

Foundations of Relational Quantum Field Theory

Scalar Quantum Fields

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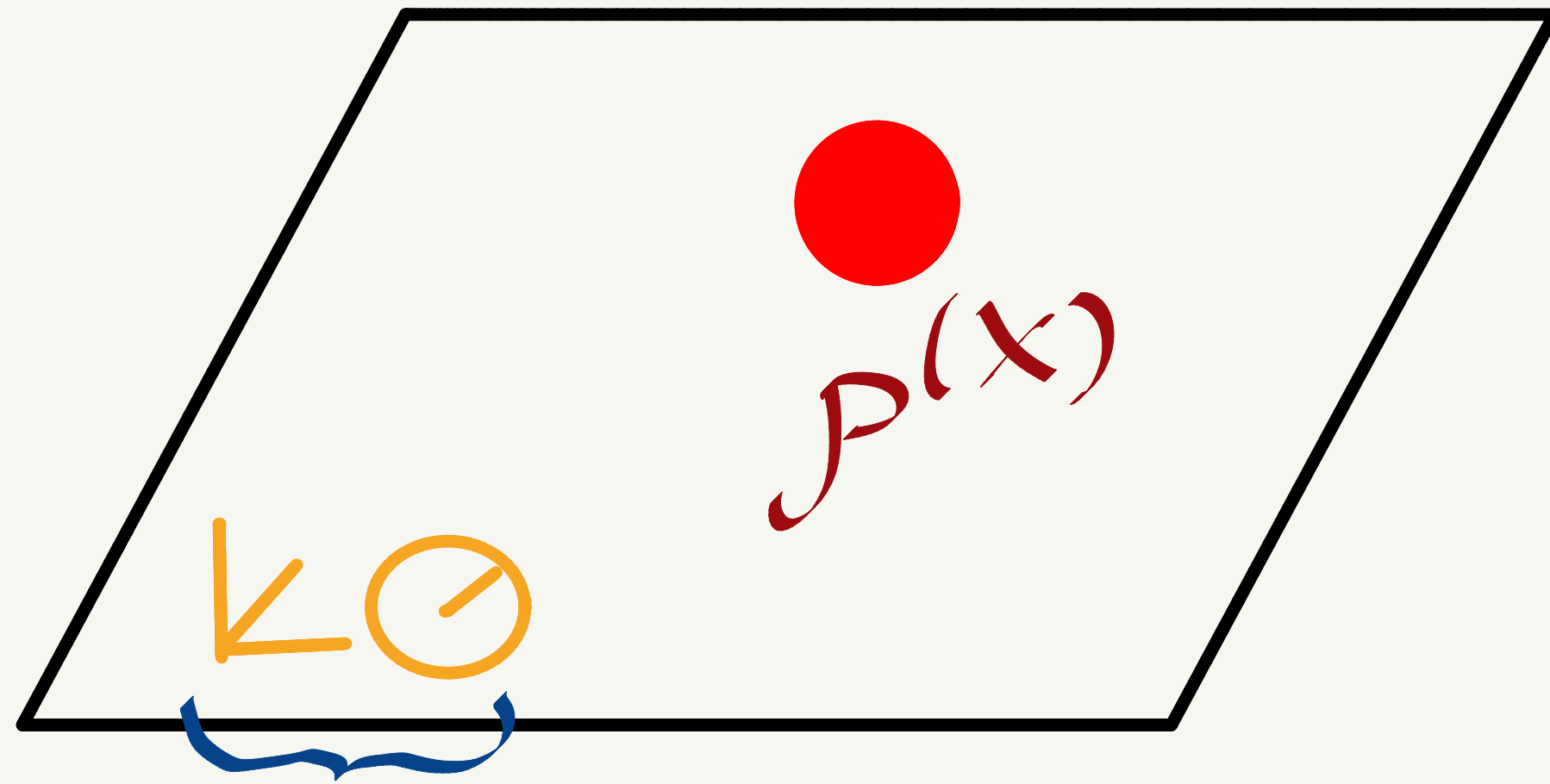
CQIF, DAMTP, University of Cambridge



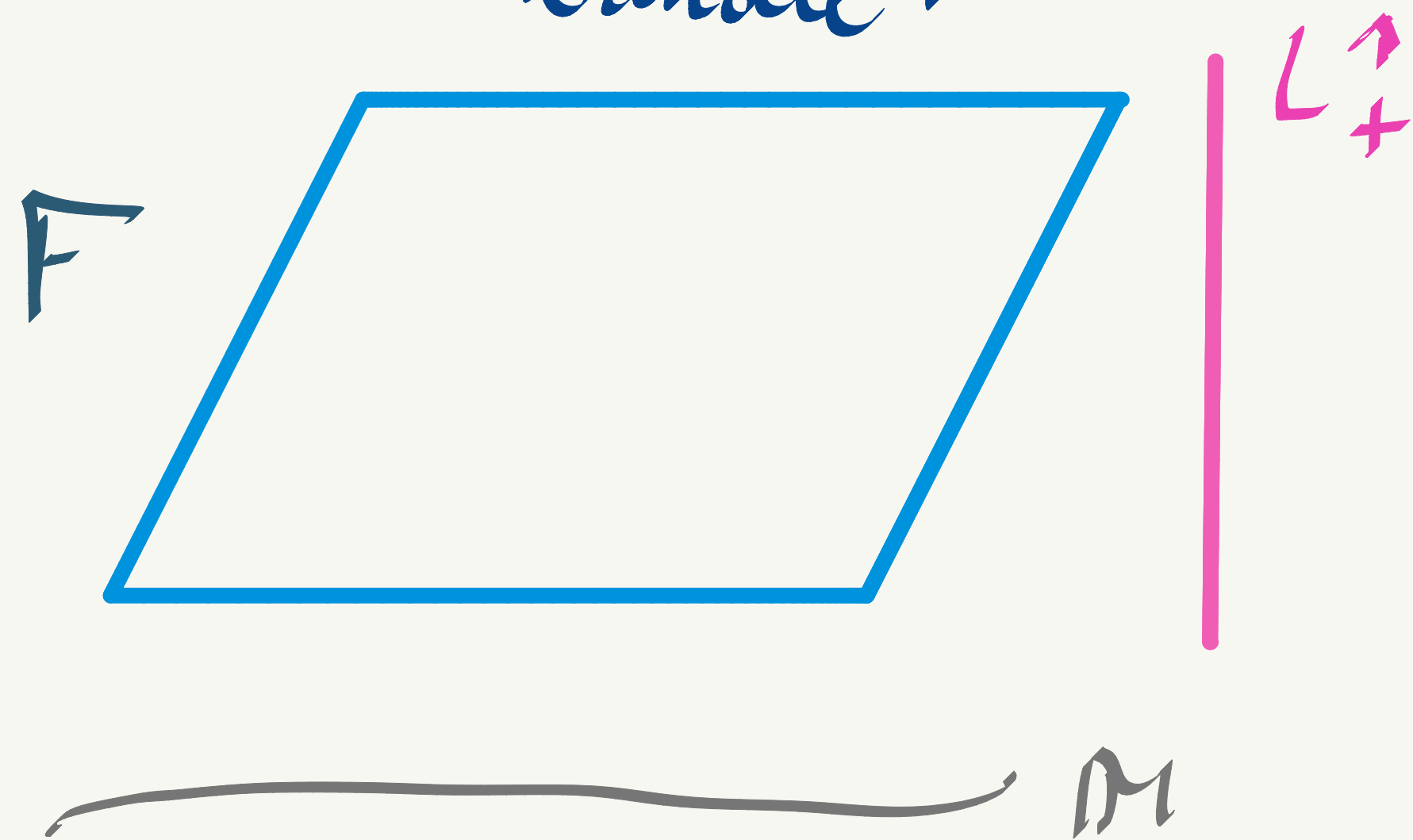
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I. Relativistic Reference Frames & Scalars

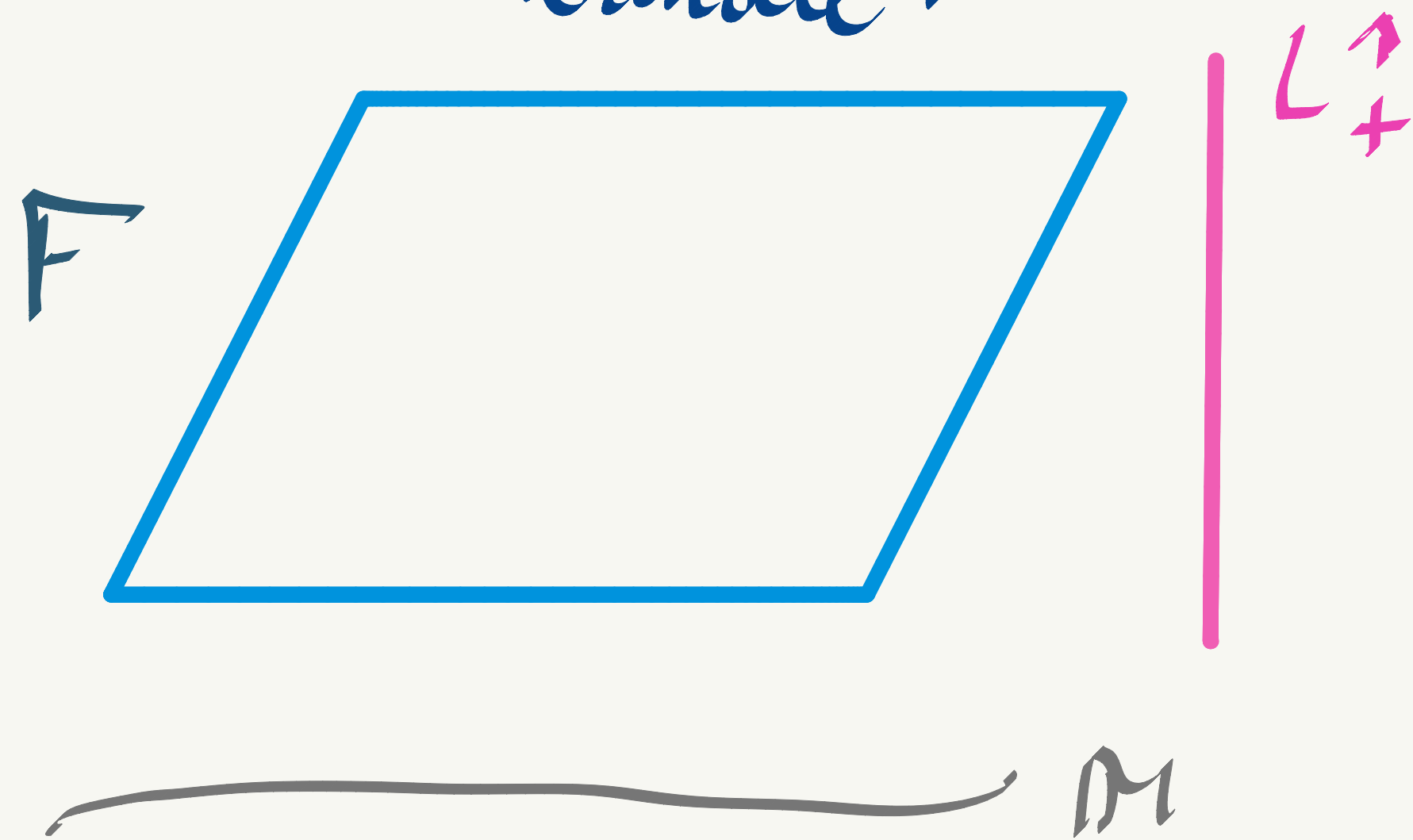
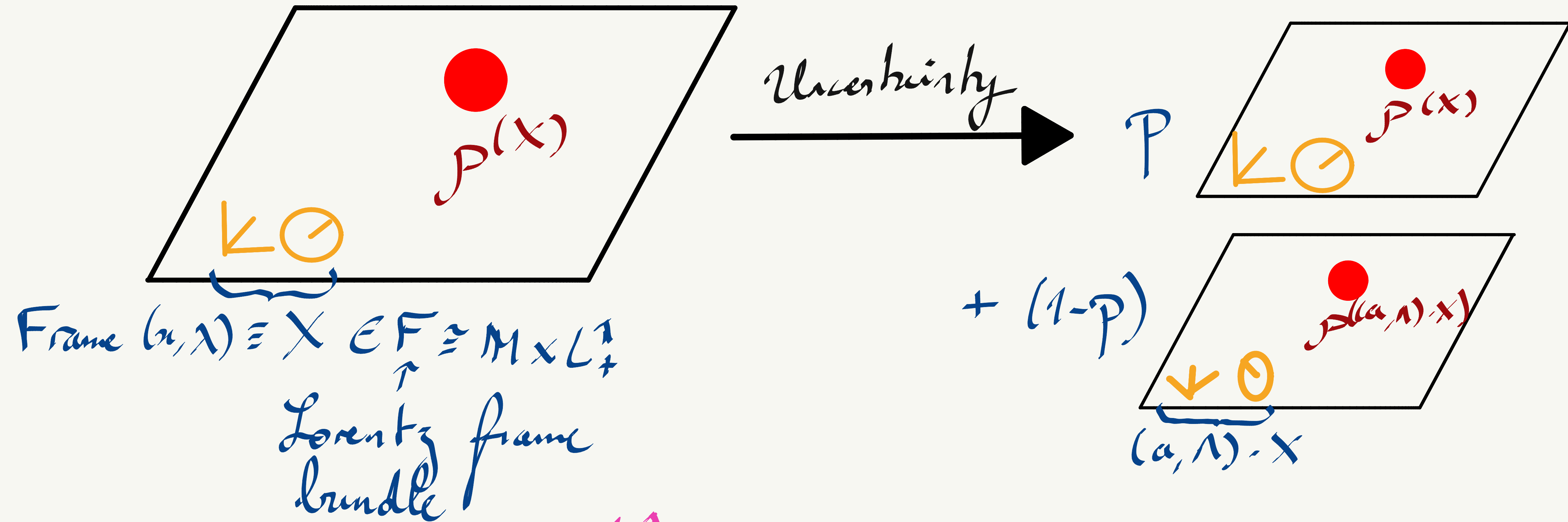


Frame $(u, \lambda) \equiv X \in F \cong M \times L_+^\uparrow$
 Lorentz frame
 bundle



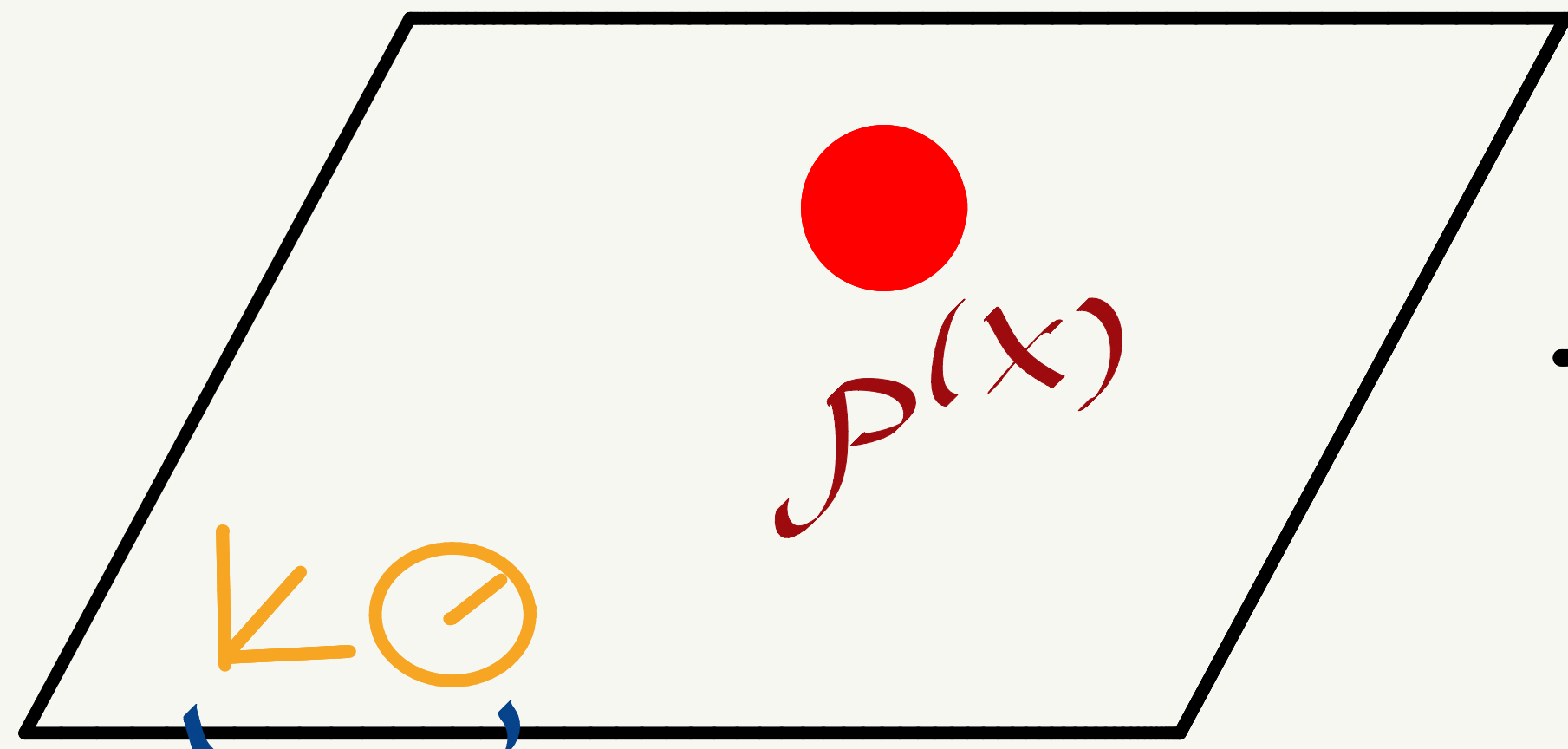
$$P_+^\uparrow \cap F : (a, \Lambda) \cdot (u, \lambda) = (\Lambda u + a, \Lambda \lambda)$$

I. Relativistic Reference Frames & Scalars

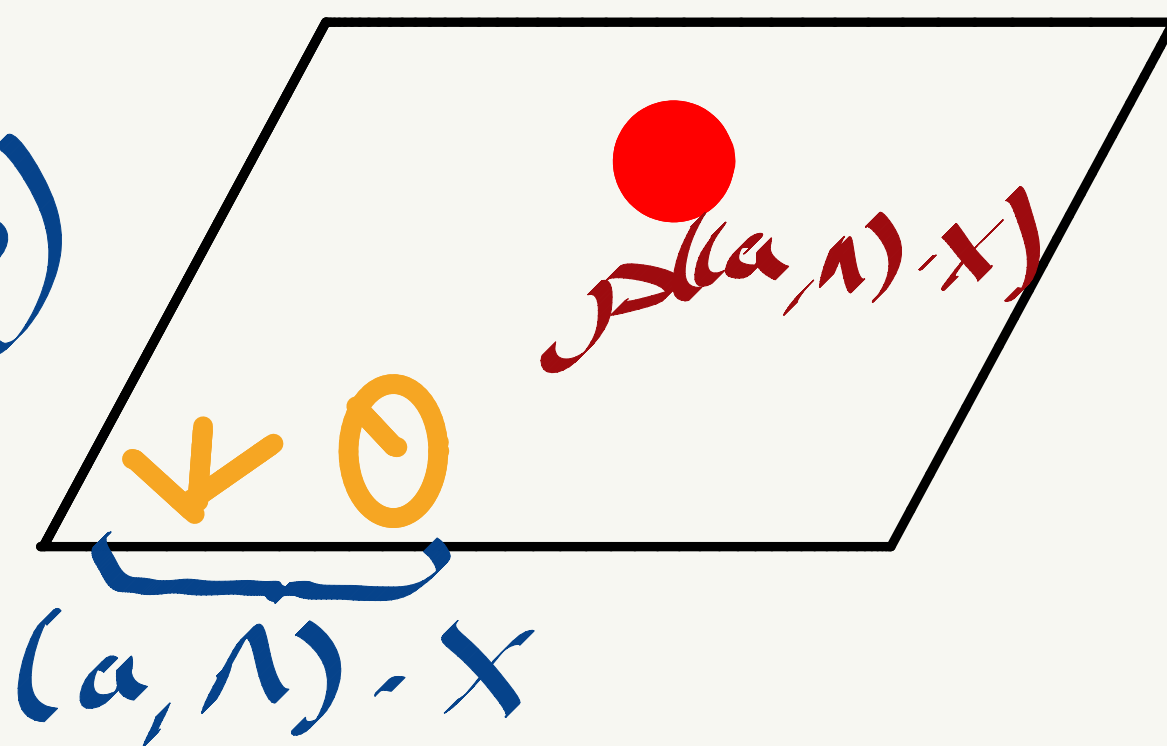
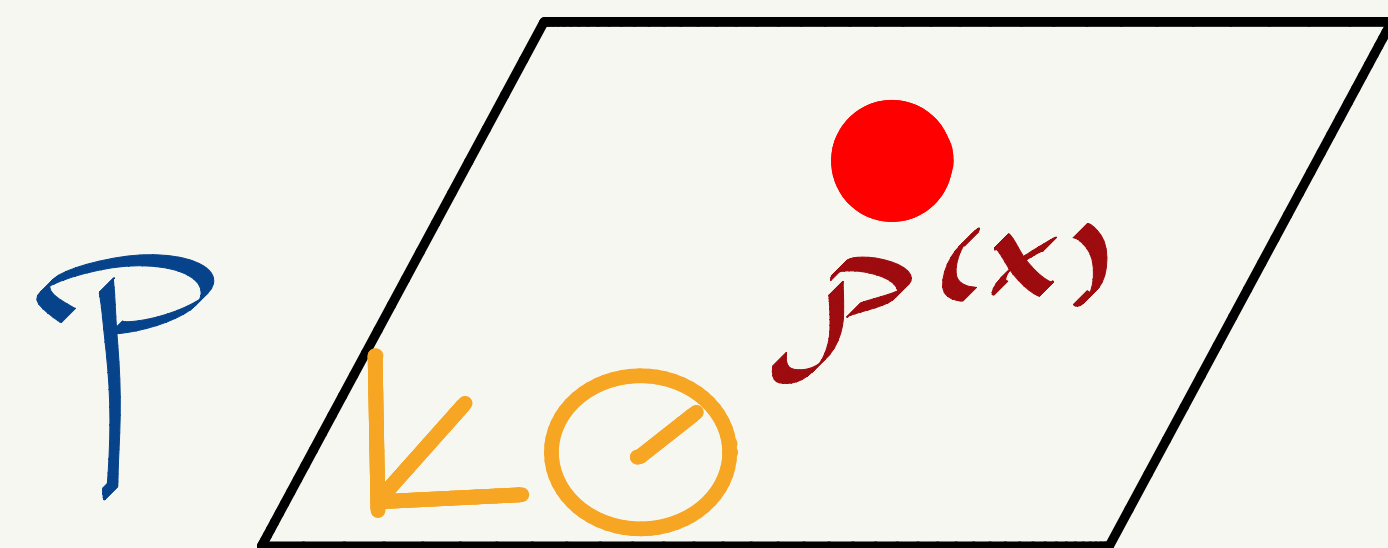


$$\mathcal{P}_+^\uparrow \cap F : (\alpha, \lambda) \cdot (\alpha, \lambda) = (\Lambda \alpha + a, \Lambda \lambda)$$

I. Relativistic Reference Frames & Scalars

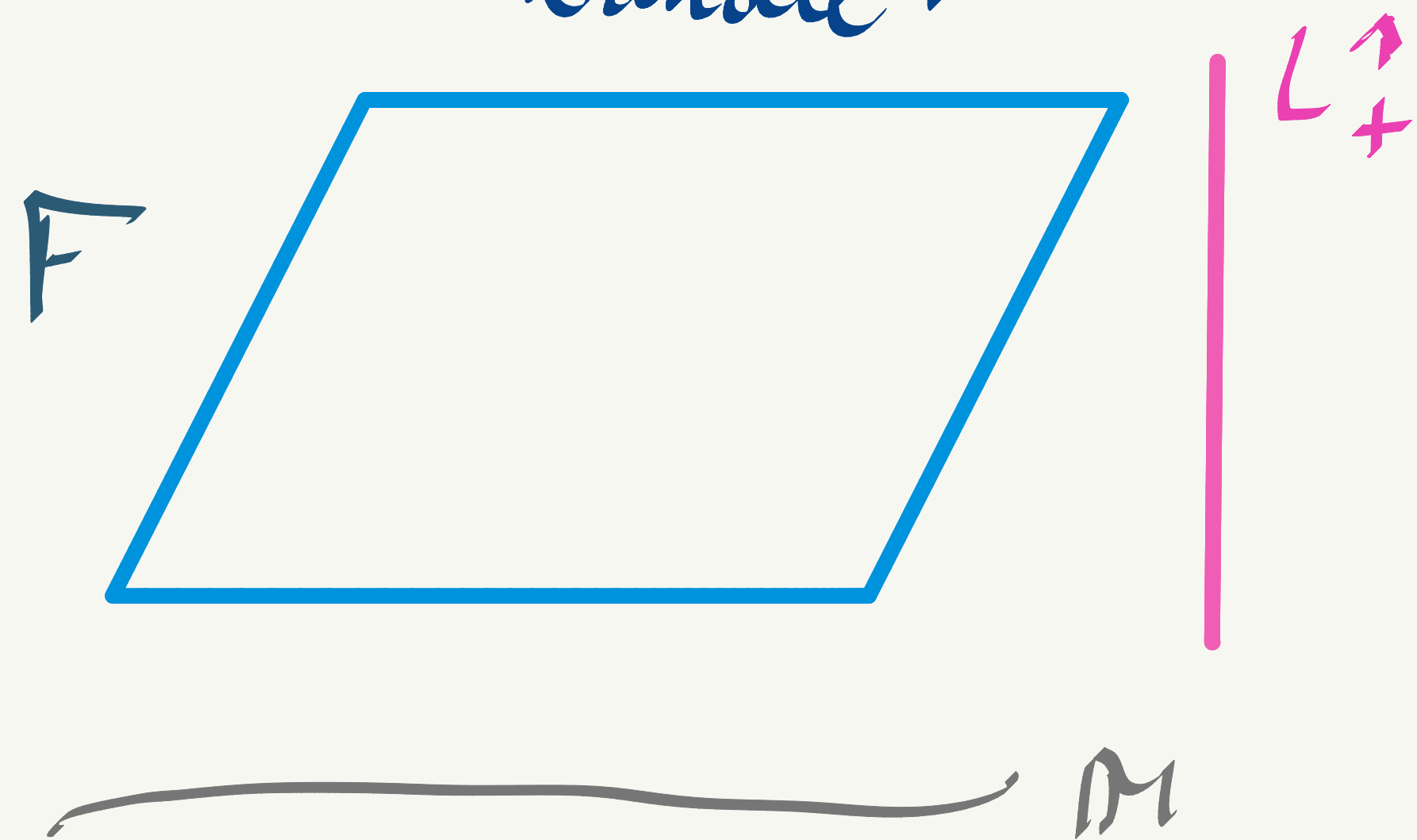


Uncertainty $\rightarrow$



+ (1-P)

Frame $(a, \lambda) \equiv x \in F \cong M \times L^1_+$
Lorentz frame
bundle



$$P^1_+ \cap F : (a, \Lambda) \cdot (x, \lambda) = (\Lambda x + a, \Lambda \lambda)$$

Probability distribution
 $\mu \in \text{Prob}(F)$

$$P^{(4)} := \int_F \text{[Diagram of a frame with red dot } P(x) \text{ and yellow frame symbol]} d\mu(x)$$

I. Relativistic Reference Frames & Scalars (continued)

By choosing* an "origin" $X_e = (0, e) \in F$, we get

$$\mathcal{P}^{(\mu)} = \int_F \mathcal{P}^{(\mu)} d\mu(X) = \int_F \mathcal{P}^{(\mu, \lambda) \cdot X_e} d\mu((x, \lambda) \cdot X_e)$$

* The construction
is independent of this
choice (non-trivial)

I. Relativistic Reference Frames & Scalars (continued)

By choosing* an "origin" $X_e = (0, e) \in F$, we get

$$\mathcal{P}^{(t)} = \int_F \mathcal{P}^{(x)} d\mu(X) = \int_F \mathcal{P}^{((x, \lambda), X_e)} d\mu((x, \lambda), X_e)$$

* The construction is independent of this choice (nontrivial)

(*) Assume the exp. val. of $\phi \in \mathcal{B}(\mathcal{H}_s)$ w.r.t. $\mathcal{P}^{((x, \lambda), X_e)}$
 $\iff$ the exp. val. of ϕ w.r.t. $\mathcal{P}^{(X_e)}$ through a unitary transformation (ultraeakly continuous)
 $\hookrightarrow \mathcal{P}_+^\uparrow \cap \mathcal{B}(\mathcal{H}_s) \xRightarrow{U_s} \mathcal{D}(\mathcal{H}_s) \cap \mathcal{P}_+^\uparrow$

Write

$$\mathcal{P} \cdot (a, \Lambda) = U_s(a, \Lambda)^* \mathcal{P} U_s(a, \Lambda) \quad (a, \Lambda) \cdot \phi = U_s(a, \Lambda) \phi U_s(a, \Lambda)^*$$

I. Relativistic Reference Frames & Scalars (continued)

$$\begin{aligned} \text{Tr}[\rho^{(4)} \phi] &= \int \text{Tr}[\rho^{(xe)}(x, \lambda) \phi] d\mu(x, \lambda) \\ &= \int_{\mathcal{F}}^{\mathcal{F}^*} \underbrace{\text{Tr}[\rho^{(xe)}(x, \lambda) \phi]}_{\equiv \mathcal{P}} d\mu(x, \lambda) \end{aligned}$$

$$\equiv \text{Tr}[\rho \phi^{(4)}]$$

* not a trivial abuse
of notation

Then

$$\phi^{(4)} = \int_{\mathcal{F}} (x, \lambda) \cdot \phi d\mu(x, \lambda)$$

I. Relativistic Reference Frames & Scalars (continued)

Then

$$\phi(\mu) = \int_F (x, \lambda) \cdot \phi \, d\mu(x, \lambda)$$

(**) Suppose reference frames are quantum mechanical $\Rightarrow$ QRFs with Hilbert space $\mathcal{H}_R$

$$\hookrightarrow E_R : \text{Bor}(F) \rightarrow \mathcal{E}(\mathcal{H}_R) \leftarrow \begin{array}{l} \text{space of} \\ \text{effects} \end{array}$$

$$\hookrightarrow P_+^{\mathcal{U}_R} \cap \mathcal{B}(\mathcal{H}_R), \quad D(\mathcal{H}_R) \cap \mathcal{B}(\mathcal{H}_R)$$

$$\hookrightarrow (\alpha, \Lambda) \cdot E_R(\mathcal{U} \times \tilde{\Lambda}) = E_R((\Lambda\mathcal{U} + \alpha) \times \Lambda\tilde{\Lambda}) \quad \begin{array}{l} \forall \mathcal{U} \in \text{Bor}(\mathcal{M}) \\ \forall \tilde{\Lambda} \in \text{Bor}(L_+^{\mathcal{U}}) \end{array}$$

E_R : quantum clocks and rods, ω : orientation (preparation)

$$\hookrightarrow \mu_{\omega}^{E_R} := \text{Tr}[\omega E_R(\cdot)] \quad \text{Born prob. measure}$$

I. Relativistic Reference Frames & Scalars (continued)

Then
$$\phi(\mu) = \int_F (x, \lambda) \cdot \phi \, d\mu(x, \lambda)$$

(**) Suppose reference frames are quantum mechanical $\Rightarrow$ QRFs with Hilbert space $\mathcal{H}_R$

$\hookrightarrow E_R : \text{Bor}(F) \rightarrow \mathcal{E}(\mathcal{H}_R) \leftarrow \begin{matrix} \text{space of} \\ \text{effects} \end{matrix}$

$\hookrightarrow P_+^{\mathcal{U}_R} \cap \mathcal{B}(\mathcal{H}_R), D(\mathcal{H}_R) \cap \mathcal{B}(\mathcal{H}_R)$

$\hookrightarrow (\alpha, \Lambda) \cdot E_R(\mathcal{U} \times \tilde{\Lambda}) = E_R((\Lambda\mathcal{U} + \alpha) \times \Lambda\tilde{\Lambda}) \quad \begin{matrix} \forall \mathcal{U} \in \text{Bor}(\mathcal{U}) \\ \forall \tilde{\Lambda} \in \text{Bor}(L_+^{\mathcal{U}}) \end{matrix}$

E_R : quantum clocks and rods, ω : orientation (preparation)

$\hookrightarrow \mu_{\omega}^{E_R} := \text{Tr}[\omega E_R(\cdot)]$ Born prob. measure

$\Rightarrow \phi_{\omega}^R := \int_F (x, \lambda) \cdot \phi \, d\mu_{\omega}^{E_R}(x, \lambda)$

Scalar relational
local observable

II. Scalar Quantum Fields

Defs. $\cdot \hat{\Phi}(\cdot) : M \times L_+^\uparrow \longrightarrow \mathcal{B}(\mathcal{H}_S) \quad \left. \begin{array}{l} \hat{\Phi}_\lambda(x) = (x, \lambda) \cdot \phi \end{array} \right\} \text{Lorentz-oriented quantum field}$
(scalar)

$\cdot \hat{\Phi} : M \longrightarrow \mathcal{B}(\mathcal{H}_S)$
 $\hat{\Phi}(x) := \hat{\Phi}_e(x) = (x, e) \cdot \phi \quad \left. \right\} \text{absolute quantum field}$
(scalar)

$$\Rightarrow \phi_\omega^R = \int_F \hat{\Phi}_\lambda(x) \, d\mu_\omega^{E_R}(x, \lambda)$$

II. Scalar Quantum Fields

Def_s. $\cdot \hat{\Phi}(\cdot) : M \times L_+^\uparrow \longrightarrow \mathcal{B}(\mathcal{H}_s)$ $\} \underline{\text{Lorentz-oriented quantum field}}$
 $\hat{\Phi}_\lambda(x) = (a, \lambda) \cdot \Phi$ $\} \text{(scalar)}$

$\cdot \hat{\Phi} : M \longrightarrow \mathcal{B}(\mathcal{H}_s)$ $\} \underline{\text{absolute quantum field}}$
 $\hat{\Phi}(x) := \hat{\Phi}_e(x) = (x, e) \cdot \Phi$ $\} \text{(scalar)}$

$$\Rightarrow \Phi_\omega^R = \int_F \hat{\Phi}_\lambda(x) d\mu_\omega^{E_R}(x, \lambda)$$

Lemma 1. $(a, \Lambda) \cdot \Phi_\omega^R = \Phi_{\omega \cdot (a, \Lambda)^{-1}}^R = \int_F \hat{\Phi}_{\Lambda\lambda}(\Lambda x + a) d\mu_\omega^{E_R}(x, \lambda)$

Important! $(a, \Lambda) \cdot \hat{\Phi}_\lambda(x) = \hat{\Phi}_{\Lambda\lambda}(\Lambda x + a) \neq \hat{\Phi}_\lambda(\Lambda x + a)$ (unless Φ is $L_+^\uparrow$ -inv)

THIS IS GOOD! Otherwise Wightman'ski would kill this formalism!

III. Relational Quantum Fields

Use disintegration of $\mu_\omega^{E_R}$:

$$d\mu_\omega^{E_R}(x, \lambda) = d\nu_\omega^{E_R}(\lambda|x) d\mu_\omega^{F_R}(x)$$

where $\cdot$ $F_R: \text{Bor}(M) \rightarrow \mathcal{E}(\mathcal{H}_R)$ is the marginal POVM

$\hookrightarrow P_+^\dagger$ covariant

$\hookrightarrow \mu_\omega^{F_R}$ marginal Born prob. measure

$\cdot$ $\nu_\omega^{E_R}$ is the conditional prob. measure

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$\hookrightarrow P_+^\uparrow$ covariant

$\hookrightarrow \mu_\omega^{F_R}$ marginal Born prob. measure

$\nu_\omega^{E_R}$ is the conditional prob. measure

Def. A relational scalar quantum field is a map

$$\hat{\Phi}_\omega^R: \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H}_S)$$

given by

$$\hat{\Phi}_\omega^R(x) := \int_{L_+^\uparrow} \hat{\Phi}_\lambda(x) d\nu_\omega^{E_R}(\lambda|x)$$

IV. Covariance

So

$$\phi_{\omega}^R = \int_M \hat{\phi}_{\omega}^R(x) d\mu_{\omega}^{F_R}(x)$$

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Thm. $(\alpha, \Lambda) \cdot \hat{\phi}_{\omega}^R(x) = \hat{\phi}_{\omega \cdot (\alpha, \Lambda)^{-1}}^R(\Lambda x + \alpha)$ for $\mu_{\omega}^{F_R}$ -a.e. $x \in M$.

$$\Rightarrow (\alpha, \Lambda) \cdot \phi_{\omega}^R = \int_M \hat{\phi}_{\omega \cdot (\alpha, \Lambda)^{-1}}^R(\Lambda x + \alpha) d\mu_{\omega}^{F_R}(x)$$

Covariance under the integral!

V Comparison with Wightman QFT

RQFT

$$\Phi^R: D(\mathcal{H}_R) \rightarrow B(\mathcal{H}_S)$$

$$\Phi^R(\omega) := \Phi_\omega^R = \int_M \hat{\Phi}_\omega^R(x) d\mu_\omega^F(x)$$

Schwartz
function
↓

WQFT

$$\Phi: S(M, \mathbb{C}) \rightarrow L(D, \mathcal{H}_S)$$

$$\Phi(f) = \int_M \hat{\Phi}(x) f(x) d^d x$$

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$$\Phi^R(\omega) := \phi_\omega^R = \int_M \hat{\phi}_\omega^R(x) d\mu_\omega^R(x)$$

$$(a, \Lambda) \cdot \Phi^R(\omega) = \Phi^R((a, \Lambda) \cdot \omega)$$

Schwartz
function
↓

WQFT

$$\Phi: \mathcal{S}(M, \mathbb{C}) \rightarrow \mathcal{L}(D, \mathcal{H}_S)$$

$$\Phi(f) = \int_M \hat{\phi}(x) f(x) d^d x$$

$$(a, \Lambda) \cdot \Phi(f) = \Phi((a, \Lambda) \cdot f)$$

$$\text{where } ((a, \Lambda) f)(x) = f(\Lambda^{-1}(x - a))$$

V Comparison with Wightman QFT

RQFT

$$\Phi^R: D(\mathcal{H}_R) \rightarrow B(\mathcal{H}_S)$$

$$\Phi^R(\omega) := \Phi_\omega^R = \int_M \hat{\Phi}_\omega^R(x) d\mu_\omega^R(x)$$

$$(a, \Lambda) \cdot \Phi^R(\omega) = \Phi^R((a, \Lambda) \cdot \omega)$$

If $\frac{d\mu_\omega^R}{d^d x} \equiv f_\omega^R$ exists & is Schwartz, $\Phi^R(\omega) = \int_M \hat{\Phi}_\omega^R(x) f_\omega^R(x) d^d x$ and

$$((a, \Lambda) \cdot f_\omega^R)(x) = f_{(a, \Lambda) \cdot \omega}^R(x) = f_\omega^R(\Lambda^{-1}(x-a))$$

Schwartz
function
↓

WQFT

$$\Phi: S(M, \mathbb{C}) \rightarrow L(D, \mathcal{H}_S)$$

$$\Phi(f) = \int_M \hat{\Phi}(x) f(x) d^d x$$

$$(a, \Lambda) \cdot \Phi(f) = \Phi((a, \Lambda) \cdot f)$$

$$\text{where } ((a, \Lambda) f)(x) = f(\Lambda^{-1}(x-a))$$

VI. Causality

Write $\omega_1 \perp^R \omega_2$ whenever $\text{supp } \rho_{\omega_1}^{F_R} \perp \text{supp } \rho_{\omega_2}^{F_R}$.

marginal
POVM $\rightarrow$

Einstein Causality:

$$\omega_1 \perp^R \omega_2 \Rightarrow [\Phi_{\omega_1}^R, \Phi_{\omega_2}^R] = [\mathbb{E}^R(\omega_1), \mathbb{E}^R(\omega_2)] = 0$$

Compare with Wightman

$$\text{supp } f_1 \perp \text{supp } f_2 \Rightarrow [\mathbb{E}(f_1), \mathbb{E}(f_2)] = 0$$

VII. Causality

Write $\omega_1 \perp^R \omega_2$ whenever $\text{supp } \rho_{\omega_1}^{F_R} \perp \text{supp } \rho_{\omega_2}^{F_R}$.

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Einstein Causality:

$$\omega_1 \perp^R \omega_2 \Rightarrow [\phi_{\omega_1}^R, \phi_{\omega_2}^R] = [\Phi^R(\omega_1), \Phi^R(\omega_2)] = 0$$

Compare with Wightman

$$\text{supp } f_1 \perp \text{supp } f_2 \Rightarrow [\Phi(f_1), \Phi(f_2)] = 0$$

We can recover EC by imposing microcausality (and/or SI conditions...) e.g.

Microcausality:

(avoids Wightman's
no-go thm because
 $\hat{\phi}_{\omega}^R$ are not $T(1,1)$ cov)

$$\omega_1 \perp^R \omega_2 \Rightarrow \begin{cases} [\hat{\phi}_{\omega_1}^R(x_1), \hat{\phi}_{\omega_2}^R(x_2)] = 0 \\ [\hat{\phi}_{\omega_1}^R(x_1)^\dagger, \hat{\phi}_{\omega_2}^R(x_2)] = 0 \end{cases} \quad \text{for } \rho_{\omega_1}^{F_R} \text{-a.e. } x_i$$

VII. Globally Oriented QRFs

Suppose (R, ω) acts as a "probabilistically smeared constant global section" in the Lorentz frame bundle:

$$v_{\omega}^{E_R}(\cdot | x) = v_{\omega}^{E_R}(\cdot | x') \Leftrightarrow \mu_{\omega}^{E_R} = \mu_{\omega}^{F_R} \otimes \mu_{\omega}^{G_R}$$

where $G_R(\Lambda) := E_R(M \times \Lambda)$ is the $L_+^{\uparrow}$ -marginal POVM.

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where $G_R(\Lambda) := E_R(M \times \Lambda)$ is the $L_+^{\uparrow}$ -marginal POVM.

Then $\cdot \supp \mu_{\omega}^{F_R} = M$

$$\cdot a \cdot \hat{\Phi}_{\omega}^R(x) = \hat{\Phi}_{\omega}^R(x+a)$$

BUT cannot impose microcausality (Wightman no-go).

VIII. Vacuum Expectation Values

(***) Suppose $\exists \Omega \in \mathcal{D}(\mathcal{H}_S)^{\mathcal{P}_+^{\uparrow}}$ ← Poincaré-invariant vacuum state
(possibly non-unique and/or mixed)

Defs: $\cdot \mathcal{W}_n^{(\mathcal{R}, \Omega)}[\omega_1, \dots, \omega_n] : \mathcal{B}(\mathcal{H}_S)^n \rightarrow \mathbb{C}$

$$\mathcal{W}_n^{(\mathcal{R}, \Omega)}[\omega_1, \dots, \omega_n](\phi_1, \dots, \phi_n) = \text{Tr} \left[\Omega \prod_{i=1}^n (\hat{\phi}_i)_{\omega_i}^{\mathcal{R}} \right]$$

$\cdot W_n^{(\mathcal{R}, \Omega)}[\omega_1, \dots, \omega_n; x_1, \dots, x_n] : \mathcal{B}(\mathcal{H}_S)^n \rightarrow \mathbb{C}$

$$W_n^{(\mathcal{R}, \Omega)}[\omega_1, \dots, \omega_n; x_1, \dots, x_n](\phi_1, \dots, \phi_n) = \text{Tr} \left[\Omega \prod_{i=1}^n (\hat{\phi}_i)_{\omega_i}^{\mathcal{R}}(x_i) \right]$$

$$\mathcal{W}_n^{(\mathcal{R}, \Omega)}[\omega_1, \dots, \omega_n] = \int \cdots \int_{\mathcal{M}^n} W_n^{(\mathcal{R}, \Omega)}[\omega_1, \dots, \omega_n; x_1, \dots, x_n] d\left(\prod_{i=1}^n \mu_{\omega_i}^{\mathcal{R}}\right)(x_1, \dots, x_n)$$

VIII. Vacuum Expectation Values

Properties: $\cdot \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1 \cdot (a, 1), \dots, \omega_n \cdot (a, 1)] = \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1, \dots, \omega_n]$

$$\cdot \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1, \dots, \omega_n](a, 1) \cdot \phi_1, \dots, (a, 1) \cdot \phi_n) \\ = \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1, \dots, \omega_n](\phi_1, \dots, \phi_n)$$

$$\cdot \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1, \dots, \omega_n](\phi_1, \dots, \phi_n)$$

$$= \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_n, \dots, \omega_1](\phi_n^+, \dots, \phi_1^+)$$

$$\cdot \omega_i \perp^{\mathbb{R}} \omega_{i+1} \Rightarrow \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1, \dots, \omega_n](\phi_1, \dots, \phi_n)$$

$$= \mathcal{Z}_n^{(\Omega, \mathbb{R})}[\omega_1, \dots, \omega_{i-1}, \omega_{i+1}, \omega_i, \omega_{i+2}, \dots, \omega_n](\phi_1, \dots, \phi_{i-1}, \phi_{i+1}, \phi_i, \phi_{i+2}, \dots, \phi_n)$$

$\vdots$

IX. Conclusion

- We get a natural notion of relational scalar quantum fields
 - We get covariance under the integral
 - Can discuss causality, statistical independence, ...
 - Vacuum expectation values have same properties as Wightman
 - Spinors can be recovered too ($L^{\uparrow}_+ \mapsto \tilde{L}^{\uparrow}_+$)
 - Relational Lagrangian density, relational Hamiltonian density
↳ Relational action
 - and a lot more - stay tuned for the paper (to appear soon on arXiv)
- (AQFT, Euclidean RQFT, vacuum polarisation, vacuum state orthogonality, ...)
- Thank you for your attention!

Supplementary Slide - Operational QRFs

Relativisation map

$$\Upsilon^R: \mathcal{B}(\mathcal{H}_S) \rightarrow \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_R)$$

$$\Upsilon^R(\phi) \equiv \phi^R = \int_{\mathcal{P}_+^\uparrow} (x, \lambda) \cdot \phi \otimes dE_R(x, \lambda)$$

Thm 1. $\phi^R \in \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_R)^{\mathcal{P}_+^\uparrow}$

Restriction map. $\Gamma_\omega: \mathcal{B}(\mathcal{H}_S \otimes \mathcal{H}_R) \rightarrow \mathcal{B}(\mathcal{H}_S) \quad , \omega \in \mathcal{D}(\mathcal{H}_R)$

$$\Gamma_\omega(A) = \text{Tr}_{\mathcal{H}_R}[\mathbb{1} \otimes \omega A]$$

Restricted relativisation map

$$\Upsilon_\omega^R: \mathcal{B}(\mathcal{H}_S) \rightarrow \mathcal{B}(\mathcal{H}_S)$$

$$\phi_\omega^R \equiv \Upsilon_\omega^R(\phi) := \Gamma_\omega \circ \Upsilon^R(\phi) = \int_{\mathcal{P}_+^\uparrow} (x, \lambda) \cdot \phi \, d\mu_\omega^{E_R}(x, \lambda)$$

Supplementary Slides - Principles

1) Universality of QM $\rightarrow$ Hilbert space $\mathcal{H}$ & standard QM

2) Relationality & Operationality $\rightarrow$ Statements should be made at the level of expectation values and nothing more (as far as possible)

$\downarrow$

$$\mathcal{H} \cong \mathcal{H}_s \otimes \mathcal{H}_R$$

$$E_R: \text{Bor}(\overline{F}) \rightarrow \mathcal{E}(\mathcal{H}_R) \Rightarrow \text{QRF } R$$

Accessible operators w.r.t. (R, ω) in $\text{Im}(\overline{F}_\omega^R)$

** modified beyond Minkowski*

3) Symmetry $\rightarrow G \cong \overline{P}_+^{\uparrow*}(\times H) \curvearrowright \mathcal{B}(\mathcal{H}_s), \mathcal{B}(\mathcal{H}_R)$

$\rightarrow$ Covariance of expectation values in $\mathcal{H}_s$ w.r.t. $\mathcal{H}_R$
 $\hookrightarrow E_R$ is covariant

4) Vacuum state orthogonality* $\rightarrow$ QRF is not prepared in vacuum state

$$5) \exists \Omega \in \mathcal{D}(\mathcal{H}_s)^{\overline{P}_+^{\uparrow*}}$$

Supplementary Slide - Independence on choice of origin

$$\mathcal{P}^{(g_1 g_2 X)} = \mathcal{P}^{(X)} \cdot (g_2 g_1) \quad , \quad S_x: \mathcal{P}_+^1 \rightarrow F, \quad S_x(g_1 g_2) = g_2^{-1} S_x(g_1)$$

$$S_x(e) = X, \quad S_x(g) = g \cdot X. \quad X' := h \cdot X \Rightarrow S_{x'}(e) = h \cdot X$$

$$S_{x'}(g) = g X' = g h X = S_x(g h) \quad , \quad S_x(g) = S_{x'}(g h^{-1})$$

$$= h \cdot S_{x'}(g) = L_h \circ S_{x'}(g)$$

$$\mu_x = \mu \circ S_x^{-1} = \mu \circ S_{x'}^{-1} \circ L_h^{-1} = \mu_{x'} \circ L_h^{-1}$$

$$\mathcal{P}_x^{(\mu)} = \int_G \mathcal{P}^{(X)} \cdot g \, d\mu_x(g) = \int_G \mathcal{P}^{(X)} \cdot g \, d(\mu_{x'} \circ L_h^{-1})(g)$$

$$= \int_G \mathcal{P}^{(X)} \cdot (hg) \, d\mu_{x'}(g) = \mathcal{P}_{x'}^{(\mu)} \quad \square$$

Supplementary slide - Integration over bundle vs group

Let $\mu_P^{(x)}: \text{Bor}(\mathcal{P}_+^\pi) \rightarrow [0,1]$, $d\mu_P^{(x)}(x,\lambda) := d\mu_F((x,\lambda), X)$

$$\begin{aligned} \Rightarrow \mathcal{P}^{(x)} &= \int_F \mathcal{P}^{(x,\lambda), X} d\mu_F((x,\lambda), X) \\ &= \int_{\mathcal{P}_+^\pi} \mathcal{P}^{(x)} \cdot (x,\lambda) d\mu_P^{(x)}(x,\lambda) \end{aligned}$$

QRFs: $E_R: \text{Bor}(F) \rightarrow \mathcal{E}(\mathcal{H}_R)$

Let $\tilde{\mu}^{(x)}(E_R, \omega): \text{Bor}(\mathcal{P}_+^\pi) \rightarrow [0,1]$, $d\tilde{\mu}^{(x)}(E_R, \omega)(x,\lambda) := d\mu_\omega^{E_R}((x,\lambda), X)$

$$\phi_\omega^R = \int_F \phi^{(x,\lambda), X} d\mu_\omega^{E_R}((x,\lambda), X) = \int_{\mathcal{P}_+^\pi} (x,\lambda) \cdot \phi^{(x)} d\tilde{\mu}^{(x)}(E_R, \omega)(x,\lambda)$$

Supplementary Slide - Lagrangian densities

Up to bounded vs unbounded, consider $\partial_\mu \phi := i[\mathcal{P}_\mu, \phi]$.

Let

$$\mathcal{L} := \frac{m^2}{2} \phi^2 + \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi) \in \mathcal{L}(\mathcal{D}, \mathcal{H}_s)$$

Defn. $\hat{\mathcal{L}}_\lambda^{(1)}: L_+^\infty \times \mathcal{M} \rightarrow \mathcal{L}(\mathcal{D}, \mathcal{H}_s)$, $\hat{\mathcal{L}}_\lambda(x) := (x, \lambda) \cdot \mathcal{L}$

↳ Absolute (oriented) Lagrangian density $= \frac{m^2}{2} \phi^2 + \frac{1}{2} (\partial_\mu \phi) (\partial^\mu \phi)$

Then $\mathbb{F}_\omega^{\mathcal{L}}(Z) = \int_{\mathcal{F}} \hat{\mathcal{L}}_\lambda(x) d\mu_\omega^{\mathcal{E}_\mathcal{L}}(x, \lambda) = \int_{\mathcal{M}} \hat{\mathcal{L}}_\omega^{\mathcal{L}}(x) d\mu_\omega^{\mathcal{E}_\mathcal{L}}(x)$

generalised $\nearrow$
Lagrangian (action operator)

where $\hat{\mathcal{L}}_\omega^{\mathcal{L}}(x) := \int_{L_+^\infty} \hat{\mathcal{L}}_\lambda(x) d\nu_\omega^{\mathcal{E}_\mathcal{L}}(\lambda | x)$

Relational Lagrangian density

Supplementary Slide - Vacuum state orthogonality

Giannitsopoulos, 1998: Let $F: \text{Bor}(M) \rightarrow \mathcal{E}(\mathcal{H})$ be a translation covariant PDDM. Suppose $\Omega \in D(\mathcal{H})^{T(1,d-1)}$. Then $\text{Tr}[\Omega F(U)] = 0 \ \forall \text{ precompact } U \in \text{Bor}(M)$. If Ω is faithful then $F(U) = 0 \ \forall \text{ precompact } U$.

Note: $T(1,d-1)$ is amenable $\Rightarrow \Omega \in D(\mathcal{H})^{T(1,d-1)}$ is (almost) guaranteed to exist.

$\Rightarrow F_R$ is not normalized $\Rightarrow E_R$ is not normalized

$\Rightarrow \text{VSD: } \text{Tr}[F_R(M)p] = 1 \ \forall p \in D(\mathcal{H}) - D(\mathcal{H})^{T(1,d-1)}$

If $\exists \tilde{\Omega} \in D(\mathcal{H}_R)^{\mathcal{P}_+^n}$ pure and U_R satisfies the spectrum condition

$\Rightarrow$ Reeh-Schlieder: $\exists U \perp \mathcal{V}_{\text{precompact}} \text{ s.t. } [F_R(U), F_R(\mathcal{O})] \neq 0$

Supplementary Slide - RQFs and Covariance

Formally, $\hat{\Phi}_{\omega}^R : \text{supp } \mu_{\omega}^{F_R} \rightarrow \mathcal{B}(\mathcal{H}_S)$

This can be extended (by abuse of notation) to

$$\hat{\Phi}_{\omega}^R : \mathbb{M} \rightarrow \mathcal{B}(\mathcal{H}_S)$$

$$\hat{\Phi}_{\omega}^R(x) = \begin{cases} \hat{\Phi}_{\omega}^R(x) & \text{if } x \in \text{supp } \mu_{\omega}^{F_R} \\ 0 & \text{otherwise} \end{cases} \Rightarrow \text{same under the integral}$$

Covariance: If $x \in \text{supp } \mu_{\omega}^{F_R}$ it may be that $\Lambda x + a \notin \text{supp } \mu_{\omega}^{F_R}$

so if $\hat{\Phi}_{\omega}^R(x) \neq 0$ it could be that $\hat{\Phi}_{\omega}^R(\Lambda x + a) = 0$

$\Rightarrow$ contradiction if we had $U(a, \Lambda) \underbrace{\hat{\Phi}_{\omega}^R(x)}_{\neq 0} U(a, \Lambda)^{\dagger} = \hat{\Phi}_{\omega}^R(\Lambda x + a) = 0$

Instead, by covariance, $x \in \text{supp } \mu_{\omega}^{F_R} \Rightarrow \Lambda x + a \in \text{supp } \mu_{\omega \cdot (a, \Lambda)^{-2}}^{F_R} \quad \checkmark$