}\, e^{v\otimes K} = e^{\theta'\otimes M}\, e^{b'\otimes P_0}\, e^{a'\otimes P_1}\, e^{v'\otimes K} \cdot e^{\theta''\otimes M}\, e^{b''\otimes P_0}\, e^{a''\otimes P_1}\, e^{v''\otimes K}\ ,$$

$$(\theta, b, a, v) = (\theta' + \theta'' + v'a'' + \tfrac{1}{2}v'^2 b'', b' + b'', a' + a'' + v'b'', v' + v'') + \mathcal{O}(\alpha)\ .$$

- The quantum algebra $U_\alpha(\mathfrak{g})$:

$$[P_0, P_1] = \mathcal{O}(\alpha)\ , \quad [K, P_0] = P_1 + \mathcal{O}(\alpha)\ , \quad [K, P_1] = M + \mathcal{O}(\alpha)\ , \quad [M, \cdot] = \mathcal{O}(\alpha)\ .$$

In the context of Hopf algebras, there is a canonical element formalizing these properties: The Hopf algebra dual form.

[Á. Ballesteros, DFS, F. Giacomini, G. Gubitosi, arXiv: 2504.00569 (2025)]

A quantum Galilei group with commutative time

The Galilei Lie group admits 26 inequivalent quantum group structures.*

Which one?

We ask 2 requirements based on the structure of QRF transformations:

- The coordinate b is a central element.
- The commutator $[a, v]$ is nonzero.

Any solution? One!

The noncommutative algebra of coordinates of this quantum Galilei group is

$$[\theta, a] = -i\kappa\alpha\theta\,, \quad [\theta, v] = \frac{1}{2}i\kappa\alpha v^2\,, \quad [a, v] = i\kappa\alpha v\,, \quad [b, \cdot] = 0\,,$$

where:

- α is the deformation parameter.
- κ is the quantization parameter.

It is compatible with the Galilei Lie group composition of coordinates.

*[A. Opanowicz, J. Phys. A **31**(41), 8387 (1998)]

The Galilei Lie algebra also turns into a quantum Galilei algebra, such that

$$[P_0, P_1] = 0\,, \quad [K, M] = \tfrac{1}{2} i\kappa\alpha e^{\frac{\kappa}{\hbar}\alpha P_1} M^2\,, \quad [K, P_0] = i\hbar \frac{1 - e^{\frac{\kappa}{\hbar}\alpha P_1}}{\frac{\kappa}{\hbar}\alpha}\,, \quad [K, P_1] = -i\hbar e^{\frac{\kappa}{\hbar}\alpha P_1} M\,.$$

According to the QRF formalism, the quantum Galilei group coordinates are realized in phase space:

$$\hat{\theta} = \tfrac{1}{4}\phi(\hat{q}_A e^{\alpha \hat{p}_A} + e^{\alpha \hat{p}_A}\hat{q}_A)\,, \quad \hat{b} = t\,, \quad \hat{a} = \hat{q}_A\,, \quad \hat{v} = \phi e^{\alpha \hat{p}_A}\,,$$

with $[\hat{q}_A, \hat{p}_A] = i\kappa \hat{1}_A$, and ϕ with dimensions of velocity.

Note that:

- The coordinate a is realized as the position operator $\hat{q}_A$.
- The coordinate v is not realized as the velocity operator $\hat{v}_A = \frac{\hat{p}_A}{m_A}$.

At first order in α,

$$\hat{v} \approx \phi(1 + \alpha m_A \hat{v}_A)\,,$$

and, by taking $\alpha = \frac{1}{m_A \phi}$,

$$\hat{v} \approx \hat{v}_A + \phi \equiv \hat{v}'_A\,.$$

Analogously, the quantum Galilei group generators are realized in phase space:

$$\hat{M} = m_B e^{-\frac{\kappa}{2\hbar}\alpha\hat{p}_B}\,, \qquad \hat{P}_0 = \frac{1}{m_B(\frac{\kappa}{2\hbar}\alpha)^2}\left(\cosh\left(\tfrac{\kappa}{2\hbar}\alpha\hat{p}_B\right) - 1\right)\,,$$

$$\hat{P}_1 = \hat{p}_B\,, \qquad \hat{K} = -\tfrac{1}{2}m_B\left(e^{\frac{\kappa}{2\hbar}\alpha\hat{p}_B}\hat{q}_B + \hat{q}_B e^{\frac{\kappa}{2\hbar}\alpha\hat{p}_B}\right) + t\hat{p}_B\,,$$

with $[\hat{q}_B, \hat{p}_B] = i\hbar\hat{1}_B$.

According to the QRF formalism, let us consider the quantum Galilei group element

$$\hat{G}_{\alpha} = e^{\frac{i}{\hbar}\hat{\theta}\otimes\hat{M}}\, e^{\frac{i}{\hbar}\hat{b}\otimes\hat{P}_0}\, e^{\frac{i}{\hbar}\hat{a}\otimes\hat{P}_1}\, e^{\frac{i}{\hbar}\hat{v}\otimes\hat{K}}\,,$$

and define the operators

$$\hat{M}^{\alpha}_{AB} \equiv \hat{\theta}\otimes\hat{M}\,, \quad \hat{P}^{\alpha}_{0\,AB} \equiv \hat{b}\otimes\hat{P}_0\,, \quad \hat{P}^{\alpha}_{1\,AB} \equiv \hat{a}\otimes\hat{P}_1\,, \quad \hat{K}^{\alpha}_{AB} \equiv \hat{v}\otimes\hat{K}\,.$$

At first order in α, with $\alpha = \frac{1}{m_A\phi}$, and with $\hat{p}'_A$ the physical momentum operator:

$$\begin{aligned}
\hat{M}^{\alpha}_{AB} &= \tfrac{1}{2}\tfrac{m_B}{m_A}\big(\hat{D}_A - \tfrac{\kappa}{2\hbar}\hat{P}_{AB}\big)\,,\\
\hat{P}^{\alpha}_{0\,AB} &= t\hat{Q}_B\,,\\
\hat{P}^{\alpha}_{1\,AB} &= \hat{P}_{AB}\,,\\
\hat{K}^{\alpha}_{AB} &= \hat{K}_{AB} - \tfrac{\kappa}{2\hbar}\tfrac{m_B}{m_A}\hat{D}_B\,.
\end{aligned}$$

Finally, $\{\hat{M}^{\alpha}_{AB}, \hat{P}^{\alpha}_{0\,AB}, \hat{P}^{\alpha}_{1\,AB}, \hat{K}^{\alpha}_{AB}\}$ closes a higher-dimensional Lie algebra, which is (isomorphic to) the dynamical Lie algebra $\mathcal{D}(7)$.

4. Outlook

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A quantum group structure underlies the Lie group of QRF transformations.

The dynamical Lie group emerges from (the 1st-order of) a quantum Galilei group.

What about the action of the quantum Galilei group transformations on quantum states?

The Poisson–Lie Galilei group arises when the QRF is in a semiclassical state.

The dynamical Lie group arises when the QRF is in a superposition of semiclassical states.

Conjecture: The all-order quantum Galilei group describes QRF transformations when the QRF is in a generic quantum state.

A 1st-step toward bridging the quantization of physical reference frames and the quantization of spacetime.

[Á. Ballesteros, DFS, F. Giacomini, G. Gubitosi, arXiv: 2504.00569 (2025)]

THANK YOU!